Unlike the previous problem set, in this one you will need to prove your claims rigorously.

1. (a) Prove Bernoulli’s inequality: \((1 + x)^n \geq 1 + nx\) for every real number \(x \geq -1\) and every \(n \in \mathbb{N}\).

(b) Define the sequence \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) by

\[
a_n = 2 - \frac{1}{2^n} \quad \text{and} \quad b_n = \frac{2n^2 + 1}{n^2 + 3n}.
\]

Prove from the definition of limits that

\[
\lim_{n \to \infty} a_n = 2 = \lim_{n \to \infty} b_n.
\]

In other words, given \(\varepsilon > 0\), you should produce a corresponding \(N \in \mathbb{N}\) that satisfies the definition.

2. Let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be sequences of real numbers.

(a) Suppose \(\lim_{n \to \infty} a_n = 0\) and \((b_n)_{n \in \mathbb{N}}\) is bounded (but not necessarily convergent). Prove that \(\lim_{n \to \infty} a_nb_n = 0\).

(b) Suppose \((a_n)_{n \in \mathbb{N}}\) is convergent and \((b_n)_{n \in \mathbb{N}}\) is divergent. Prove that \((a_n + b_n)_{n \in \mathbb{N}}\) is divergent.

3. Let \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) be sequences of real numbers. Are the following statements true or false? You need to prove the statement or give a counterexample.

(a) If \((a_n)_{n \in \mathbb{N}}\) is convergent and \((b_n)_{n \in \mathbb{N}}\) is divergent, then \((a_nb_n)_{n \in \mathbb{N}}\) is divergent.

(b) If \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are both divergent, then \((a_n + b_n)_{n \in \mathbb{N}}\) is divergent.

(c) If \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are both divergent, then \((a_nb_n)_{n \in \mathbb{N}}\) is divergent.

4. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers. Define the sequence \((s_n)_{n \in \mathbb{N}}\) by

\[
s_n := \frac{a_1 + a_2 + \cdots + a_n}{n}
\]

for every \(n \in \mathbb{N}\).

(a) Prove that if \(\lim_{n \to \infty} a_n = a\), then \(\lim_{n \to \infty} s_n = a\).

(b) Give an example to show that the converse is not always true.