MATH 104: INTRODUCTORY ANALYSIS
SPRING 2008/09
PROBLEM SET 3

Unlike the previous problem set, in this one you will need to prove your claims rigorously.

1. (a) Prove Bernoulli's inequality: $(1+x)^n \geq 1 + nx$ for every real number $x \geq -1$ and every $n \in \mathbb{N}$.

(b) Define the sequence $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ by

$$a_n = 2 - \frac{1}{2^n} \quad \text{and} \quad b_n = \frac{2n^2+1}{n^2+3n}.$$

Prove from the definition of limits that

$$\lim_{n\to\infty} a_n = 2 = \lim_{n\to\infty} b_n.$$

In other words, given $\varepsilon > 0$, you should produce a corresponding $N \in \mathbb{N}$ that satisfies the definition.

2. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers.

(a) Suppose $\lim_{n\to\infty} a_n = 0$ and $(b_n)_{n\in\mathbb{N}}$ is bounded (but not necessarily convergent). Prove that $\lim_{n\to\infty} a_n b_n = 0$.

(b) Suppose $(a_n)_{n\in\mathbb{N}}$ is convergent and $(b_n)_{n\in\mathbb{N}}$ is divergent. Prove that $(a_n+b_n)_{n\in\mathbb{N}}$ is divergent.

3. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers. Are the following statements true or false? You need to prove the statement or give a counterexample.

(a) If $(a_n)_{n\in\mathbb{N}}$ is convergent and $(b_n)_{n\in\mathbb{N}}$ is divergent, then $(a_n b_n)_{n\in\mathbb{N}}$ is divergent.

(b) If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are both divergent, then $(a_n + b_n)_{n\in\mathbb{N}}$ is divergent.

(c) If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are both divergent, then $(a_n b_n)_{n\in\mathbb{N}}$ is divergent.

4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define the sequence $(s_n)_{n\in\mathbb{N}}$ by

$$s_n := \frac{a_1 + a_2 + \cdots + a_n}{n}$$

for every $n \in \mathbb{N}$.

(a) Prove that if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} s_n = a$.

(b) Give an example to show that the converse is not always true.

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