MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 4 SOLUTIONS

3.22: Prove that the following limits do not exist:

$$\lim_{x\to 0}\frac{|x|}{x}, \qquad \lim_{x\to 0} e^x \cos\frac{1}{x}, \quad \lim_{x\to 0} e^{-1/x}.$$

SOLUTION. If these limits exist and equal $L_1, L_2, L_3 \in \mathbb{R}$ respectively, then by Theorem **3.6**,

$$L_1 = \lim_{n\to\infty}\frac{|1/n|}{1/n} = 1 \quad \text{and} \quad L_1 = \lim_{n\to\infty}\frac{|-1/n|}{-1/n} = -1,$$

$$L_2 = \lim_{n\to\infty} e^{1/2n\pi}\cos\frac{1}{1/2n\pi} = 1 \quad \text{and} \quad L_2 = \lim_{n\to\infty} e^{1/(2n+\frac{1}{2})\pi}\cos\frac{1}{1/(2n+\frac{1}{2})\pi} = 0,$$

$$L_3 = \lim_{n\to\infty} e^{-1/(1/n)} = \lim_{n\to\infty} e^{-n} = 0 \quad \text{and} \quad L_3 = \lim_{n\to\infty} e^{-1/(-1/n)} = \lim_{n\to\infty} e^{n} = \infty,$$

which give contradictions in all cases since the values of L_1, L_2, L_3, if exist, must be unique by Theorem **3.3**.

3.24: Prove: $\lim_{x\to a} f(x) = L$ implies $\lim_{x\to a}|f(x)| = |L|$.
SOLUTION. Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenver $0 < |x - a| < \delta$. By triangle inequality, $||f(x)| - |L|| \le |f(x) - L| < \varepsilon$ whenver $0 < |x - a| < \delta$. Hence $\lim_{x\to a}|f(x)| = |L|$.

3.26: Prove: $\lim_{x\to a}|f(x)| = 0$ implies $\lim_{x\to a} f(x) = 0$.
SOLUTION. Let $\varepsilon > 0$. Since $\lim_{x\to a}|f(x)| = 0$ there exists $\delta > 0$ such that $||f(x)| - 0| < \varepsilon$ whenver $0 < |x - a| < \delta$. But $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \varepsilon$ whenver $0 < |x - a| < \delta$. Hence $\lim_{x\to a} f(x) = 0$.

3.27: Prove: If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ and for all $x \in N^*_\delta(a)$, $f(x) \le g(x) \le h(x)$ then $\lim_{x\to a} g(x) = L$.
SOLUTION. Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that $|f(x) - L| < \varepsilon$ whenver $0 < |x - a| < \delta_1$. There exists $\delta_2 > 0$ such that $|h(x) - L| < \varepsilon$ whenver $0 < |x - a| < \delta_2$. In particular

$$-\varepsilon < f(x) - L \quad \text{and} \quad h(x) - L < \varepsilon.$$

Set $\delta' := \min(\delta_1, \delta_2, \delta)$. So when $0 < |x - a| < \delta'$, we get

$$-\varepsilon < f(x) - L \le g(x) - L \le h(x) - L < \varepsilon$$

and so $|g(x) - L| < \varepsilon$. Hence $\lim_{x\to a} g(x) = L$.

3.29: Prove: $\lim_{x\to a}[(\sin x)/x] = 1$. *Hint*: First give a geometric argument to show that $0 < \sin x < x < \tan x$ for every $x \in (0, \pi/2)$ and then observe that $(\sin x)/x$ is an even function. Finally apply the result of Exercise **3.27**.
SOLUTION. By the remark in the solution to Exercise **3.1** in Homework **3**, we have $0 < \sin x < x < \tan x$ for every $x \in (0, \pi/2)$. Since $\sin x > 0$ for $x \in (0, \pi/2)$, we may divide the

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inequality by $\sin x$ to get

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x}.$$

Now observe that this holds for any $x \in (-\pi/2, 0) \cup (0, \pi/2)$ since $(-x)/\sin(-x) = x/\sin x$. Finally applying Exercise **3.27** with $a = 0$, $\delta = \pi/2$, $f(x) = 1$, $g(x) = x/\sin x$, $h(x) = 1/\cos x$, and $L = 1$, we get

$$\lim_{x\to 0} \frac{x}{\sin x} = 1.$$

The required limit then follows from Theorem **3.4**(c).

3.31: Prove: If $\lim_{x\to a} f(x) = 0$ and g is bounded on some deleted neighborhood of $x = a$ then $\lim_{x\to a}[f(x) \cdot g(x)] = 0$.
SOLUTION. There exists $M > 0$ such that $|g(x)| \le M$ for $x \in N^*_{\delta_1}(a)$. Let $\varepsilon > 0$. Then there exists $\delta_2 > 0$ such that $|f(x) - 0| < \varepsilon/M$ whenver $0 < |x - a| < \delta_2$. Set $\delta = \min(\delta_1, \delta_2)$. Then when $0 < |x - a| < \delta$, we get

$$|f(x) \cdot g(x) - 0| = |f(x)||g(x)| < \frac{\varepsilon}{M} \times M = \varepsilon.$$

Hence $\lim_{x\to a}[f(x) \cdot g(x)] = 0$.

3.32: Suppose f is defined on (α, β) and f is unbounded on $[a, b] \subset (\alpha, \beta)$. Prove that for some $x_0 \in [a, b]$, $\lim_{x\to x_0} f(x)$ does not exist.
SOLUTION. Suppose not and $\lim_{x\to x_0} f(x)$ exists for all $x_0 \in [a, b]$. By the Lemma on pp. 72, f is bounded at x_0 according to the definition in the last paragraph on pp. 61. So since $[a, b]$ is compact, f is bounded on $[a, b]$ by Theorem **3.2**, a contradiction.

3.33: Using the definitions, prove the following one-sided limits:

$$\lim_{x\to 1^+} [\text{sgn}(1 - x^2)] = -1, \qquad \lim_{x\to 0^-} \frac{1}{\lfloor x \rfloor} = -1, \qquad \lim_{x\to 1^+} e^{-1/(x+1)} = 0.$$

SOLUTION. Let $\varepsilon > 0$.

$$\begin{aligned}
|\text{sgn}(1 - x^2) - (-1)| < \varepsilon \quad &\Leftarrow \quad |\text{sgn}[(1 - x)(1 + x) + 1| < \varepsilon \\
&\Leftarrow \quad |\text{sgn}[(1 - x)(1 + x) + 1| < \varepsilon \quad \text{and} \quad x > 1 \\
&\Leftarrow \quad |-1 + 1| < \varepsilon \quad \text{and} \quad x > 1 \\
&\Leftarrow \quad 0 < \varepsilon \quad \text{and} \quad x > 1 \\
&\Leftarrow \quad 0 < x - 1 < \delta
\end{aligned}$$

where any $\delta > 0$ would work.
Let $\varepsilon > 0$.

$$\begin{aligned}
\left|\frac{1}{\lfloor x \rfloor} - (-1)\right| < \varepsilon \quad \text{and} \quad x \ne 0 \quad &\Leftarrow \quad \left|\frac{1}{\lfloor x \rfloor} + 1\right| < \varepsilon \quad \text{and} \quad -1 < x < 0 \\
&\Leftarrow \quad \left|\frac{1}{-1} + 1\right| < \varepsilon \quad \text{and} \quad -1 < x < 0 \\
&\Leftarrow \quad 0 < \varepsilon \quad \text{and} \quad -1 < x - 0 < 0 \\
&\Leftarrow \quad -1 < x - 0 < 0 \\
&\Leftarrow \quad -\delta < x - 0 < 0
\end{aligned}$$

where we have set $\delta = 1$.

Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $e^{-N} < \varepsilon$.

$$\begin{aligned}
\left|e^{-1/(x+1)} - 0\right| < \varepsilon \quad \text{and} \quad x \neq -1 \quad &\Leftarrow \quad e^{-1/(x+1)} < \varepsilon \quad \text{and} \quad x \neq -1 \\
&\Leftarrow \quad e^{-1/(x+1)} < \varepsilon \quad \text{and} \quad 0 < x+1 < \frac{1}{N} \\
&\Leftarrow \quad 0 < x - (-1) < \delta
\end{aligned}$$

where we have set $\delta = 1/N$. Note that when $0 < x+1 < 1/N$, we get $1/(x+1) > N$ and so $-1/(x+1) < -N$ and so $e^{-1/(x+1)} < e^{-N} < \varepsilon$.

3.40: Using Definition **3.5**, prove:

$$\lim_{x\to 2} \frac{x}{|x-2|} = \infty, \qquad \lim_{x\to -1} \frac{-1}{(x+1)^2} = -\infty, \qquad \lim_{x\to 0} \frac{1}{\sqrt{|x|}} = \infty, \qquad \lim_{x\to -2} \frac{x-2}{x^3+4x^2+4x} = \infty.$$

Solution. Let $M > 0$.

$$\begin{aligned}
\frac{x}{|x-2|} > M \quad \text{and} \quad x \neq 2 \quad &\Leftarrow \quad \frac{x}{|x-2|} > M \quad \text{and} \quad 0 < |x-2| < 1 \\
&\Leftarrow \quad \frac{x}{|x-2|} > M \quad \text{and} \quad \begin{cases} 0 < x-2 < 1 \\ -1 < x-2 < 0 \end{cases} \\
&\Leftarrow \quad \frac{x}{|x-2|} > M \quad \text{and} \quad \begin{cases} 2 < x < 3 \\ 1 < x < 2 \end{cases} \\
&\Leftarrow \quad \frac{1}{|x-2|} > M \quad \text{and} \quad \begin{cases} 2 < x < 3 \\ 1 < x < 2 \end{cases} \\
&\Leftarrow \quad \frac{1}{|x-2|} > M \quad \text{and} \quad 0 < |x-2| < 1 \\
&\Leftarrow \quad |x-2| < \frac{1}{M} \quad \text{and} \quad 0 < |x-2| < 1 \\
&\Leftarrow \quad 0 < |x-2| < \min\left(1, \frac{1}{M}\right) \\
&\Leftarrow \quad 0 < |x-2| < \delta
\end{aligned}$$

where we have set $\delta = \min(1, 1/M)$.

Let $M > 0$.

$$\begin{aligned}
\frac{-1}{(x+1)^2} < -M \quad \textit{and} \quad x \neq -1 \quad &\Leftarrow \quad \frac{1}{(x+1)^2} > M \quad \textit{and} \quad x \neq -1 \\
&\Leftarrow \quad (x+1)^2 < \frac{1}{M} \quad \textit{and} \quad x \neq -1 \\
&\Leftarrow \quad |x+1| < \frac{1}{\sqrt{M}} \quad \textit{and} \quad x \neq -1 \\
&\Leftarrow \quad 0 < |x+1| < \frac{1}{\sqrt{M}} \\
&\Leftarrow \quad 0 < |x-(-1)| < \delta
\end{aligned}$$

where we have set $\delta = 1/\sqrt{M}$.

Let $M > 0$.

$$\begin{aligned}
\frac{1}{\sqrt{|x|}} > M \quad \text{and} \quad x \neq 0 \quad &\Leftarrow \quad \frac{1}{|x|} > M^2 \quad \text{and} \quad x \neq 0 \\
&\Leftarrow \quad |x| < \frac{1}{M^2} \quad \text{and} \quad x \neq 0 \\
&\Leftarrow \quad 0 < |x| < \frac{1}{M^2} \\
&\Leftarrow \quad 0 < |x - 0| < \delta
\end{aligned}$$

where we have set $\delta = 1/M^2$.

Let $M > 0$.

$$\begin{aligned}
\frac{x-2}{x^3+4x^2+4x} > M \quad \text{and} \quad x \neq 0, -2 \quad &\Leftarrow \quad \frac{x-2}{x(x+2)^2} > M \quad \text{and} \quad x \neq 0, -2 \\
&\Leftarrow \quad \frac{x-2}{x(x+2)^2} > M \quad \text{and} \quad 0 < |x+2| < 1 \\
&\Leftarrow \quad \frac{x-2}{x(x+2)^2} > M \quad \text{and} \quad \begin{cases} 0 < x+2 < 1 \\ -1 < x+2 < 0 \end{cases} \\
&\Leftarrow \quad \frac{x-2}{x} \times \frac{1}{(x+2)^2} > M \quad \text{and} \quad \begin{cases} -2 < x < -1 \\ -3 < x < -2 \end{cases} \\
&\Leftarrow \quad \left(1 - \frac{2}{x}\right) \times \frac{1}{(x+2)^2} > M \quad \text{and} \quad \begin{cases} -2 < x < -1 \\ -3 < x < -2 \end{cases} \\
&\Leftarrow \quad \left(1 + \frac{2}{3}\right) \times \frac{1}{(x+2)^2} > M \quad \text{and} \quad \begin{cases} -2 < x < -1 \\ -3 < x < -2 \end{cases} \\
&\Leftarrow \quad \frac{5}{3} \times \frac{1}{(x+2)^2} > M \quad \text{and} \quad 0 < |x+2| < 1 \\
&\Leftarrow \quad (x+2)^2 < \frac{3}{5}M \quad \text{and} \quad 0 < |x+2| < 1 \\
&\Leftarrow \quad |x+2| < \sqrt{\frac{3}{5}M} \quad \text{and} \quad 0 < |x+2| < 1 \\
&\Leftarrow \quad 0 < |x+2| < \sqrt{\frac{3}{5}M} \\
&\Leftarrow \quad 0 < |x - (-2)| < \delta
\end{aligned}$$

where we have set $\delta = \sqrt{3M/5}$.

3.42: Prove: If $\lim_{x \to a} f(x) = \infty$ and $g(x) \geq f(x)$ on some deleted neighborhood of $x = a$ then $\lim_{x \to a} g(x) = \infty$.

SOLUTION. Suppose $g(x) \geq f(x)$ for $x \in N^*_{\delta_1}(a)$. Let $M > 0$. Then there exists $\delta_2 > 0$ such that $f(x) > M$ whenver $0 < |x - a| < \delta_2$. Set $\delta = \min(\delta_1, \delta_2)$. Then when $0 < |x - a| < \delta$, we get

$$g(x) \geq f(x) > M.$$

Hence $\lim_{x \to a} g(x) = \infty$.

3.43: Prove: (a) If $\lim_{x \to a} f(x) = \infty$ then $\lim_{x \to a}[1/f(x)] = 0$. (b) If $\lim_{x \to a} f(x) = 0$ and $f(x) \neq 0$ for each $x \in N^*_\delta(a)$ then $\lim_{x \to a}[1/|f(x)|] = \infty$.

SOLUTION. (a) Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = \infty$, for $M = 1/\varepsilon$, there exists $\delta > 0$ such that $f(x) > 1/\varepsilon$ whenever $0 < |x - a| < \delta$. In particular $f(x) > 0$ and

$$\left|\frac{1}{f(x)} - 0\right| = \frac{1}{f(x)} < \varepsilon$$

whenever $0 < |x - a| < \delta$. Hence $\lim_{x\to a}[1/f(x)] = 0$.

(b) Let $M > 0$. Since $\lim_{x\to a} f(x) = 0$, for $\varepsilon = 1/M$, there exists $\delta_1 > 0$ such that $|f(x) - 0| < 1/M$ whenever $0 < |x - a| < \delta_1$. Set $\delta' = \min(\delta, \delta_1)$. Then when $0 < |x - a| < \delta$, $1/f(x)$ is well-defined and

$$\frac{1}{|f(x)|} > M.$$

Hence $\lim_{x\to a}[1/|f(x)|] = \infty$.

3.57: Prove that if $\lim_{x\to\infty} f(x) = \infty$ and $g(x) \geq \alpha > 0$ on some set $[a, \infty)$, where $a \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) \cdot g(x) = \infty$.

SOLUTION. Let $M > 0$. Since $\lim_{x\to\infty} f(x) = \infty$, for $M_1 = M/\alpha$, there exists $K_1 > 0$ such that $f(x) > M_1 = M/\alpha$ whenever $x > K_1$. Set $K = \max(K_1, a)$. Then when $x > K$,

$$f(x) \cdot g(x) \geq \frac{M}{\alpha} \times \alpha = M.$$

Hence $\lim_{x\to\infty} f(x) \cdot g(x) = \infty$.