MATH 104: INTRODUCTORY ANALYSIS
SPRING 2008/09
PROBLEM SET 2

Feel free to use results about sequences that you learnt in Math **1B**. There is no need to prove anything from the definition of limit in this homework — just use mathematical induction, squeezing/sandwich lemma, bounded monotone convergence, etc.

1. (a) Find the following limits and prove your answers:

$$\lim_{n\to\infty} \frac{n!}{n^n}, \quad \lim_{n\to\infty} \sqrt[n]{n}, \quad \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}.$$

(b) Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. Let $k \in \mathbb{N}$. Find the following limits and prove your answers:

$$\lim_{n\to\infty} \frac{\alpha^n}{n!}, \quad \lim_{n\to\infty} \frac{n^k}{\alpha^n}, \quad \lim_{n\to\infty} \sqrt[n]{\alpha}.$$

2. Define the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ by

$$a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2},$$
$$b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}.$$

Prove that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 1.$$

3. Prove that the following sequences converge and find the value of their limits.

(a) $(a_n)_{n\in\mathbb{N}}$ where

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2+\sqrt{2}}, \quad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots.$$

(b) $(b_n)_{n\in\mathbb{N}}$ where

$$b_1 = 1, \quad b_2 = 1 + \frac{1}{1}, \quad b_3 = 1 + \frac{1}{1+\frac{1}{1}}, \quad b_4 = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}, \ldots.$$

4. Let $a_1, b_1 \in \mathbb{R}$ and $a_1 > b_1 > 0$. Define the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ by

$$a_n = \frac{a_{n-1}+b_{n-1}}{2}, \quad b_n = \sqrt{a_{n-1}b_{n-1}}$$

for all $n \in \mathbb{N}$. Prove that both sequences converge and that the limits satisfy

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Note that a_2 and b_2 are respectively the arithmetic mean and geometric mean of a_1 and b_1. The comon limit above is called the *arithmetic-geometric mean* of a_1 and b_1.

5. (a) Prove that for all $n \in \mathbb{N}$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Date: February 17, 2009 (Version 1.0); due: February 25, 2009.

(b) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of 2×2 matrices where

$$A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}.$$

If the sequences of real numbers $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$ converge to $a, b, c, d \in \mathbb{R}$ respectively, the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is called the *limit matrix* of $(A_n)_{n\in\mathbb{N}}$ and is denoted $A = \lim_{n\to\infty} A_n$.
Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. For all $n \in \mathbb{N}$, we define

$$A_n := \begin{bmatrix} 1 & -\frac{\alpha}{n} \\ \frac{\alpha}{n} & 1 \end{bmatrix}^n$$

and let $\lim_{n\to\infty} A_n = A$. Show that

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

You may assume that $\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$ for any $z \in \mathbb{C}$. [*Hint*: Let $\frac{\alpha}{n} = \tan\theta_n$ and apply part (a) and De Moivre's theorem.]