1. (a) Find the following limits and prove your answers:

\[
\lim_{n \to \infty} \frac{n!}{n^n}, \quad \lim_{n \to \infty} \sqrt[n]{n}, \quad \lim_{n \to \infty} \frac{1}{\sqrt[n]{n!}}.
\]

(b) Let \( \alpha \in \mathbb{R} \) and \( \alpha > 0 \). Let \( k \in \mathbb{N} \). Find the following limits and prove your answers:

\[
\lim_{n \to \infty} \frac{\alpha^n}{n!}, \quad \lim_{n \to \infty} \frac{n^k}{\alpha^n}, \quad \lim_{n \to \infty} \sqrt[n]{\alpha}.
\]

2. Define the sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) by

\[
a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2},
\]

\[
b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}.
\]

Prove that

\[
\lim_{n \to \infty} a_n = 0 \quad \text{and} \quad \lim_{n \to \infty} b_n = 1.
\]

3. Prove that the following sequences converge and find the value of their limits.

(a) \( (a_n)_{n \in \mathbb{N}} \) where

\[
a_1 = \sqrt{2}, \quad a_2 = \sqrt{2 + \sqrt{2}}, \quad a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots
\]

(b) \( (b_n)_{n \in \mathbb{N}} \) where

\[
b_1 = 1, \quad b_2 = 1 + \frac{1}{1}, \quad b_3 = 1 + \frac{1}{1 + \frac{1}{1}}, \quad b_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \ldots
\]

4. Let \( a_1, b_1 \in \mathbb{R} \) and \( a_1 > b_1 > 0 \). Define the sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) by

\[
a_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad b_n = \sqrt{a_{n-1}b_{n-1}}
\]

for all \( n \in \mathbb{N} \). Prove that both sequences converge and that the limits satisfy

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.
\]

Note that \( a_2 \) and \( b_2 \) are respectively the arithmetic mean and geometric mean of \( a_1 \) and \( b_1 \). The common limit above is called the arithmetic-geometric mean of \( a_1 \) and \( b_1 \).

5. (a) Prove that for all \( n \in \mathbb{N} \),

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}^n = \begin{bmatrix}
\cos n\theta & -\sin n\theta \\
\sin n\theta & \cos n\theta
\end{bmatrix}.
\]

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(b) Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of \(2 \times 2\) matrices where

\[
A_n = \begin{bmatrix}
a_n & b_n \\
c_n & d_n
\end{bmatrix}.
\]

If the sequences of real numbers \((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}, (d_n)_{n \in \mathbb{N}}\) converge to \(a, b, c, d \in \mathbb{R}\) respectively, the matrix

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

is called the limit matrix of \((A_n)_{n \in \mathbb{N}}\) and is denoted \(A = \lim_{n \to \infty} A_n\).

Let \(\alpha \in \mathbb{R}\) and \(\alpha > 0\). For all \(n \in \mathbb{N}\), we define

\[
A_n := \begin{bmatrix} 1 & -\alpha \\
\alpha & 1
\end{bmatrix}^n
\]

and let \(\lim_{n \to \infty} A_n = A\). Show that

\[
A = \begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix}
\]

You may assume that \(\lim_{n \to \infty} (1 + z/n)^n = e^z\) for any \(z \in \mathbb{C}\). [Hint: Let \(\frac{\alpha}{n} = \tan \theta_n\) and apply part (a) and De Moivre’s theorem.]