MATH 104: INTRODUCTORY ANALYSIS
SPRING 2008/09
PROBLEM SET 9

1. Let f be continuous on $[a, b]$ and differentiable on (a, b).

(a) Suppose
$$f(a) = f(b) = 0.$$
Show that for a given $\alpha \in \mathbb{R}$, there exists $x_0 \in (a, b)$ such that
$$\alpha f(x_0) + f'(x_0) = 0.$$

(b) Suppose
$$f(b)^2 - f(a)^2 = b^2 - a^2.$$
Show that there exists $x_0 \in (a, b)$ such that
$$f'(x_0)f(x_0) = x_0.$$

(c) Suppose $a > 0$ and
$$\frac{f(a)}{a} = \frac{f(b)}{b}.$$
Show that there exists $x_0 \in (a, b)$ such that
$$x_0 f'(x_0) = f(x_0).$$

(d) Suppose $a > 0$. Show that there exists $x_0 \in (a, b)$ such that
$$\frac{bf(a) - af(b)}{b - a} = f(x_0) - x_0 f'(x_0).$$

2. (a) Show that each of the following equations has exactly one real root.
$$x^{13} + 7x^3 - 5 = 0, \qquad 3^x + 4^x = 5^x.$$

(b) Let $a_1, \ldots, a_n \in \mathbb{R}$ be non-zero. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be distinct, i.e. $\alpha_i \neq \alpha_j$ for $i \neq j$. Prove that the equation
$$a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n} = 0$$
has at most $n - 1$ roots in $(0, \infty)$. Hence or otherwise, prove that
$$a_1 e^{\alpha_1 x} + \cdots + a_n e^{\alpha_n x} = 0$$
has at most $n - 1$ roots in $\mathbb{R}$.

3. Let f be differentiable on $[a, b]$ and twice differentiable on (a, b).

(a) Suppose
$$f(a) = f'(a) = f(b) = 0.$$
Show that there exists $x_0 \in (a, b)$ such that
$$f''(x_0) = 0.$$

(b) Suppose
$$f(a) = f(b) \quad \text{and} \quad f'(a) = 0 = f'(b).$$
Show that there exists $x_0, x_1 \in (a, b)$, $x_0 \neq x_1$, such that
$$f''(x_0) = f''(x_1).$$

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(c) Suppose $a = 0$, $b = 2$, and

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 2.$$

Show that there exists $x_0 \in (0, 2)$ such that

$$f''(x_0) = 0.$$

4. (a) Suppose f is differentiable and f' is continuous in a neighborhood of $x = a$. Prove that

$$\lim_{h \to 0} \frac{f(a + h/2) - f(a - h/2)}{h} = f'(a).$$

(b) Suppose f is twice differentiable and f'' is continuous in a neighborhood of $x = a$. Prove that

$$\lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2} = f''(a).$$