MATH 104: INTRODUCTORY ANALYSIS
SPRING 2008/09
PROBLEM SET 6

In each of the problems where you have been instructed to use the ε-δ definition, you are required to produce a δ that will satisfy the definition for any given $\varepsilon > 0$.

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers where $a_n > 0$ for all $n \in \mathbb{N}$. Suppose

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Is the follow series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Prove your answer.

2. Use the ε-δ definition of limits to prove the following.

$$\lim_{x\to 2}(3x-1) = 5, \quad \lim_{x\to -2} \frac{x^2+x-2}{x+2} = -3, \quad \lim_{x\to 1/2} \frac{1}{x^2} = 4,$$

$$\lim_{x\to 1} x^3 = 1, \quad \lim_{x\to 0} \frac{1}{8x+1} = 1, \quad \lim_{x\to 8} \sqrt[3]{x} = 2.$$

3. Use the ε-δ definition of limits to prove the following.
(a) $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ for any $a > 0$.
(b) $\lim_{x\to a} 1/x = 1/a$ for any $a \neq 0$.
(c) $\lim_{x\to a} 1/\sqrt{x} = 1/\sqrt{a}$ for any $a > 0$.

4. Prove the following limits do not exists.

$$\lim_{x\to 0} \frac{|x|}{x}, \quad \lim_{x\to 0} e^x \cos\frac{1}{x}, \quad \lim_{x\to 0} e^{-1/x}.$$

5. Let $a > 0$ and let $f : (-a, 0) \cup (0, a) \to \mathbb{R}$.
(a) Suppose $f(x) > 0$ for all $0 < |x| < a$ and

$$\lim_{x\to 0} \left[f(x) + \frac{1}{f(x)} \right] = 2.$$

Prove that $\lim_{x\to 0} f(x)$ exists and find its value.
(b) Suppose $f(x) \neq 0$ for all $0 < |x| < a$ and

$$\lim_{x\to 0} \left[f(x) + \frac{1}{|f(x)|} \right] = 0.$$

Prove that $\lim_{x\to 0} f(x)$ exists and find its value.

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