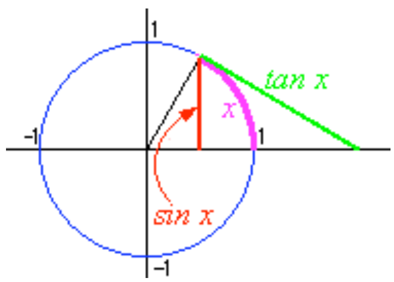
MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 3 SOLUTIONS

3.1: Give a geometrical argument to verify that $|\sin x| \le |x|$ for every real number x.
SOLUTION. If $|x| \ge 1$, then $|\sin x| \le 1 \le |x|$. So we just need to show the case where $|x| < 1$. But since $\sin(-x) = -\sin x$, it is enough (why?) to show that $\sin x \le x$ for $0 \le x < 1$. Note that if $0 \le x < 1$, then in particular $x \in [0, \pi/2)$ (since $1 < \pi/2$) and so the following triangle can always be drawn:



Consider a circle of radius 1. It follows from definition that an angle of x radians must be subtended by an arc of length x. By the definiton of sine (opposite over hypotenuse), the opposite edge has length $\sin x$ since the hypotenuse has length 1. Since the length of the opposite edge cannot be more than the length of the arc, we have $\sin x \le x$. [*Remark*: The figure also shows that $x \le \tan x$ for $x \in [0, \pi/2)$ but this is not required here. I downloaded this picture from the web.]

3.2: Prove that if f is bounded on A and f is also bounded on B then f is bounded on $A \cup B$.
SOLUTION. Since f is bounded above on A, there exists $M_1 > 0$ such that $f(x) \le M_1$ for all $x \in A$. Since f is bounded above on B, there exists $M_2 > 0$ such that $f(x) \le M_2$ for all $x \in B$. Let $M = \max(M_1, M_2)$. Then $M > 0$ and $f(x) \le M$ for all $x \in A \cup B$. So f is bounded above on $A \cup B$. Likewise if 'above' is replaced by 'below'.

3.3: Prove that if f and g are each bounded above (below) on A then $f + g$ is bounded above (below) on A.
SOLUTION. Since f is bounded above on A, there exists $M_1 > 0$ such that $f(x) \le M_1$ for all $x \in A$. Since g is bounded above on A, there exists $M_2 > 0$ such that $g(x) \le M_2$ for all $x \in A$. Let $M = M_1 + M_2$. Then $M > 0$ and $f(x) + g(x) \le M_1 + M_2 = M$ for all $x \in A$. So $f + g$ is bounded above on A. Likewise if 'above' is replaced by 'below'.

3.4: Prove: If f is bounded above (below) on A and $k > 0$ then $k \cdot f$ is bounded above (below) on A; if f is bounded above (below) on A and $k < 0$ then $k \cdot f$ is bounded below (above) on A.
SOLUTION. Since f is bounded above on A, there exists $M_1 > 0$ such that $f(x) \le M_1$ for all $x \in A$. Let $M = kM_1$. Then $kM > 0$ (as $k > 0$) and $kf(x) \le kM_1 = M$ for all $x \in A$. So $k \cdot f$ is bounded above on A. Likewise if 'above' is replaced by 'below' and $k > 0$ replaced by $k < 0$.

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3.6: Prove that each polynomial function

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

is bounded on every bounded interval I.

SOLUTION. Since I is a bounded interval, $I \subseteq [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$. Let $M = \max(|a|, |b|)$. Then $[a, b] \subseteq [-M, M]$ and so $I \subseteq [-M, M]$. If we can show that p is bounded on $[-M, M]$, then it must be bounded on I. Since $|x| \le M$ for all $x \in [-M, M]$, by the triangle inequality,

$$\begin{aligned} |p(x)| &\le |a_0x^n| + |a_1x^{n-1}| + \cdots + |a_{n-1}x| + |a_n| \\ &= |a_0||x|^n + |a_1||x|^{n-1} + \cdots + |a_{n-1}||x| + |a_n| \\ &\le |a_0|M^n + |a_1|M^{n-1} + \cdots + |a_{n-1}|M + |a_n|, \end{aligned}$$

and so p is bounded on $[-M, M]$. Alternatively, apply Theorem **3.1** repeatedly.

3.8: Prove that a nonconstant polynomial cannot be bounded on an unbounded interval.

SOLUTION. Let I be an unbounded interval. WOLG, suppose I is unbounded above, i.e. $I = (a, \infty)$ for some $a \in \mathbb{R} \cup \{-\infty\}$. Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be nonconstant, i.e. $a_0 \ne 0$ and $n > 0$. By the triangle inequality,

$$\begin{aligned} |p(x)| &\ge |a_0x^n| - |a_1x^{n-1} + \cdots + a_{n-1}x + a_n| \\ &\ge |a_0x^n| - |a_1x^{n-1}| - \cdots - |a_{n-1}x| - |a_n| \\ &= |a_0||x|^n - |a_1||x|^{n-1} - \cdots - |a_{n-1}||x| - |a_n| \\ &= |x|^n \left(|a_0| - \frac{|a_1|}{|x|} - \cdots - \frac{|a_{n-1}|}{|x|^{n-1}} - \frac{|a_n|}{|x|^n} \right) \\ &\ge |x|^n \left(|a_0| - \frac{|a_1|}{|x|} - \cdots - \frac{|a_{n-1}|}{|x|^{n-1}} - \frac{|a_n|}{|x|^n} \right). \end{aligned} \tag{1}$$

Since $a_0 \ne 0$, we may let

$$b := 3 \max \left(\frac{|a_1|}{|a_0|}, \sqrt[2]{\frac{|a_2|}{|a_0|}}, \ldots, \sqrt[n-1]{\frac{|a_{n-1}|}{|a_0|}}, \sqrt[n]{\frac{|a_n|}{|a_0|}} \right). \tag{2}$$

Note that for $x \in (b, \infty)$, we have $|x| > b$ and so for all $k = 1, \ldots, n$,

$$-\frac{|a_k|}{|x|^k} > -\frac{|a_0|}{3^k}.$$

(*Remark*: In fact this is how I got the b in (2) — by working backwards from this inequality that I want). Hence if $x \in (b, \infty)$, then the last expression in (1) can be further bounded below

$$\begin{aligned} |p(x)| &\ge |x|^n \left(|a_0| - \frac{|a_1|}{|x|} - \cdots - \frac{|a_{n-1}|}{|x|^{n-1}} - \frac{|a_n|}{|x|^n} \right) \\ &\ge |x|^n \left(|a_0| - \frac{|a_0|}{3} - \cdots - \frac{|a_0|}{3^{n-1}} - \frac{|a_0|}{3^n} \right) \\ &= |x|^n|a_0| \left[1 - \frac{1}{3} \left(1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}} \right) \right] \\ &\ge |x|^n|a_0| \left[1 - \frac{1}{3} \left(1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}} + \cdots \right) \right] \\ &= \frac{1}{2}|x|^n|a_0|. \end{aligned}$$

Suppose $p(x)$ is bounded on (a,∞) by some $M>0$. Choose

$$x > \max\left(a, b, \sqrt[n]{\frac{2M}{|a_0|}}\right),$$

then $x \in (a,\infty)$ and

$$|p(x)| \geq \frac{1}{2}|x|^n|a_0| > M,$$

a contradiction.

3.16: A rational number $r = p/q$ where $p, q \in \mathbb{Z}$ and $q \neq 0$, is said to be *properly reduced* if $\gcd(p,q) = 1$ and $q > 0$. Define the function f as follows:

$$f(x) = \begin{cases} q & \text{if } x = p/q,\ \gcd(p,q) = 1,\ q > 0, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that for every real number x_0, f fails to be bounded at x_0.
Solution. Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. If f is bounded on $(x_0 - \varepsilon, x_0 + \varepsilon)$, then for all $p/q \in (x_0 - \varepsilon, x_0 + \varepsilon)$ with $\gcd(p,q) = 1$ and $q > 0$, the denominators q would be bounded, and hence the numerators p would be too. But this would permit only finitely many rational numbers in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, contradicting Exercise **1.43** in Homework **1**. Therefore f must be unbounded on $(x_0 - \varepsilon, x_0 + \varepsilon)$.

3.17: Prove, using Definition **3.2**, that if $f(x) = k$, a constant function, and $g(x) = x$, the identity function, then for any real number a, $\lim_{x\to a} f(x) = k$ and $\lim_{x\to a} g(x) = a$.
Solution. Let $\varepsilon > 0$. Then $|f(x) - k| = |k - k| = 0 < \varepsilon$ — this holds independent of the condition $0 < |x - a| < \delta$ and therefore holds for any choice of $\delta > 0$ and any $a \in \mathbb{R}$.
Let $\varepsilon > 0$. Then $|g(x) - a| = |x - a| < \varepsilon$ — this holds for any x satisfying $0 < |x - a| < \delta$ if we pick $\delta = \varepsilon$.

3.19: Prove that for every real number a, $\lim_{x\to a} \cos x = \cos a$.
Solution. Let $\varepsilon > 0$. Then

$$\begin{aligned} |\cos x - \cos a| &= \left|-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right| = 2\left|\sin\left(\frac{x+a}{2}\right)\right|\left|\sin\left(\frac{x-a}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-a}{2}\right)\right| \leq 2\left|\frac{x-a}{2}\right| = |x-a| < \varepsilon, \end{aligned}$$

the last inequality holds for any x satisfying $0 < |x - a| < \delta$ if we pick $\delta = \varepsilon$. Note that we have used the inequality in Problem **3.1**.

3.21: Let

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \in \mathbb{Q}, \\ 5 - x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to a} f(x)$ does not exist if $a \neq 2$.
Solution. Let $\varepsilon > 0$. Then

$$|f(x) - 3| = \begin{cases} |2x - 1 - 3| & \text{if } x \in \mathbb{Q}, \\ |5 - x - 3| & \text{if } x \notin \mathbb{Q}, \end{cases} = \begin{cases} 2|x - 2| & \text{if } x \in \mathbb{Q}, \\ |x - 2| & \text{if } x \notin \mathbb{Q}, \end{cases} \leq 2|x - 2| < \varepsilon,$$

the last inequality holds for any x satisfying $0 < |x - 2| < \delta$ if we pick $\delta = \varepsilon/2$.
Let $a \neq 2$. Note that for any $n \in \mathbb{N}$, the interval $(a, a+1/n)$ must contain a rational number

x_n and an irrational number y_n by Theorems **1.9** and **1.10**. Note that $a < x_n < a + 1/n$ and $a < y_n < a + 1/n$, so by Squeezing Lemma,

$$\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n.$$

If $\lim_{x\to a} f(x)$ exists, then by Theorem **3.6**,

$$\lim_{x\to a} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (2x_n - 1) = 2 \lim_{n\to\infty} x_n - 1 = 2a - 1,$$

and also

$$\lim_{x\to a} f(x) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} (5 - y_n) = 5 - \lim_{n\to\infty} y_n = 5 - a.$$

Now by Theorem **3.3**, we must have $2a - 1 = 5 - a$ and so $a = 2$, contradicting our original assumption. Hence $\lim_{x\to a} f(x)$ does not exist.