1. Let \( f \) be continuous on \([a, b]\) and differentiable on \((a, b)\).
   (a) Suppose \( f(a) = f(b) = 0 \).
   Show that for a given \( \alpha \in \mathbb{R} \), there exists \( x_0 \in (a, b) \) such that
   \[ \alpha f(x_0) + f'(x_0) = 0. \]
   (b) Suppose \( f(b)^2 - f(a)^2 = b^2 - a^2 \).
   Show that there exists \( x_0 \in (a, b) \) such that
   \[ f'(x_0)f(x_0) = x_0. \]
   (c) Suppose \( a > 0 \) and
   \[ \frac{f(a)}{a} = \frac{f(b)}{b}. \]
   Show that there exists \( x_0 \in (a, b) \) such that
   \[ x_0f'(x_0) = f(x_0). \]
   (d) Suppose \( a > 0 \). Show that there exists \( x_0 \in (a, b) \) such that
   \[ \frac{bf(a) - af(b)}{b - a} = f(x_0) - x_0f'(x_0). \]

2. (a) Show that each of the following equations has exactly one real root.
   \[ x^{13} + 7x^3 - 5 = 0, \quad 3^x + 4^x = 5^x. \]
   (b) Let \( a_1, \ldots, a_n \in \mathbb{R} \) be non-zero. Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) be distinct, i.e. \( \alpha_i \neq \alpha_j \) for \( i \neq j \). Prove that the equation
   \[ a_1x^{\alpha_1} + \cdots + a_nx^{\alpha_n} = 0 \]
   has at most \( n - 1 \) roots in \((0, \infty)\). Hence or otherwise, prove that
   \[ a_1e^{\alpha_1x} + \cdots + a_ne^{\alpha_nx} = 0 \]
   has at most \( n - 1 \) roots in \( \mathbb{R} \).

3. Let \( f \) be differentiable on \([a, b]\) and twice differentiable on \((a, b)\).
   (a) Suppose \( f(a) = f'(a) = f(b) = 0 \).
   Show that there exists \( x_0 \in (a, b) \) such that
   \[ f''(x_0) = 0. \]
   (b) Suppose \( f(a) = f(b) \) and \( f'(a) = 0 = f'(b) \).
   Show that there exists \( x_0, x_1 \in (a, b) \), \( x_0 \neq x_1 \), such that
   \[ f''(x_0) = f''(x_1). \]
(c) Suppose $a = 0$, $b = 2$, and
\[
    f(0) = 0, \quad f(1) = 1, \quad f(2) = 2.
\]
Show that there exists $x_0 \in (0, 2)$ such that
\[
    f''(x_0) = 0.
\]

4. (a) Suppose $f$ is differentiable and $f'$ is continuous in a neighborhood of $x = a$. Prove that
\[
    \lim_{h \to 0} \frac{f(a + h/2) - f(a - h/2)}{h} = f'(a).
\]
(b) Suppose $f$ is twice differentiable and $f''$ is continuous in a neighborhood of $x = a$. Prove that
\[
    \lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2} = f''(a).
\]