MATH 104: INTRODUCTORY ANALYSIS
SPRING 2008/09
PROBLEM SET 7 SOLUTIONS

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. We know from Theorem **2.1** in the lectures that if $(x_n)_{n\in\mathbb{N}}$ is convergent, then

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

But suppose $(x_n)_{n\in\mathbb{N}}$ is not convergent.

(a) Find counterexamples to show that the following equalities do not always hold:

$$\limsup_{n\to\infty} f(x_n) = f(\limsup_{n\to\infty} x_n) \quad \text{and} \quad \liminf_{n\to\infty} f(x_n) = f(\liminf_{n\to\infty} x_n).$$

Solution. Take $f(x) = -x$ and $x_n = (-1)^n$. Then

$$\liminf_{n\to\infty} f(x_n) = -1 < 1 = f(\liminf_{n\to\infty} x_n)$$

and

$$\limsup_{n\to\infty} f(x_n) = 1 > -1 = f(\limsup_{n\to\infty} x_n).$$

(b) Prove that

$$\limsup_{n\to\infty} f(x_n) \geq f(\limsup_{n\to\infty} x_n) \quad \text{and} \quad \liminf_{n\to\infty} f(x_n) \leq f(\liminf_{n\to\infty} x_n).$$

Solution. Let $\liminf_{n\to\infty} x_n = a$. By the continuity of f, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for} \quad |x - a| < \delta. \tag{1.1}$$

By Homework **5**, Problem **1**(a), there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to a and so there exists $N \in \mathbb{N}$ such that $k > N$ implies

$$|x_{n_k} - a| < \delta.$$

Now by (1.1) we get

$$|f(x_{n_k}) - f(a)| < \varepsilon \quad \text{for} \quad k > N.$$

Hence we have shown that

$$\lim_{k\to\infty} f(x_{n_k}) = f(a).$$

In other words, there exists a subsequence $(f(x_{n_k}))_{k\in\mathbb{N}}$ of the sequence $(f(x_n))_{n\in\mathbb{N}}$ that converges to $f(a)$. By Homework **5**, Problem **1**(b),

$$\liminf_{n\to\infty} f(x_n) \leq \lim_{k\to\infty} f(x_{n_k}) = f(a),$$

and so we obtain

$$\liminf_{n\to\infty} f(x_n) \leq f(\liminf_{n\to\infty} x_n).$$

The proof for limit superior is similar.

2. Let f and g be functions such that

$$\lim_{x\to a} f(x) = b \quad \text{and} \quad \lim_{y\to b} g(y) = c.$$

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(a) Show that the following is not necessarily true

$$\lim_{x\to a} g(f(x)) = c.$$

SOLUTION. In fact the limit of the composition need not even exist. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \sin x & \text{otherwise,} \end{cases} \quad \text{and} \quad g(y) = \begin{cases} 0 & \text{if } y = 0, \\ \dfrac{\sin y}{y} & \text{otherwise.} \end{cases}$$

Then

$$g(f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \text{ or } x = k\pi, k \in \mathbb{Z}, \\ \dfrac{\sin(\sin x)}{\sin x} & \text{otherwise.} \end{cases}$$

Note that

$$\lim_{x\to 0} f(x) = 0, \quad \lim_{y\to 0} g(y) = 1,$$

but $\lim_{x\to 0} g(f(x))$ does not exist.

(b) Show that it is true if f is continuous at a and g is continuous at b (which must be equal to $f(a)$), i.e.

$$\lim_{x\to a} g(f(x)) = g(f(a)).$$

SOLUTION. Since f is continuous at a,

$$f(a) = \lim_{x\to a} f(x) = b.$$

Let $\varepsilon > 0$ be given. Since g is continuous at the point $b = f(a)$, there is an $\eta > 0$ such that if $|y - f(a)| < \eta$ then $|g(y) - g(f(a))| < \varepsilon$. But since f is continuous at a, there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \eta$. Consequently, for each x satisfying $|x - a| < \delta$ we have $|g(f(x)) - g(f(a))| < \varepsilon$. Therefore

$$\lim_{x\to a} g(f(x)) = g(f(a)).$$

3. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \frac{x + |x|}{2}, \quad f(x) = \begin{cases} x & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$

For which values of x are f and g continuous? Write down an expression for the composite function $h(x) = g(f(x))$ and determine the values of x for which it is continuous.

SOLUTION. Note that

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

So when $x < 0$, $h(x) = g(f(x)) = g(x) = 0$ and when $x \geq 0$, $h(x) = g(f(x)) = g(x^2) = x^2$. Hence

$$h(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$

It is easy to see that f, g, h are continuous everywhere on $\mathbb{R}$.

(b) Let $f : [-1, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x} & \text{if } -1 \leq x < 0, \\ \alpha & \text{if } x = 0, \\ \dfrac{\log(1+x)}{x} & \text{if } x > 0. \end{cases}$$

Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. For which values of α is f left- or right-continuous at 0?

SOLUTION. Note that

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \\ &= \lim_{x\to 0^-} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} \\ &= \lim_{x\to 0^-} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\log(1+x)}{x} \\ &= \lim_{x\to 0^+} \log(1+x)^{1/x} \\ &= \log\left[\lim_{x\to 0^+} (1+x)^{1/x}\right] \\ &= \log e^1 \\ &= 1\end{aligned}$$

where the third equality follows from the continuity of log on $(0,\infty)$. Hence setting $\alpha = 1$ makes f a continuous function, i.e. both left- and right-continuous.

4. (a) Prove that the equation $(1-x)\cos x = \sin x$ has at least one solution in $(0,1)$.

SOLUTION. Consider the continuous function $f(x) = (1-x)\cos x - \sin x$. Then $f(0) = 1 > 0$ and $f(1) = -\sin 1 < 0$. Therefore there is $x_0 \in (0,1)$ satisfying $f(x_0) = 0$ by the intermediate value theorem.

(b) Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions where $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $c \in (a,b)$ such that

$$f(c) = g(c).$$

Hence or otherwise show that a continuous $f : [0,1] \to [0,1]$ must have a fixed point, i.e. $c \in [0,1]$ such that $f(c) = c$.

SOLUTION. Consider the continuous function $h(x) = f(x) - g(x)$. Observe that $h(a) < 0$ and $h(b) > 0$. By the intermediate value theorem, there exists $c \in (a,b)$ such that

$$f(c) = g(c).$$

Let $g(x) = x$. If either $f(0) = 0$ or $f(1) = 1$, we would have a fixed point. Otherwise $f(0) > 0 = g(0)$ and $f(1) < 1 = g(1)$ implies that we have a fixed point $c \in (0,1)$ by the earlier part.

(c) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that for any $x_1, x_2, \ldots, x_n \in (a,b)$, there exists $x_0 \in (a,b)$ such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Hence or otherwise show that there exists $0 < \theta < \pi/2$ such that

$$\sin\theta = \frac{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}{10}.$$

SOLUTION. By the extreme value theorem there exists $x_{\min}, x_{\max} \in [a,b]$ such that $f(x_{\min}) = m := \min\{f(x) \mid x \in [a,b]\}$ and $f(x_{\max}) = M := \max\{f(x) \mid x \in [a,b]\}$. If f is constant,

then we are done. Assume f is nonconstant. Then $x_{\min} \neq x_{\max}$. Clearly,

$$m \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq M.$$

By the intermediate value theorem, f assumes every value between m and M, so there exists $x_0 \in (x_{\min}, x_{\max})$ or $(x_{\max}, x_{\min})$ (depending on whether $x_{\min} < x_{\max}$ or $x_{\max} < x_{\min}$) such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Let $f(x) = \sin x$ and let $x_k = k\pi/12 \in (0, \pi/2)$, $k = 1, 2, 3, 4, 5$. Since

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{1}{2\sqrt{2}}(\sqrt{3} - 1),$$

$$\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{1}{2\sqrt{2}}(\sqrt{3} + 1),$$

we have

$$\frac{1}{5}\left(\sin\frac{\pi}{12} + \sin\frac{\pi}{6} + \sin\frac{\pi}{4} + \sin\frac{\pi}{3} + \sin\frac{5\pi}{12}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + \sqrt{6}}{10}.$$

The existence of the required $\theta \in (0, \pi/2)$ then follows from the earlier part.