MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 2 SOLUTIONS

2.2: Use the definition of limits to prove that

$$\lim_{n\to\infty} \frac{1}{n+n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{2n^3 - 3n^2 + 1}{n + n^3} = 2.$$

SOLUTION. Let $\varepsilon > 0$. We want

$$\left| \frac{1}{n+n^2} - 0 \right| = \frac{1}{n+n^2} < \frac{1}{n} < \varepsilon$$

for all $n > N$. So pick $N = \lceil 1/\varepsilon \rceil$.

For the second limit,

$$\left| \frac{2n^3 - 3n^2 + 1}{n + n^3} - 2 \right| = \left| \frac{-3n^2 - 2n + 1}{n + n^3} \right| = \left| \frac{(n+1)(3n-1)}{(n^2+1)n} \right|$$
$$= \left| \frac{n+1}{n^2+1} \right| \left| \frac{3n-1}{n} \right| < \frac{n+1}{n^2+1} \times 3 \leq \frac{3}{n-1}$$

where the last inequality follows from $(n+1)/(n^2+1) \leq 1/(n-1) \Leftrightarrow n^2 - 1 \leq n^2 + 1 \Leftrightarrow -1 \leq 1$, which is evidently true provided $n > 1$. We want

$$\left| \frac{2n^3 - 3n^2 + 1}{n + n^3} - 2 \right| < \frac{3}{n-1} < \varepsilon$$

for all $n > N$. So pick $N = \lceil 3/\varepsilon \rceil + 1$.

2.4: Prove that if $(a_n)_{n\in\mathbb{N}}$ converges and $(b_n)_{n\in\mathbb{N}}$ diverges, then $(a_n + b_n)_{n\in\mathbb{N}}$ diverges.
SOLUTION. Suppose not and $(a_n + b_n)_{n\in\mathbb{N}}$ converges. Then by Theorem **2.3**(a) and (b) with $k = -1$, the sequence $(b_n)_{n\in\mathbb{N}} = (-1 \cdot a_n + (a_n + b_n))_{n\in\mathbb{N}}$ converges, a contradiction.

2.6: Prove the following.

(a) If $\lim_{n\to\infty} a_n = \infty$, $a_n \neq 0$ for all $n \in \mathbb{N}$, and $(b_n)_{n\in\mathbb{N}}$ is a bounded sequence, then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

SOLUTION. Let $\varepsilon > 0$. Since $(b_n)_{n\in\mathbb{N}}$ is bounded, there exists $K > 0$ such that $|b_n| < K$ for all $n \in \mathbb{N}$. Let $M = K/\varepsilon$. Since $\lim_{n\to\infty} a_n = \infty$, there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$. Hence

$$\left| \frac{b_n}{a_n} - 0 \right| = \frac{|b_n|}{a_n} < \frac{K}{M} = \varepsilon$$

for all $n > N$, as required.

(b) If $\lim_{n\to\infty} a_n = \infty$ and if there exists $\eta > 0$ and $N \in \mathbb{N}$ such that $b_n > \eta$ whenver $n > N$ then

$$\lim_{n\to\infty} a_n b_n = \infty.$$

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SOLUTION. Let $M > 0$. Since $M/\eta > 0$ and $\lim_{n\to\infty} a_n = \infty$, there exists $N' \in \mathbb{N}$ such that $a_n > M/\eta$ for all $n > N'$. Hence

$$a_n b_n > \frac{M}{\eta} \times \eta = M$$

for all $n > \max(N', N)$, as required.

(c) If $\lim_{n\to\infty}|a_n| = \infty$, $a_n \neq 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

SOLUTION. Let $\varepsilon > 0$. Let $M = 1/\varepsilon$. Since $\lim_{n\to\infty}|a_n| = \infty$, there exists $N \in \mathbb{N}$ such that $|a_n| > M$ for all $n > N$. Hence

$$\left|\frac{1}{a_n} - 0\right| = \frac{1}{|a_n|} < \frac{1}{M} = \varepsilon$$

for all $n > N$, as required.

2.10: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define the sequence $(s_n)_{n\in\mathbb{N}}$ by

$$s_n := \frac{a_1 + a_2 + \cdots + a_n}{n}$$

for every $n \in \mathbb{N}$.

(a) Prove that if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} s_n = a$.

SOLUTION. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = a$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon/2$$

whenever $n > N_1$. Now by the Archimedean property, there exists $N_2 \in \mathbb{N}$ such that

$$\frac{|a_1 - a| + |a_2 - a| + \cdots + |a_{N_1} - a|}{N_2} < \frac{\varepsilon}{2}.$$

Hence

$$\frac{|a_1 - a| + |a_2 - a| + \cdots + |a_{N_1} - a|}{n} < \frac{\varepsilon}{2}$$

whenever $n > N_2$. Now for $n > \max\{N_1, N_2\}$,

$$\begin{aligned}
|s_n - a| &= \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - a\right| \\
&= \left|\frac{(a_1 - a) + (a_2 - a) + \cdots + (a_n - a)}{n}\right| \\
&\leq \frac{|a_1 - a| + |a_2 - a| + \cdots + |a_n - a|}{n} \\
&= \frac{|a_1 - a| + \cdots + |a_{N_1} - a|}{n} + \frac{|a_{N_1+1} - a| + \cdots + |a_n - a|}{n} \\
&< \frac{\varepsilon}{2} + \frac{n - N_1}{n}\frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Hence $\lim_{n\to\infty} s_n = a$.

(b) Give an example to show that the converse is not always true.

SOLUTION. Let $a_n = (-1)^n$. Then $(a_n)_{n\in\mathbb{N}}$ is divergent. However observe that

$$s_n = \begin{cases} -\dfrac{1}{n} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even,} \end{cases}$$

and so $(s_n)_{n\in\mathbb{N}}$ is convergent. In fact $\lim_{n\to\infty} s_n = 0$.

2.11: Prove that if $(a_n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded below then $(a_n)_{n\in\mathbb{N}}$ converges and

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n.$$

SOLUTION. Either emulate the proof of Theorem **2.5** or observe that $(-a_n)_{n\in\mathbb{N}}$ is monotone increasing and bounded above and so converges to $\sup_{n\in\mathbb{N}}(-a_n)$ by Theorem **2.5**, i.e.

$$\lim_{n\to\infty}(-a_n) = \sup_{n\in\mathbb{N}}(-a_n).$$

Now note that $\sup_{n\in\mathbb{N}}(-a_n) = -\inf_{n\in\mathbb{N}} a_n$ (cf. Homework **1** Problem **3** from Spring '09) and $\lim_{n\to\infty}(-a_n) = -\lim_{n\to\infty} a_n$ by Theorem **2.3**(b). Multiplying by -1 yields the required result.

2.12: Prove that if $(a_n)_{n\in\mathbb{N}}$ has one subsequence converging to A and a second subsequence converging to B and $A \neq B$ then $(a_n)_{n\in\mathbb{N}}$ diverges.

SOLUTION. We will show that $(a_n)_{n\in\mathbb{N}}$ is not Cauchy and therefore not convergent (cf. discussion after Definition **2.2**). Since $A \neq B$, let $\varepsilon = |A-B|/3 > 0$. Let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence converging to A and $(a_{n_l})_{l\in\mathbb{N}}$ be a subsequence converging to B. So there exists $K \in \mathbb{N}$ such that

$$|a_{n_k} - A| < \varepsilon$$

for all $k > K$ and there exists $L \in \mathbb{N}$ such that

$$|a_{n_l} - B| < \varepsilon$$

for all $l > L$. Now

$$\begin{aligned} |a_{n_k} - a_{n_l}| &= |A - B - (A - B - a_{n_k} + a_{n_l})| \\ &\geq |A-B| - |a_{n_k} - A + B - a_{n_l}| \\ &\geq |A-B| - |a_{n_k} - A| - |B - a_{n_l}| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon \end{aligned}$$

for all $k > K$ and $l > L$. Since $\lim_{k\to\infty} n_k = \infty$, for any $N \in \mathbb{N}$ there exists $K' \in \mathbb{N}$ such that $n_k > N$ for all $k > K'$; likewise, there exists $L' \in \mathbb{N}$ such that $n_l > N$ for all $l > L'$. So for $\varepsilon = |A-B|/3$, given any $N \in \mathbb{N}$, if $k > \max(K, K')$ and $l > \max(L, L')$, we get $n_k, n_l > N$ with

$$|a_{n_k} - a_{n_l}| > \varepsilon.$$

This shows that $(a_n)_{n\in\mathbb{N}}$ is not Cauchy.

2.14: Prove that if $\lim_{n\to\infty} a_n = A$ then every subsequence of $(a_n)_{n\in\mathbb{N}}$ is convergent with limit A.

SOLUTION. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$|a_n - A| < \varepsilon$$

for all $n > N$. Let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$. Since $\lim_{k\to\infty} n_k = \infty$, there exists $K \in \mathbb{N}$ such that $n_k > N$ for all $k > K$. Hence

$$|a_{n_k} - A| < \varepsilon$$

for all $k > K$. In other words $\lim_{k\to\infty} a_{n_k} = A$, as required.

2.16: Prove that a monotone sequence can have at most one cluster point.

SOLUTION. Let $(a_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence. If $(a_n)_{n\in\mathbb{N}}$ has no cluster point, then we are done. If $(a_n)_{n\in\mathbb{N}}$ has at least one cluster point, we will show that it is bounded above and so by Theorem **2.5**, it is convergent; and so by Exercise **2.12** above, it cannot have more than one cluster point. Now if $(a_n)_{n\in\mathbb{N}}$ has a cluster point, then it has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ (that converges to that cluster point). By the Lemma on pp. 40, $(a_{n_k})_{k\in\mathbb{N}}$ is bounded and in particular bounded above, i.e. there exists $M > 0$ such that

$$a_{n_k} < M$$

for all $k \in \mathbb{N}$. For any $n \in \mathbb{N}$, since $\lim_{k\to\infty} n_k = \infty$, there exists $n_k > n$, and since $(a_n)_{n\in\mathbb{N}}$ is monotone,

$$a_n < a_{n_k} < M.$$

So $(a_n)_{n\in\mathbb{N}}$ is also bounded above by M.

2.17: Prove that if $(a_n)_{n\in\mathbb{N}}$ is monotone and x is a cluster point of $(a_n)_{n\in\mathbb{N}}$ then $\lim_{n\to\infty} a_n = x$.

Solution. Let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence that converges to x. Let $\varepsilon > 0$. Then there exists $K \in \mathbb{N}$ such that

$$|a_{n_k} - A| < \varepsilon$$

for all $k > K$. In other words,

$$A - \varepsilon < a_{n_k} < A + \varepsilon$$

for all $k > K$. Let $N = n_{K+1}$. Since $(a_n)_{n\in\mathbb{N}}$ is monotone, for any $n > N$,

$$a_n \geq a_N = a_{n_{K+1}} > A - \varepsilon.$$

Since $\lim_{k\to\infty} n_k = \infty$, there exists $L > K$ with $n_L > n$, and by monotonicity again,

$$a_n \leq a_{n_L} < A + \varepsilon.$$

Hence

$$A - \varepsilon < a_n < A + \varepsilon$$

for all $n > N$, as required.

2.19: Suppose $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence. Prove that $\limsup_{n\to\infty} a_n$ is the largest cluster point of $(a_n)_{n\in\mathbb{N}}$ and $\liminf_{n\to\infty} a_n$ is the smallest cluster point of $(a_n)_{n\in\mathbb{N}}$.

Solution. Let C be the set of cluster points of $(a_n)_{n\in\mathbb{N}}$. Let $L = \limsup_{n\to\infty} a_n$. By definition

$$L = \limsup_{n\to\infty} a_n = \sup C$$

and so L is an upper bound of C. If we can show that L is a cluster point, i.e. $L \in C$, then it is automatically the largest cluster point. We will show this by constructing a subsequence of $(a_n)_{n\in\mathbb{N}}$ that converges to L. The easiest way to do this is to first show that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup_{k\geq n} a_n),$$

i.e. $\limsup_{n\to\infty} a_n$ is the limit of another sequence $(s_n)_{n\in\mathbb{N}}$ whose terms are

$$s_n = \sup\{a_k \mid k \geq n\}.$$

It is then much easier to show that there is a subsequence of $(a_n)_{n\in\mathbb{N}}$ converging to the limit of $(s_n)_{n\in\mathbb{N}}$. [Note: I'll add this part later].