1. (20 pts) In this question we solve the 2D Poisson equation of electrostatics using finite differences. Consider that the square \([0, 1]^2\) is made of a dielectric material with permittivity 1. It is subjected to (1) a given pattern of electric potential \(f(y)\) on its left side, (2) it is grounded on its right side, and (3) it is insulated on its top and bottom sides. The resulting equation for the potential \(u(x,y)\) inside the square, as a function of the excitation \(f(y)\), is

\[-\Delta u(x,y) = 0, \quad x \in [0, 1]^2,\]

\[u(0,y) = f(y), \quad u(1,y) = 0, \quad 0 \leq y \leq 1, \quad \text{(Dirichlet conditions)}\]

\[\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0, \quad 0 \leq x \leq 1, \quad \text{(Neumann conditions)}\]

Note that the electric current is \(\nabla u\). As usual, \(-\Delta\) is minus the Laplacian. Unless otherwise stated, assume that \(f(y) = \cos(2\pi y)\).

(a) Propose a second-order finite difference discretization for minus the Laplacian, which takes into account the boundary conditions. Detail your choice for the number of grid points that you use in \(x\) vs. \(y\), and whether your cells are square or rectangular. [Hint: I strongly recommend to use the ghost point method for the Neumann conditions, i.e., in 1D, write the centered scheme \(U_{-1} - U_1 = 0\) with a ghost point \(x_{-1}\) outside the domain, and eliminate this point by evaluating the equation one more time at the boundary. In that case the Laplacian, written as a block matrix, should be of the form

\[-\tilde{\Delta} = A \otimes I + I \otimes B,\]

where \(A\) is a matrix for minus the 1D Dirichlet Laplacian, \(B\) is a matrix for minus the 1D Neumann Laplacian, and \(\otimes\) is the Kronecker product. You already know \(A\); this exercise is about finding \(B\) and specifying the sizes of \(A, B\), and the two identities.]

(b) Using this FD scheme, implement a solver for \(u(x,y)\). Solve the linear system by a method of your choice, such as Matlab’s backslash. Illustrate the convergence of your numerical scheme in a log-log plot of the maximum norm of the error vs. the grid spacing \(h\). Check that the slope is approximately 2 in this graph. [Hint: either find the exact solution as a basis for comparison, or use a numerical solution on a very fine grid for that purpose. I strongly recommend that you use nested grids for the different values of \(h\), so that the points on a coarse grid are a subset of the points on a finer grid.]

(c) Argue (very briefly; in one sentence perhaps) the consistency of your scheme, i.e., how does the local truncation error depend on \(h\). It is fine to assume that the solution \(u\) is infinitely differentiable, for now.

(d) What are the eigenvalues and eigenvectors of \(A \otimes I + I \otimes B\) as a function of those of \(A\) and \(B\)? Justify your answer.

(e) The rows of \(B\) that correspond to the Neumann conditions can always be rescaled by an arbitrary nonzero number, since the corresponding components in the right-hand-side are zero. Perform such a rescaling to ensure that \(B\) is a symmetric matrix. Then prove that \(B\) is positive semidefinite, i.e., all its eigenvalues are nonnegative. [Hint: one eigenvalue must be zero, and its eigenvector is constant.]

(f) Use the results in (d) and (e) to show that there is an eigenvalue gap:

\[\lambda_{\min}(-\tilde{\Delta}) \geq C > 0,\]

with \(C\) independent of \(h\). This ensures stability, so the order of convergence must be the same as the order of consistency.
(g) Replace $f(y)$ given above by

$$\tilde{f}(y) = \text{sgn}(\cos(2\pi y)),$$

for the left boundary condition, where $\text{sgn}(x) = 1$ if $x \geq 0$, and $-1$ if $x < 0$. Run your code again. Empirically, what does the order of convergence become? In one sentence, how do you explain this behavior?
Concerning bonus questions: totals will be rounded down to 100.