MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 1 SOLUTIONS

1.5: Show that if $f : X \to Y$ is one-to-one and $g : Y \to Z$ is one-to-one then $g \circ f : X \to Z$ is one-to-one.
SOLUTION. $g \circ f(x_1) = g \circ f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

1.16: Prove $2^{n-1} \le n!$ for each $n \in \mathbb{N}$.
SOLUTION. Let $A := \{n \in \mathbb{N} \mid 2^{n-1} \le n!\}$. Since $2^{1-1} = 1 \le 1!$, so $1 \in A$. If $p \in A$, then $2^{p-1} \le p!$ and so

$$2^{(p+1)-1} = 2 \cdot 2^{p-1} \le 2 \cdot p! \le (p+1) \cdot p! = (p+1)!$$

as $2 \le p+1$ for all $p \in \mathbb{N}$, hence $p+1 \in A$. By **P5**, $A = \mathbb{N}$.

1.20: Prove the cancelation laws in $\mathbb{Z}$:

(a) $j + k = j + l$ implies $k = l$ for all $j, k, l \in \mathbb{Z}$.
SOLUTION. Let $j = [(a, b)], k = [(c, d)], l = [(e, f)]$ where $a, \dots, f \in \mathbb{N}$. Then

$$\begin{aligned}
[(a,b)] + [(c,d)] &= [(a,b)] + [(e,f)] \\
[(a+c, b+d)] &= [(a+e, b+f)] \\
(a+c) + (b+f) &= (a+e) + (b+d) \\
(a+b) + (c+f) &= (a+b) + (e+d) \\
(c+f) &= (e+d)
\end{aligned}$$

where the last step is by the additive cancelation law in $\mathbb{N}$. Since $c + f = e + d$, we see that

$$k = [(c,d)] = [(e,f)] = l.$$

(b) $j \cdot k = j \cdot l$ implies $k = l$ for all $j, k, l \in \mathbb{Z}$ with $j \ne 0$.
SOLUTION. Let $j = [(a, b)], k = [(c, d)], l = [(e, f)]$ where $a, \dots, f \in \mathbb{N}$ and $a \ne b$. Then

$$\begin{aligned}
[(a,b)] \cdot [(c,d)] &= [(a,b)] \cdot [(e,f)] \\
[(ac+bd, ad+bc)] &= [(ae+bf, af+be)] \\
(ac+bd) + (af+be) &= (ad+bc) + (ae+bf)
\end{aligned}$$

Since $a \ne b$, we must have $a < b$ or $b < a$ by trichotomy. Suppose $a < b$. Then there exists $n \in \mathbb{N}$ such that $a + n = b$. The last equation above becomes

$$ac + ad + nd + af + ae + ne = ad + ac + nc + ae + af + nf$$

and applying the additive cancelation law in $\mathbb{N}$ we get

$$\begin{aligned}
nd + ne &= nc + nf \\
n(d+e) &= n(c+f)
\end{aligned}$$

and applying the multiplicative cancelation law in $\mathbb{N}$ we get

$$c + f = d + e$$

and so $k = [(c,d)] = [(e,f)] = l$, as required.

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1.25: Prove that the order properties in $\mathbb{Q}$:

(a) $u < v$ implies $u + w < v + w$ for every $w \in \mathbb{Q}$.

SOLUTION. Let $\mathbb{Q}_+$ denote the subset of positive rationals. Since $u < v$, there exists $p \in \mathbb{Q}_+$ such that $u + p = v$. Let $w \in \mathbb{Q}$, by associativity and commutativity, $(u + w) + p = (u + p) + w = v + w$. Hence $u + w < v + w$ for every $w \in \mathbb{Q}$.

(b) $u < v$ and $w > 0$ implies $u \cdot w < v \cdot w$ for every $w \in \mathbb{Q}$.

SOLUTION. Since $u < v$ and $0 < w$, there exists $p \in \mathbb{Q}_+$ such that $u + p = v$ and $q \in \mathbb{Q}_+$ such that $0 + q = w$. In particular $w = q$. Now $v \cdot w = (u + p) \cdot w = u \cdot w + p \cdot w = u \cdot w + p \cdot q$. Since the product of two positive rationals is positive, $p \cdot q \in \mathbb{Q}_+$ and hence $u \cdot w + p \cdot q = v \cdot w$ implies that $u \cdot w < v \cdot w$.

1.28: Prove that for any $r \in \mathbb{Q}$ the set $\{x \in \mathbb{Q} \mid x > r\}$ is a ray in $\mathbb{Q}$.

SOLUTION. Let $U = \{x \in \mathbb{Q} \mid x > r\}$. Since $r \in \mathbb{Q}$, we have $r + 1 \in \mathbb{Q}$ and so $r + 1 \in U$ and so $U \neq \varnothing$; also since $r \notin U$, $U \neq \mathbb{Q}$. Hence U is a nonempty proper subset of $\mathbb{Q}$. If $x \in U$ and $y > x$, then $y > x > r$ and so $y > r$ by transitivity, so $y \in U$. Suppose $x_0 \in U$ is a first element, then $x_0 \leq x$ for all $x \in U$ and $x_0 > r$. Let $y_0 = (x_0 + r)/2$. Then $y_0 > r$ and $y_0 \in \mathbb{Q}$ and so $y_0 \in U$. But $y_0 < x_0$ and so x_0 cannot be a first element, contradicting our assumption. Hence U has no first element. These show that U is a ray in $\mathbb{Q}$.

1.33: Prove that a nonempty set S of real numbers is bounded if and only if there is a nonnegative real number K such that $-K \leq x \leq K$ for every $x \in S$.

SOLUTION. ($\Rightarrow$) Let $l, u \in \mathbb{R}$ be lower and upper bounds of S. Then $l \leq x \leq u$ for every $x \in S$. Let $K = \max\{|l|, |u|\}$. Note that $|l| \leq K$ and so $-K \leq l$; also $|u| \leq K$ and so $u \leq K$. Hence

$$-K \leq l \leq x \leq u \leq K \quad \text{for every } s \in S,$$

as required.

($\Leftarrow$) If $-K \leq x \leq K$ for every $x \in S$, then $l = -K$ is a lower bound for S and $u = K$ is an upper bound for S. So S is bounded.

1.37: Prove that, if they exist, the least upper bound and the greatest lower bound of a nonempty set $S \subset \mathbb{R}$ are unique.

SOLUTION. Let M_1 and M_2 both be least upper bounds of S. Since M_1 is an upper bound, and M_2 is a least upper bound, we must have $M_1 \leq M_2$. Since M_2 is an upper bound, and M_1 is a least upper bound, we must have $M_2 \leq M_1$. Hence $M_1 = M_2$. Ditto for greatest lower bound.

1.38: Prove Bernoulli's inequality: $(1 + x)^n \geq 1 + nx$ for every real number $x \geq -1$ and every $n \in \mathbb{N}$.

SOLUTION. Let $A = \{n \in \mathbb{N} \mid (1+x)^n \geq 1+nx \text{ for all } x \in (-1, \infty)\}$. Since $(1+x)^1 \geq 1+1 \cdot x$ for all $x \geq -1$, $1 \in A$. If $p \in A$, then $(1 + x)^p \geq 1 + px$ for all $x \geq -1$ and so

$$(1 + x)^{p+1} = (1 + x)(1 + x)^p \geq (1 + x)(1 + px) = 1 + (p + 1)x + px^2 \geq 1 + (p + 1)x$$

for all $x \geq -1$, since $px^2 \geq 0$ for all $p \in \mathbb{N}$, hence $p + 1 \in A$. By **P5**, $A = \mathbb{N}$.

1.43: Prove that every interval of real numbers contains infinitely many rational and irrational numbers.

SOLUTION. Let I be an interval of real numbers and let $a = \inf I$ and $b = \sup I$. Then $(a, b) \subseteq I$. If $a = -\infty$, then I contains an infinite number of disjoint intervals $I_n = (b - n, b - n + 1)$, $n \in \mathbb{N}$. If $b = \infty$, then I contains an infinite number of disjoint intervals $I_n = (a + n - 1, a + n)$, $n \in \mathbb{N}$. If $-\infty < a < b < \infty$, then I contains an infinite number of disjoint intervals $I_n = ([a + (2^{n-1} - 1)b]/2^{n-1}, [a + (2^n - 1)b]/2^n)$. In all three cases, each I_n contains a rational and an irrational by Theorems **1.9** and **1.10**. Since $I_n \cap I_m = \varnothing$

if $n \neq m$, the rationals/irrationals in these intervals are distinct. So $I \supseteq \cup_{n\in\mathbb{N}} I_n$ contains infinitely many rationals and irrationals.