In each of the problems where you have been instructed to use the $\varepsilon$-$\delta$ definition, you are required to produce a $\delta$ that will satisfy the definition for any given $\varepsilon > 0$.

1. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers where $a_n > 0$ for all $n \in \mathbb{N}$. Suppose

$$\sum_{n=1}^{\infty} a_n < \infty.$$ 

Is the follow series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Prove your answer.

2. Use the $\varepsilon$-$\delta$ definition of limits to prove the following.

$$\lim_{x \to 2} (3x - 1) = 5,$$

$$\lim_{x \to -2} \frac{x^2 + x - 2}{x + 2} = -3,$$

$$\lim_{x \to 1/2} \frac{1}{x^2} = 4,$$

$$\lim_{x \to 1} \frac{1}{2} x^3 = 1,$$

$$\lim_{x \to 8} 3\sqrt[3]{x} = 2.$$ 

3. Use the $\varepsilon$-$\delta$ definition of limits to prove the following.

(a) $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ for any $a > 0$.

(b) $\lim_{x \to a} \frac{1}{x} = \frac{1}{a}$ for any $a \neq 0$.

(c) $\lim_{x \to a} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{a}}$ for any $a > 0$.

4. Prove the following limits do not exists.

$$\lim_{x \to 0} \frac{|x|}{x}, \quad \lim_{x \to 0} e^x \cos \frac{1}{x}, \quad \lim_{x \to 0} e^{-1/x}.$$ 

5. Let $a > 0$ and let $f : (-a, 0) \cup (0, a) \to \mathbb{R}$.

(a) Suppose $f(x) > 0$ for all $0 < |x| < a$ and

$$\lim_{x \to 0} \left[ f(x) + \frac{1}{f(x)} \right] = 2.$$ 

Prove that $\lim_{x \to 0} f(x)$ exists and find its value.

(b) Suppose $f(x) \neq 0$ for all $0 < |x| < a$ and

$$\lim_{x \to 0} \left[ f(x) + \frac{1}{|f(x)|} \right] = 0.$$ 

Prove that $\lim_{x \to 0} f(x)$ exists and find its value.