1. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and let \((x_n)_{n \in \mathbb{N}}\) be a bounded sequence. We know from Theorem 2.1 in the lectures that if \((x_n)_{n \in \mathbb{N}}\) is convergent, then
\[
\lim_{n \to \infty} f(x_n) = f\left( \lim_{n \to \infty} x_n \right).
\]
But suppose \((x_n)_{n \in \mathbb{N}}\) is not convergent.
(a) Find counterexamples to show that the following equalities do not always hold:
\[
\limsup_{n \to \infty} f(x_n) = f\left( \limsup_{n \to \infty} x_n \right) \quad \text{and} \quad \liminf_{n \to \infty} f(x_n) = f\left( \liminf_{n \to \infty} x_n \right).
\]
Solution. Take \(f(x) = -x\) and \(x_n = (-1)^n\). Then
\[
\liminf_{n \to \infty} f(x_n) = -1 < 1 = f\left( \liminf_{n \to \infty} x_n \right)
\]
and
\[
\limsup_{n \to \infty} f(x_n) = 1 > -1 = f\left( \limsup_{n \to \infty} x_n \right).
\]
(b) Prove that
\[
\limsup_{n \to \infty} f(x_n) \geq f\left( \limsup_{n \to \infty} x_n \right) \quad \text{and} \quad \liminf_{n \to \infty} f(x_n) \leq f\left( \liminf_{n \to \infty} x_n \right).
\]
Solution. Let \(\liminf_{n \to \infty} x_n = a\). By the continuity of \(f\), for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that
\[
|f(x) - f(a)| < \varepsilon \quad \text{for} \quad |x - a| < \delta.
\]
By Homework 5, Problem 1(a), there exists a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) that converges to \(a\) and so there exists \(N \in \mathbb{N}\) such that \(k > N\) implies
\[
|x_{n_k} - a| < \delta.
\]
Now by (1.1) we get
\[
|f(x_{n_k}) - f(a)| < \varepsilon \quad \text{for} \quad k > N.
\]
Hence we have shown that
\[
\lim_{k \to \infty} f(x_{n_k}) = f(a).
\]
In other words, there exists a subsequence \((f(x_{n_k}))_{k \in \mathbb{N}}\) of the sequence \((f(x_n))_{n \in \mathbb{N}}\) that converges to \(f(a)\). By Homework 5, Problem 1(b),
\[
\liminf_{n \to \infty} f(x_n) \leq \lim_{k \to \infty} f(x_{n_k}) = f(a),
\]
and so we obtain
\[
\liminf_{n \to \infty} f(x_n) \leq f\left( \liminf_{n \to \infty} x_n \right).
\]
The proof for limit superior is similar.

2. Let \(f\) and \(g\) be functions such that
\[
\lim_{x \to a} f(x) = b \quad \text{and} \quad \lim_{y \to b} g(y) = c.
\]

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(a) Show that the following is not necessarily true
\[ \lim_{x \to a} g(f(x)) = c. \]

**Solution.** In fact the limit of the composition need not even exist. Consider the functions
\( f, g : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \\ \sin x & \text{otherwise}, \end{cases}
\]
and
\[
g(y) = \begin{cases} 0 & \text{if } y = 0, \\ \frac{\sin y}{y} & \text{otherwise.} \end{cases}
\]
Then
\[
g(f(x)) = \begin{cases} 0 & \text{if } x = \frac{1}{n}, n \in \mathbb{N}, \text{ or } x = k\pi, k \in \mathbb{Z}, \\ \sin(\sin x) & \text{otherwise.} \end{cases}
\]
Note that
\[ \lim_{x \to 0} f(x) = 0, \quad \lim_{y \to 0} g(y) = 1, \]
but \( \lim_{x \to 0} g(f(x)) \) does not exist.

(b) Show that it is true if \( f \) is continuous at \( a \) and \( g \) is continuous at \( b \) (which must be equal to \( f(a) \)), i.e.
\[ \lim_{x \to a} g(f(x)) = g(f(a)). \]

**Solution.** Since \( f \) is continuous at \( a \),
\[ f(a) = \lim_{x \to a} f(x) = b. \]
Let \( \varepsilon > 0 \) be given. Since \( g \) is continuous at the point \( b = f(a) \), there is an \( \eta > 0 \) such that if \( |y - f(a)| < \eta \) then \( |g(y) - g(f(a))| < \varepsilon \). But since \( f \) is continuous at \( a \), there is a \( \delta > 0 \) such that if \( |x - a| < \delta \) then \( |f(x) - f(a)| < \eta \). Consequently, for each \( x \) satisfying \( |x - a| < \delta \) we have \( |g(f(x)) - g(f(a))| < \varepsilon \). Therefore
\[ \lim_{x \to a} g(f(x)) = g(f(a)). \]

3. (a) Let \( f, g : \mathbb{R} \to \mathbb{R} \) be defined by
\[
g(x) = \frac{x + |x|}{2}, \quad f(x) = \begin{cases} x & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}
\]
For which values of \( x \) are \( f \) and \( g \) continuous? Write down an expression for the composite function \( h(x) = g(f(x)) \) and determine the values of \( x \) for which it is continuous.

**Solution.** Note that
\[
g(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}
\]
So when \( x < 0 \), \( h(x) = g(f(x)) = g(x) = 0 \) and when \( x \geq 0 \), \( h(x) = g(f(x)) = g(x^2) = x^2 \).
Hence
\[
h(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}
\]
It is easy to see that \( f, g, h \) are continuous everywhere on \( \mathbb{R} \).

(b) Let \( f : [-1, \infty) \to \mathbb{R} \) be defined by
\[
f(x) = \begin{cases} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} & \text{if } -1 \leq x < 0, \\ \alpha & \text{if } x = 0, \\ \frac{\log(1 + x)}{x} & \text{if } x > 0. \end{cases}
\]
Find \( \lim_{x \to 0^-} f(x) \) and \( \lim_{x \to 0^+} f(x) \). For which values of \( \alpha \) is \( f \) left- or right-continuous at 0?

**Solution.** Note that

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \\
= \lim_{x \to 0^-} \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} \\
= \lim_{x \to 0^-} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} \\
= 1
\]

and

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\log(1+x)}{x} \\
= \lim_{x \to 0^+} \log(1+x)^{1/x} \\
= \log \left( \lim_{x \to 0^+} (1+x)^{1/x} \right) \\
= \log e^1 \\
= 1
\]

where the third equality follows from the continuity of \( \log \) on \((0, \infty)\). Hence setting \( \alpha = 1 \) makes \( f \) a continuous function, i.e. both left- and right-continuous.

4. (a) Prove that the equation \((1-x) \cos x = \sin x\) has at least one solution in \((0, 1)\).

**Solution.** Consider the continuous function \( f(x) = (1-x) \cos x - \sin x \). Then \( f(0) = 1 > 0 \) and \( f(1) = -\sin 1 < 0 \). Therefore there is \( x_0 \in (0, 1) \) satisfying \( f(x_0) = 0 \) by the intermediate value theorem.

(b) Let \( f, g : [a, b] \to \mathbb{R} \) be continuous functions where \( f(a) < g(a) \) and \( f(b) > g(b) \). Prove that there exists \( c \in (a, b) \) such that

\[
f(c) = g(c).
\]

Hence or otherwise show that a continuous \( f : [0, 1] \to [0, 1] \) must have a fixed point, i.e. \( c \in [0, 1] \) such that \( f(c) = c \).

**Solution.** Consider the continuous function \( h(x) = f(x) - g(x) \). Observe that \( h(a) < 0 \) and \( h(b) > 0 \). By the intermediate value theorem, there exists \( c \in (a, b) \) such that

\[
f(c) = g(c).
\]

Let \( g(x) = x \). If either \( f(0) = 0 \) or \( f(1) = 1 \), we would have a fixed point. Otherwise \( f(0) > 0 = g(0) \) and \( f(1) < 1 = g(1) \) implies that we have a fixed point \( c \in (0, 1) \) by the earlier part.

(c) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Prove that for any \( x_1, x_2, \ldots, x_n \in (a, b) \), there exists \( x_0 \in (a, b) \) such that

\[
f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.
\]

Hence or otherwise show that there exists \( 0 < \theta < \pi/2 \) such that

\[
\sin \theta = \frac{1 + \sqrt{2} + \sqrt{3} + \sqrt{6}}{10}.
\]

**Solution.** By the extreme value theorem there exists \( x_{\min}, x_{\max} \in [a, b] \) such that \( f(x_{\min}) = m := \min \{f(x) \mid x \in [a, b]\} \) and \( f(x_{\max}) = M := \max \{f(x) \mid x \in [a, b]\} \). If \( f \) is constant,
then we are done. Assume \( f \) is nonconstant. Then \( x_{\min} \neq x_{\max} \). Clearly,

\[
m \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq M.
\]

By the intermediate value theorem, \( f \) assumes every value between \( m \) and \( M \), so there exists \( x_0 \in (x_{\min}, x_{\max}) \) or \((x_{\max}, x_{\min})\) (depending on whether \( x_{\min} < x_{\max} \) or \( x_{\max} < x_{\min} \)) such that

\[
f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n},
\]

Let \( f(x) = \sin x \) and let \( x_k = k\pi/12 \in (0, \pi/2), \ k = 1, 2, 3, 4, 5 \). Since

\[
\sin \frac{\pi}{12} = \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{1}{2\sqrt{2}} (\sqrt{3} - 1),
\]

\[
\sin \frac{5\pi}{12} = \sin \left( \frac{\pi}{4} + \frac{\pi}{6} \right) = \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{1}{2\sqrt{2}} (\sqrt{3} + 1),
\]

we have

\[
\frac{1}{5} \left( \sin \frac{\pi}{12} + \sin \frac{\pi}{6} + \sin \frac{\pi}{4} + \sin \frac{\pi}{3} + \sin \frac{5\pi}{12} \right) = \frac{1 + \sqrt{2} + \sqrt{3} + \sqrt{6}}{10}.
\]

The existence of the required \( \theta \in (0, \pi/2) \) then follows from the earlier part.