In this problem set you will need to prove your claims rigorously.

1. Let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be a sequence of real numbers that satisfy
\[ a_n \leq b_n \quad \text{for all } n > N \]
for some \(N \in \mathbb{N}\).
(a) Prove that if \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are convergent, then
\[ \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n. \]

**Solution.** Suppose not. Let \(a = \lim_{n \to \infty} a_n\) and \(b = \lim_{n \to \infty} b_n\) and suppose \(a > b\). Set \(\varepsilon = (a - b)/2 > 0\). Then there exists \(N_1 \in \mathbb{N}\) such that \(|a_n - a| < \varepsilon/2\) and \(N_2 \in \mathbb{N}\) such that \(|b_n - b| < \varepsilon/2\). So if \(n > \max\{N, N_1, N_2\}\), then
\[ a_n > a - \varepsilon = a - \left(\frac{a - b}{2}\right) = b + \left(\frac{a - b}{2}\right) = b + \varepsilon > b_n. \]
That is, \(a_n > b_n\) for some \(n > N\), which contradicts the given condition.
(b) Prove that if \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) satisfy
\[ a_n \leq b_n \quad \text{for all } n > N \]
for some \(N \in \mathbb{N}\), then
\[ \liminf_{n \to \infty} a_n \leq \liminf_{n \to \infty} b_n \quad \text{and} \quad \limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n. \]

**Solution.** By the given condition, for \(n > N\), we must have
\[ \sup_{k \geq n} a_k \leq \sup_{k \geq n} b_k. \]
We now apply part (a) to the sequences \((\sup_{k \geq n} a_k)_{n \in \mathbb{N}}\) and \((\sup_{k \geq n} b_k)_{n \in \mathbb{N}}\) to get that
\[ \lim_{n \to \infty} \sup_{k \geq n} a_k \leq \lim_{n \to \infty} \sup_{k \geq n} b_k \]
and thus
\[ \limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n. \]
Likewise, for \(n > N\), we have
\[ \inf_{k \geq n} a_k \leq \inf_{k \geq n} b_k. \]
We now apply part (a) to the sequences \((\inf_{k \geq n} a_k)_{n \in \mathbb{N}}\) and \((\inf_{k \geq n} b_k)_{n \in \mathbb{N}}\) to get that
\[ \lim_{n \to \infty} (\inf_{k \geq n} a_k) \leq \lim_{n \to \infty} (\inf_{k \geq n} b_k) \]
and thus
\[ \liminf_{n \to \infty} a_n \leq \liminf_{n \to \infty} b_n. \]

2. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers that may or may not be convergent.
(a) Prove that
\[ \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n. \]

**SOLUTION.** Since for all \( n \in \mathbb{N} \), we must have
\[ \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k, \]
applying Problem 1(a) to the sequences \( (\inf_{k \geq n} a_k)_{n \in \mathbb{N}} \) and \( (\sup_{k \geq n} a_k)_{n \in \mathbb{N}} \) yields
\[ \liminf_{n \to \infty} a_k \leq \limsup_{n \to \infty} a_k \]
and thus
\[ \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n. \]

(b) Prove that \( (a_n)_{n \in \mathbb{N}} \) is bounded above iff \( \limsup_{n \to \infty} a_n < \infty \) and is bounded below iff \( \liminf_{n \to \infty} a_n > -\infty \).

**SOLUTION.** If \( \limsup_{n \to \infty} a_n = a < \infty \), then
\[ \lim_{n \to \infty} \sup_{k \geq n} a_k = a \]
and so \( (\sup_{k \geq n} a_k)_{n \in \mathbb{N}} \) is a convergent sequence and therefore bounded by our result in the lectures. In particular, the first term \( \sup_{k \geq 1} a_k =: M \) is finite and therefore \( \{a_k \mid k \in \mathbb{N}\} \) is bounded by \( M \).

Conversely, if \( \{a_k \mid k \in \mathbb{N}\} \) is bounded above by some \( M > 0 \), then since for all \( n \in \mathbb{N} \),
\[ \{a_k \mid k > n\} \subseteq \{a_k \mid k \in \mathbb{N}\}, \]
we see that
\[ \sup_{k \geq n} a_k \leq \sup_{k \geq 1} a_k \leq M. \]

Applying Problem 1(a) to the sequence \( (\sup_{k \geq n} a_k)_{n \in \mathbb{N}} \) and the constant sequence \( (M)_{n \in \mathbb{N}} \), we see that
\[ \lim_{n \to \infty} \sup_{k \geq n} a_k \leq \lim_{n \to \infty} M \]
and so
\[ \limsup_{n \to \infty} a_n \leq M. \]

The second equality may be similarly proved or may be deduced from part (c).

(c) Prove that
\[ -\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} (-a_n) \quad \text{and} \quad -\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} (-a_n). \]

**SOLUTION.** Let \( n \in \mathbb{N} \). By Homework 1, Problem 3,
\[ \inf\{-a_k \mid k > n\} = \inf\{-a_k \mid k > n\} = \sup\{a_k \mid k > n\}. \]
Since \( n \in \mathbb{N} \) is arbitrary, we may take limits to get
\[ \lim_{n \to \infty} \inf_{k \geq n} (-a_k) = \lim_{n \to \infty} (-\sup_{k \geq n} a_k) = -\lim_{n \to \infty} \sup_{k \geq n} a_k \]
and therefore
\[ \lim_{n \to \infty} (-a_n) = -\limsup_{n \to \infty} a_n. \]

The second inequality follows from applying the first to the sequence \( (-a_n)_{n \in \mathbb{N}} \) to get
\[ \liminf_{n \to \infty} a_n = -\limsup_{n \to \infty} (-a_n). \]
3. Let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be sequences of real numbers. Prove that
\[
\liminf_{n \to \infty} (a_n + b_n) \geq \liminf_{n \to \infty} a_n + \liminf_{n \to \infty} b_n
\]
and
\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.
\]

**SOLUTION.** Let \(n \in \mathbb{N}\). Note that
\[
\{a_k + b_k \mid k > n\} \subseteq \{a_k + b_l \mid k, l > n\}
\]
and so
\[
\sup\{a_k + b_k \mid k > n\} \leq \sup\{a_k + b_l \mid k, l > n\}.
\]
But
\[
\{a_k + b_l \mid k, l > n\} = \{a_k \mid k > n\} + \{b_l \mid l > n\},
\]
(recall that the LHS is the Minkowski sum) and by Homework 1, Problem 3,
\[
\sup(\{a_k \mid k > n\} + \{b_l \mid l > n\}) = \sup\{a_k \mid k > n\} + \sup\{b_l \mid l > n\}.
\]
Hence
\[
\sup\{a_k + b_k \mid k > n\} \leq \sup\{a_k \mid k > n\} + \sup\{b_k \mid k > n\}
\]
for any \(n \in \mathbb{N}\). Applying Problem 1(a) to the sequences \((\sup_{k \geq n}(a_k + b_k))_{n \in \mathbb{N}}\) and \((\sup_{k \geq n} a_k + \sup_{k \geq n} b_k)_{n \in \mathbb{N}}\), we get
\[
\lim_{n \to \infty} \sup_{k \geq n}(a_k + b_k) \leq \lim_{n \to \infty} \left[ \sup_{k \geq n} a_k + \sup_{k \geq n} b_k \right]
= \lim_{n \to \infty} \sup_{k \geq n} a_k + \lim_{n \to \infty} \sup_{k \geq n} b_k.
\]
In other words,
\[
\lim_{n \to \infty} \sup_{k \geq n}(a_k + b_k) \leq \lim_{n \to \infty} \sup_{k \geq n} a_k + \lim_{n \to \infty} \sup_{k \geq n} b_k.
\]
Now apply Problem 2(c) and the above to get
\[
\lim_{n \to \infty} \inf_{n \to \infty} (a_n + b_n) = -\lim_{n \to \infty} \sup_{n \to \infty} -(a_n + b_n)
\geq -\lim_{n \to \infty} \sup_{n \to \infty} (-a_n) - \lim_{n \to \infty} \sup_{n \to \infty} (-b_n)
= \lim_{n \to \infty} \inf_{n \to \infty} a_n + \lim_{n \to \infty} \inf_{n \to \infty} b_n.
\]

4. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers that satisfy
\[
\lim_{n \to \infty} (2a_{n+1} - a_n) = a
\]
and \(a \in \mathbb{R}\). Prove that \((a_n)_{n \in \mathbb{N}}\) is a convergent sequence and find its limit. [Hint: Use superior and inferior limits.]

**SOLUTION.** Since \((2a_{n+1} - a_n)_{n \in \mathbb{N}}\) is convergent, it is bounded and so there exists \(M > 0\) be such that
\[
|2a_{n+1} - a_n| \leq M
\]
for all \(n \in \mathbb{N}\). We will first show by induction that \((a_n)_{n \in \mathbb{N}}\) is bounded by \(M' = \max\{M, |a_1|\}\). Clearly, \(|a_1| \leq M'\). Suppose \(|a_k| \leq M'\), then
\[
|a_{k+1}| = \left| \frac{2a_{k+1} - a_k + a_k}{2} \right| \leq \frac{1}{2} |2a_{k+1} - a_k| + \frac{1}{2} |a_k| \leq \frac{1}{2} M + \frac{1}{2} M' \leq M'.
\]
Hence \(|a_n| \leq M'\) for all \(n \in \mathbb{N}\). Since \((a_n)_{n \in \mathbb{N}}\) is bounded, by Problem 2(b), \(\limsup_{n \to \infty} a_n = b\) and \(\liminf_{n \to \infty} a_n = c\) for some \(b, c \in \mathbb{R}\). Now observe that
\[
a_{n+1} = \frac{a_n + (2a_{n+1} - a_n)}{2}, \quad (4.1)
\]
Taking superior limit and using Problem 3, we get
\[
\lim_{n \to \infty} a_n = \frac{1}{2} \lim_{n \to \infty} \left[ \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} (2a_{n+1} - a_n) \right]
\]
and so \( b \leq \frac{b + a}{2} \)

and so \( b \leq a \). Likewise taking inferior limit and using Problem 3, we get
\[
\lim_{n \to \infty} a_n = \frac{1}{2} \lim_{n \to \infty} \left[ \liminf_{n \to \infty} a_n + \liminf_{n \to \infty} (2a_{n+1} - a_n) \right]
\]
which gives
\[
c \geq \frac{c + a}{2}
\]
and so \( c \geq a \). In other words, \( b \leq a \leq c \) and so
\[
\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} a_n.
\]
But the reverse equality is always true by Problem 2(a) and hence
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n.
\]
Therefore \( \lim_{n \to \infty} a_n = l \) exists. To find the value of \( l \), we take limit in (4.1),
\[
\lim_{n \to \infty} a_{n+1} = \frac{\lim_{n \to \infty} a_n + \lim_{n \to \infty} (2a_{n+1} - a_n)}{2}
\]
which gives
\[
l = \frac{l + a}{2}
\]
and so \( l = a \).

5. Define the sequence \((a_n)_{n \in \mathbb{N}}\) by \( a_1 = 1, \ a_2 = 2, \)
\[
a_{n+1} = a_n + a_{n-1}
\]
for \( n \geq 2 \). Prove that the sequence
\[
\left( \frac{a_{n+1}}{a_n} \right)_{n \in \mathbb{N}}
\]
is convergent and find its limit. [Hint: Prove that it is Cauchy.]

SOLUTION. We will show that \((a_{n+1}/a_n)_{n \in \mathbb{N}}\) is a Cauchy sequence. Note that
\[
\left| \frac{a_{n+1}}{a_n} - \frac{a_n}{a_{n-1}} \right| = \left| \frac{a_n + a_{n-1}}{a_{n-1} + a_{n-2}} - \frac{a_n}{a_{n-1}} \right| = \left| \frac{a_{n-1} - a_n a_{n-2}}{a_{n-1} + a_{n-1} a_{n-2}} \right|,
\]
Clearly \((a_n)_{n \in \mathbb{N}}\) is a monotone increasing sequence and all \( a_n > 0 \). So
\[
a_{n-1}(a_{n-1} - a_{n-2}) \geq 0
\]
and so
\[
a_{n-1}^2 - a_{n-1} a_{n-2} \geq 0.
\]
Adding $2a_{n-1}a_{n-2}$ to both sides gives
\[ a_{n-1}^2 + a_{n-1}a_{n-2} \geq 2a_{n-1}a_{n-2}. \]  
(5.3)

Applying (5.3) to (5.2), we get
\[ \left| \frac{a_{n+1}}{a_n} - \frac{a_n}{a_{n-1}} \right| \leq \left| \frac{a_{n+1}}{a_n} - \frac{a_{n-1}}{a_{n-2}} \right| = \frac{1}{2} \left| \frac{a_n}{a_{n-1}} - \frac{a_{n-1}}{a_{n-2}} \right|. \]

Applying this inequality recursively, we get
\[ \left| \frac{a_{m+1}}{a_m} - \frac{a_{n+1}}{a_n} \right| \leq \frac{1}{2} \left| \frac{a_{m+1}}{a_m} - \frac{a_{m-1}}{a_{m-2}} \right| + \frac{1}{2} \left| \frac{a_{m-1}}{a_{m-2}} - \frac{a_{m-2}}{a_{m-3}} \right| + \cdots + \frac{1}{2} \left| \frac{a_{n+1}}{a_n} - \frac{a_1}{a_0} \right| \]
\[ \leq \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2}\]
\[ = \frac{1}{2^{m-1}} \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}} \right) \]
\[ \leq \frac{1}{2^{m-1}} \sum_{k=1}^{\infty} \frac{1}{2^k} \]
\[ = \frac{1}{2^{m-1}} \]
\[ = \frac{1}{(1+1)^{n-1}} \]
\[ \leq \frac{1}{n} \]

where the last step follows from Bernoulli’s inequality. Hence for any given $\varepsilon > 0$, pick $N \in \mathbb{N}$ so that
\[ \frac{1}{N} < \varepsilon. \]

Then when $m, n > N$, we have that
\[ \left| \frac{a_{m+1}}{a_m} - \frac{a_{n+1}}{a_n} \right| \leq \max \left\{ \frac{1}{n+1}, \frac{1}{m+1} \right\} < \frac{1}{N} < \varepsilon. \]

This shows that $(a_{n+1}/a_n)_{n \in \mathbb{N}}$ is a Cauchy. By Theorem 1.17 in the lectures, it is convergent. Let
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l. \]

Then since
\[ \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}, \]
we may take limits to get
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 + \lim_{n \to \infty} \frac{a_{n-1}}{a_n} = 1 + \frac{1}{\lim_{n \to \infty} a_n/a_{n-1}}. \]

Hence
\[ l = 1 + \frac{1}{l} \]

or
\[ l^2 - l - 1 = 0. \]
The quadratic equation has one positive root

\[ l = \frac{1 + \sqrt{5}}{2}. \]

Observe that this constitutes a rigorous proof of Homework 2, Problem 3(b).