1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence. We know from Theorem 2.1 in the lectures that if $(x_n)_{n \in \mathbb{N}}$ is convergent, then

$$\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n).$$

But suppose $(x_n)_{n \in \mathbb{N}}$ is not convergent.

(a) Find counterexamples to show that the following equalities do not always hold:

$$\limsup_{n \to \infty} f(x_n) = f(\limsup_{n \to \infty} x_n) \quad \text{and} \quad \liminf_{n \to \infty} f(x_n) = f(\liminf_{n \to \infty} x_n).$$

(b) Prove that

$$\limsup_{n \to \infty} f(x_n) \geq f(\limsup_{n \to \infty} x_n) \quad \text{and} \quad \liminf_{n \to \infty} f(x_n) \leq f(\liminf_{n \to \infty} x_n).$$

2. Let $f$ and $g$ be functions such that

$$\lim_{x \to a} f(x) = b \quad \text{and} \quad \lim_{y \to b} g(y) = c.$$

(a) Show that the following is not necessarily true

$$\lim_{x \to a} g(f(x)) = c.$$

(b) Show that it is true if $f$ is continuous at $a$ and $g$ is continuous at $b$ (which must be equals to $f(a)$), i.e.

$$\lim_{x \to a} g(f(x)) = g(f(a)).$$

3. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \frac{x + |x|}{2}, \quad f(x) = \begin{cases} x & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$

For which values of $x$ are $f$ and $g$ continuous? Write down an expression for the composite function $h(x) = g(f(x))$ and determine the values of $x$ for which it is continuous.

(b) Let $f : [-1, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{x} & \text{if } -1 \leq x < 0, \\ \alpha & \text{if } x = 0, \\ \frac{\log(1 + x)}{x} & \text{if } x > 0. \end{cases}$$

Find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$. For which values of $\alpha$ is $f$ left- or right-continuous at 0?

4. (a) Prove that the equation $(1 - x) \cos x = \sin x$ has at least one solution in $(0, 1)$. 

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(b) Let \( f, g : [a, b] \to \mathbb{R} \) be continuous functions where \( f(a) < g(a) \) and \( f(b) > g(b) \). Prove that there exists \( c \in (a, b) \) such that 

\[
    f(c) = g(c).
\]

Hence or otherwise show that a continuous \( f : [0, 1] \to [0, 1] \) must have a fixed point, i.e. \( c \in [0, 1] \) such that \( f(c) = c \).

(c) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Prove that for any \( x_1, x_2, \ldots, x_n \in (a, b) \), there exists \( x_0 \in (a, b) \) such that 

\[
    f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.
\]

Hence or otherwise show that there exists \( 0 < \theta < \pi/2 \) such that 

\[
    \sin \theta = \frac{1 + \sqrt{2} + \sqrt{3} + \sqrt{6}}{10}.
\]