MATH 104: INTRODUCTORY ANALYSIS
SPRING 2008/09
PROBLEM SET 7

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. We know from Theorem **2.1** in the lectures that if $(x_n)_{n\in\mathbb{N}}$ is convergent, then

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

But suppose $(x_n)_{n\in\mathbb{N}}$ is not convergent.

(a) Find counterexamples to show that the following equalities do not always hold:

$$\limsup_{n\to\infty} f(x_n) = f(\limsup_{n\to\infty} x_n) \quad \text{and} \quad \liminf_{n\to\infty} f(x_n) = f(\liminf_{n\to\infty} x_n).$$

(b) Prove that

$$\limsup_{n\to\infty} f(x_n) \geq f(\limsup_{n\to\infty} x_n) \quad \text{and} \quad \liminf_{n\to\infty} f(x_n) \leq f(\liminf_{n\to\infty} x_n).$$

2. Let f and g be functions such that

$$\lim_{x\to a} f(x) = b \quad \text{and} \quad \lim_{y\to b} g(y) = c.$$

(a) Show that the following is not necessarily true

$$\lim_{x\to a} g(f(x)) = c.$$

(b) Show that it is true if f is continuous at a and g is continuous at b (which must be equals to $f(a)$), i.e.

$$\lim_{x\to a} g(f(x)) = g(f(a)).$$

3. (a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \frac{x + |x|}{2}, \quad f(x) = \begin{cases} x & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$$

For which values of x are f and g continuous? Write down an expression for the composite function $h(x) = g(f(x))$ and determine the values of x for which it is continuous.

(b) Let $f : [-1, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x} & \text{if } -1 \leq x < 0, \\ \alpha & \text{if } x = 0, \\ \dfrac{\log(1+x)}{x} & \text{if } x > 0. \end{cases}$$

Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$. For which values of α is f left- or right-continuous at 0?

4. (a) Prove that the equation $(1 - x)\cos x = \sin x$ has at least one solution in $(0, 1)$.

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(b) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions where $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $c \in (a, b)$ such that

$$f(c) = g(c).$$

Hence or otherwise show that a continuous $f : [0, 1] \to [0, 1]$ must have a fixed point, i.e. $c \in [0, 1]$ such that $f(c) = c$.

(c) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that for any $x_1, x_2, \ldots, x_n \in (a, b)$, there exists $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Hence or otherwise show that there exists $0 < \theta < \pi/2$ such that

$$\sin\theta = \frac{1 + \sqrt{2} + \sqrt{3} + \sqrt{6}}{10}.$$