RANDOM WALK ON A SURFACE GROUP: BOUNDARY BEHAVIOR OF THE GREEN’S FUNCTION AT THE SPECTRAL RADIUS

STEVEN P. LALLEY

ABSTRACT. It is proved that the Green’s function of the simple random walk on a surface group of large genus decays exponentially at the spectral radius. It is also shown that Ancona’s inequalities extend to the spectral radius $R$, and therefore that the Martin boundary for $R$—potentials coincides with the natural geometric boundary $S^1$.

1. INTRODUCTION

1.1. Ancona’s inequalities. A countable group $\Gamma$ is word-hyperbolic if for some (and therefore every) finite symmetric set of generators $A$ the Cayley graph $G^\Gamma$ relative to the generating set $A$ has thin triangles. See [6] and [7] for background on hyperbolic group theory. Every Fuchsian group is hyperbolic; so is the fundamental group of any compact negatively curved manifold. To every hyperbolic group is attached a natural geometric (Gromov) boundary ([10]); for a co-compact Fuchsian group, this coincides with the circle $S^1$ at infinity of the hyperbolic plane. It is natural to ask how the geometric boundary is related to the Martin and Poisson boundaries of random walks on $\Gamma$. For co-compact Fuchsian groups, Series [15] showed that the Martin boundary and the geometric boundary coincide; Ancona [2] later showed that this holds for all hyperbolic groups, and Kaimanovich [9] showed that the Poisson boundary also coincides with the geometric boundary.

If $\Gamma$ is a nonelementary hyperbolic group — that is, $\Gamma$ is neither finite nor does it contain $\mathbb{Z}$ as a finite-index subgroup — then it is nonamenable, and so by a theorem of Kesten [12], every symmetric, nearest neighbor random walk $X_n$ on $\Gamma$ (that is, a random walk whose step distribution $p(x, y) = p(x^{-1} y) = p(y^{-1} x)$ is symmetric and has support $A$) has spectral radius $R > 1$. Thus, the Green’s function

$$G_r(x, y) := \sum_{n=0}^{\infty} P_x^\pi \{X_n = y\} r^n = G_r(1, x^{-1} y)$$

has radius of convergence $R$ for every pair of group elements $x, y \in \Gamma$ (here $P_x^\pi$ denotes the probability measure on path space that makes $X_n$ a random walk with the specified step distribution and initial point $X_0 = x$). Furthermore, (cf. [17], Th. 7.8) nonamenability also implies that the random walk is $R$—transient, that is,

$$G_R(x, y) < \infty \quad \forall \ x, y.$$
Thus, for every \( r \leq R \) there is a Martin boundary for \( r \)–potentials. Ancona proved that for \( r < R \) this coincides with the geometric boundary. To do so he proved that the Green’s function is nearly submultiplicative in the following sense: for each \( r < R \) there is a constant \( C_r < \infty \) such that for every geodesic segment \( x_0 x_1 \cdots x_m \) in \( \Gamma \),

\[
G_r(x_0, x_m) \leq C_r G_r(x_0, x_k) G_r(x_k, x_m) \quad \forall 1 \leq k \leq m.
\]

His argument depends in an essential way on the hypothesis \( r < R \) (cf. his Condition \((*)\)) and therefore leaves open the structure of the Martin boundary for \( R \)–potentials (and examples \([1]\) show that in related problems the Martin boundary at the spectral radius may be quite different from the geometric boundary). The purpose of this paper is to show that in the special case of simple random walk on a surface group, Ancona’s inequalities extend to \( R \), and therefore that the Martin boundary for \( R \)–potentials coincides with the geometric boundary. Denote by \( \Gamma_g \) the surface group of genus \( g \), and let \( A = A_g \) is the standard symmetric set of generators for \( \Gamma_g \): thus,

\[
A_g = \{a_i^\pm, b_i^\pm\}_{1 \leq i \leq g},
\]

and these generators satisfy the fundamental relation

\[
\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1.
\]

**Theorem 1.** If the genus \( g \) is sufficiently large and the step distribution is the uniform distribution on \( A_g \), then

(A) The Green’s function \( G_R(1, x) \) decays exponentially in \( |x| := d(1, x) \);

(B) Ancona’s inequalities \((3)\) hold for all \( r \leq R \), with a constant \( C \) independent of \( r \); and

(C) The Martin boundary for \( R \)–potentials coincides with the geometric boundary \( S^1 \).

**Note 2.** Here and throughout the paper \( d(x, y) \) denotes the distance between the vertices \( x \) and \( y \) in the Cayley graph \( \Gamma^\top \), equivalently, distance relative to the word metric. **Exponential decay** of the Green’s function means *uniform* exponential decay in all directions, that is, there are constants \( C < \infty \) and \( g < 1 \) such that for all \( x, y \in \Gamma_g \),

\[
G_R(x, y) \leq C g^{d(x, y)}.
\]

A very simple argument, given in section \( 2.6 \) below, shows that for random walk on any nonamenable group \( G_R(1, x) \rightarrow 0 \) as \( |x| = d(1, x) \rightarrow \infty \). Given this, it is routine to show that exponential decay of the Green’s function follows from Ancona’s inequalities. Nevertheless, an independent — and simpler — proof of exponential decay is given in section \( 4.5 \).

**Note 3.** Theorem \([1]\)(A) is a discrete analogue of the main result of Hamenstaedt \([8]\) concerning the Green’s function of the Laplacean on the universal cover of a compact negatively curved manifold. Unfortunately, Hamenstaedt’s proof appears to have a serious error\([8]\) The approach taken here is quite different than that of \([8]\).

\[1\]The error is in the proof of Lemma 3.1: The claim is made that a lower bound on a finite measure implies a lower bound for its Hausdorff-Billingsley dimension relative to another measure. This is false — in fact such a lower bound on measure implies an *upper* bound on its Hausdorff-Billingsley dimension.
Assertions (A)–(B) of Theorem 1 are proved in section 4 below. The argument leans on both the planarity of the Cayley graph $G^\Gamma$ of a surface group and the large isoperimetric constant of $\Gamma_g$ for large genus. In addition, it requires certain a priori estimates on the Green’s function, established in section 3, specifically (see Proposition 15), that

$$\lim_{g \to \infty} \sup_{x \neq 1} G_{R_g}(1, x) = 0.$$  

For these, the fact that the step distribution is uniform on the generating set $A_g$ is used, together with a bound for the spectral radius $R_g = R_g$ of the simple random walk on $\Gamma_g$ due to Zuk [18] (see also Bartholdi et al [5] and Nagnibeda [14]):

$$R_g > \sqrt{g}.$$  

It is conceivable that the estimate (7) could be established more generally, without the symmetry hypothesis on the step distribution and without appealing to Zuk’s inequality on the spectral radius. If so, the argument of section 4 would then imply the conclusions of Theorem 1 for a broader class of random walks.

1.2. Martin boundary. Assertion (C) of Theorem 1 follows from the Ancona inequalities, roughly by the same argument as used in [1], [2] for the Martin boundary at $r < R$. (See Theorems 5.1–5.2 of [2], Theorem 6 of [1], and of [3].) For fixed $r \leq R$, the statement that the Martin boundary for $r-$potentials coincides with the geometric boundary means that (1) for every geodesic ray $y_0, y_1, y_2, \ldots$ converging to $\zeta \in \partial \Gamma$, and every $x \in \Gamma$,

$$\lim_{n \to \infty} \frac{G_r(x, y_n)}{G_r(1, y_n)} = K_r(x, \zeta) = K(x, \zeta)$$

exists; (2) for each $\zeta \in \partial \Gamma$ the function $K_{\zeta}(x) := K(x, \zeta)$ is minimal positive harmonic in $x$; (3) for distinct points $\zeta, \zeta' \in \partial \Gamma$ the functions $K_{\zeta}$ and $K_{\zeta'}$ are different; and (4) the topology of pointwise convergence on $\{K_{\zeta}\}_{\zeta \in \partial \Gamma}$ coincides with the usual topology on $\partial \Gamma = S^1$. An argument of Anderson & Schoen ([3], Theorems 5.1–6.2) shows that for each fixed $x \in G$ and $r < R$ the function $\zeta \mapsto K_r(x, \zeta)$ is in fact Hölder continuous. It is not difficult to adapt this argument to show that this holds uniformly in $r \leq R$:

**Theorem 4.** There exists $\varrho < 1$ such that for every $r \leq R$ and every geodesic ray $1 = y_0, y_1, y_2, \ldots$ converging to a point $\zeta \in \partial \Gamma$,

$$\left| \frac{G_r(x, y_n)}{G_r(1, y_n)} - K_r(x, \zeta) \right| \leq C_x g^n$$

for constants $C_x < \infty$ depending on $x \in \Gamma$ but not on $r \leq R$. Consequently, for each $x \in \Gamma$ the functions $\zeta \mapsto K_r(x, \zeta)$ are uniformly Hölder continuous in $\zeta$ for some exponent not depending on $r \leq R$, and $r \mapsto K_r(x, \cdot)$ is continuous in the Hölder norm.

Since the proof involves no new ideas it is omitted.

2. **Green’s function and related generating functions: preliminaries**

Throughout this section, $X_n$ is a symmetric, nearest neighbor random walk on a finitely generated, nonamenable group $\Gamma$ with (symmetric) generating set $A$. 


2.1. Green’s function as a sum over paths. The Green’s function $G_r(x, y)$ defined by (1) has an obvious interpretation as a sum over paths from $x$ to $y$. (Note: Here and in the sequel a path in $\Gamma$ is just the sequence of vertices visited by a path in the Cayley graph $G^\Gamma$, that is, a sequence of group elements such that any two successive elements differ by right-multiplication by a generator $a \in A$.) Denote by $R(x, y)$ the set of all paths $\gamma$ from $x$ to $y$, and for any such path $\gamma = (x_0, x_1, \ldots, x_m)$ define the weight

$$w_r(\gamma) := r^m \prod_{i=0}^{m-1} p(x_i, x_{i+1}).$$

Then

$$G_r(x, y) = \sum_{\gamma \in R(x, y)} w_r(\gamma).$$

Since the step distribution $p(a) = p(a^{-1})$ is symmetric with respect to inversion, so is the weight function $\gamma \mapsto w_r(\gamma)$: if $\gamma^R$ is the reversal of the path $\gamma$, then $w_r(\gamma^R) = w_r(\gamma)$. Consequently, the Green’s function is symmetric in its arguments:

$$G_r(x, y) = G_r(y, x).$$

Also, the weight function is multiplicative with respect to concatenation of paths, that is, $w_r(\gamma \gamma') = w_r(\gamma)w_r(\gamma')$. Since the step distribution $p(a) > 0$ is strictly positive on the generating set $A$, it follows that the Green’s function satisfies a system of Harnack inequalities: There exists a constant $C < \infty$ such that for each $0 < r \leq R$ and all group elements $x, y, z$,

$$G_r(x, z) \leq C^{d(y, z)}G_r(x, y).$$

2.2. First-passage generating functions. Other useful generating functions can be obtained by summing path weights over different sets of paths. Two classes of such generating functions that will be used below are the restricted Green’s functions and the first-passage generating functions (called the balayage by Ancona [2]) defined as follows. Fix a region $\Omega \subset G^\Gamma$, and for any two vertices $x, y \in G^\Gamma$ let $P(x, y; \Omega)$ be the set of all paths from $x$ to $y$ that remain in the region $\Omega$ at all except the initial and final points. Define

$$G_r(x, y; \Omega) = \sum_{\gamma \in P(x, y; \Omega)} w_r(\gamma), \quad \text{and} \quad F_r(x, y) = G_r(x, y; G^\Gamma \setminus \{y\}).$$

Thus, $F_r(x, y)$, the first-passage generating function, is the sum over all paths from $x$ to $y$ that first visit $y$ on the last step. This generating function has the alternative representation

$$F_r(x, y) = \mathbb{E}^x r^\tau(y)$$

where $\tau(y)$ is the time of the first visit to $y$ by the random walk $X_n$, and the expectation extends only over those sample paths such that $\tau(y) < \infty$. Note that the restricted Green’s functions $G_r(\cdot, \cdot; \Omega)$ obey Harnack inequalities similar to (14), but with the distance $d(y, z)$ replaced by the distance $d_\Omega(y, z)$ in the set $\Omega$. Finally, since any visit to $y$ by a path started at $x$ must follow a first visit to $y$,

$$G_r(x, y) = F_r(x, y)G_r(1, 1).$$

Therefore, since $G_r$ is symmetric in its arguments, so is $F_r$. 
2.3. Renewal equation for the Green’s function. The representation (1) suggests that \( G_r(1,1) \) can be interpreted as the expected “discounted” number of visits to the root 1 by the random walk, where the discount factor is \( r \). Any such visit must either occur at time \( n = 0 \) or after the first step, which must be to a generator \( x \in A \). Conditioning on the first step and using the Markov property, together with the symmetry \( F_r(1,x) = F_r(x,1) \), yields the renewal equation

\[
G_r(1,1) = 1 + \sum_{x \in A} p_x r F_r(1,x) G_r(1,1),
\]

which may be rewritten in the form

\[
G_r(1,1) = \frac{1}{1 - \sum_{x \in A} p_x r F_r(1,x)}.
\]

Since \( G_R(1,1) < \infty \) (recall that the group \( \Gamma \) is nonamenable), it follows that

\[
\sum_{x \in A} p_x RF_r(1,x) < 1.
\]

2.4. Retracing inequality. The first-passage generating function \( F_r(1,x) \) is the sum of weights of all paths that first reach \( x \) at the last step. For \( x \in A \) this may occur in one of two ways: either the path jumps from 1 to \( x \) at its first step, or it first jumps to some \( y \neq x \) and then later finds its way to \( x \). The latter will occur if the path returns to the root 1 from \( y \) without visiting \( x \), and then finds its way from 1 to \( x \). This leads to a simple bound for \( F_r(1,x) \) in terms of the avoidance generating function \( A_r(1;x) \) defined by

\[
A_r(1;x) := \sum_{y \neq x} p_y RF_r(y,1;\{x\},\{1\}) = G_r(1,1;\Gamma \setminus \{x\}).
\]

**Lemma 5.** The avoidance generating function satisfies \( A_R(1;x) < 1 \) for every \( x \in A \), and for every \( r \leq R \),

\[
F_r(1,x) \geq p_x R / (1 - A_r(1;x)).
\]

For simple random walk on the surface group \( \Gamma_g \), the inequality is strict.

**Proof.** A path \( \gamma \) that starts at the root 1 can reach \( x \) by jumping directly from 1 to \( x \), on the first step, or by jumping from 1 to \( x \) after an arbitrary number \( n \geq 1 \) of returns to 1 without first visiting \( x \). Hence,

\[
F_r(1,x) \geq p_x R \left\{ 1 + \sum_{n=1}^{\infty} A_r(1;x)^n \right\}.
\]

The inequality is strict for random walk on the surface group because in this case there are positive-probability paths from 1 to \( x \) that do not end in a jump from 1 to \( x \). Since \( F_R(1,x) < \infty \), by (17) and (2), it must be that \( A_R(1;x) < \infty \). □
2.5. Renewal inequality. The avoidance generating functions can be used to reformulate the renewal equation \((18)\) in a way that leads to a useful upper bound for the Green’s function. Recall that the renewal equation was obtained by splitting paths that return to the root 1 at the time of their first return. Consider a path \(\gamma\) starting at 1 that first returns to 1 only at its last step: such a path must either avoid \(x \in A\) altogether, or it must visit \(x\) before the first return to 1, and then subsequently find its way back to 1. Thus, for any generator \(x \in A\),
\[
G_r(1, 1) = 1 + A_r(1; x)G_r(1, 1) + F_r(1, x; \Omega \setminus \{1\})F_r(x, 1)G_r(1, 1)
\]
\[
\leq 1 + A_r(1; x)G_r(1, 1) + F_r(x, 1)^2G_r(1, 1).
\]
Solving for \(G_r(1, 1)\) gives the following renewal inequality:
\[
G_r(1, 1) \leq \{1 - A_r(1; x) - F_r(1, x)^2\}^{-1}.
\]

2.6. Backscattering. A very simple argument shows that the Green’s function \(G_R(1, x)\) converges to 0 as \(|x| \to \infty\). Observe that if \(\gamma\) is a path from 1 to \(x\), and \(\gamma'\) a path from \(x\) to 1, then the concatenation \(\gamma\gamma'\) is a path from 1 back to 1. Furthermore, since any path from 1 to \(x\) or back must make at least \(|x|\) steps, the length of \(\gamma\gamma'\) is at least \(2|x|\). Consequently, by symmetry,
\[
F_R(1, x)^2G_R(1, 1) \leq \sum_{n=2|x|}^{\infty} P^1\{X_n = 1\}R^n
\]
Since \(G_R(1, 1) < \infty\), by nonamenable of the group \(\Gamma\), the tail-sum on the right side of inequality \((23)\) converges to 0 as \(|x| \to \infty\). Several variations on this argument will be used later.

2.7. Subadditivity and the random walk metric. The concatenation of a path from \(x\) to \(y\) with a path from \(y\) to \(z\) is, obviously, a path from \(x\) to \(z\). Consequently, by the Markov property (or alternatively the path representation \((12)\) and the multiplicativity of the weight function \(w_r\)) the function \(-\log F_r(x, y)\) is subadditive:

**Lemma 6.** For each \(r \leq R\) the first-passage generating functions \(F_r(x, y)\) and \(F_r(x, y; \Omega)\) are super-multiplicative, that is, for any group elements \(x, y, z\),
\[
F_r(x, z) \geq F_r(x, y)F_r(y, z) \quad \text{and} \quad F_r(x, y; \Omega) \geq F_r(x, y; \Omega)F_r(y, z; \Omega).
\]
Together with Kingman’s subadditive ergodic theorem, this implies that the Green’s function \(G_r(1, x)\) must decay (or grow) at a fixed exponential rate along suitably chosen trajectories. For instance, if
\[
Y_n = \xi_1\xi_2\cdots\xi_n
\]
where \(\xi_n\) is an ergodic Markov chain on the alphabet \(A\), or on the set \(A^K\) of words of length \(K\), then Kingman’s theorem implies that
\[
\lim n^{-1}\log G_r(1, Y_n) = \alpha \quad \text{a.s.}
\]
where \(\alpha\) is a constant depending only on the the transition probabilities of the underlying Markov chain. More generally, if \(\xi_n\) is a suitable ergodic stationary process, then \((26)\) will hold. Super-multiplicativity of the Green’s function also implies the following.
Corollary 7. The function $d_{RW}(x, y) := \log F_R(x, y)$ is a metric on $\Gamma$.

Proof. The triangle inequality is immediate from Lemma 6 and symmetry $d_{RW}(x, y) = d_{RW}(y, x)$ follows from the corresponding symmetry property (13) of the Green’s function. Thus, to show that $d_{RW}$ is a metric (and not merely a pseudo-metric) it suffices to show that if $x \neq y$ then $F_R(x, y) < 1$. But this follows from the fact (2) that the Green’s function is finite at the spectral radius, because the path representation implies that

$$G_R(x, x) \geq 1 + F_R(x, y)^2 + F_R(x, y)^4 + \cdots.$$ 

□

Call $d_{RW}$ the random walk metric. The Harnack inequalities imply that the random walk metric $d_{RW}$ is dominated by a constant multiple of the word metric $d$. In general, there is no domination in the other direction. However:

Proposition 8. If the Green’s function decays exponentially in $d(x, y)$ (that is, if inequality (6) holds for all $x, y \in \Gamma$), then the random walk metric $d_{RW}$ and the word metric $d$ on $\Gamma$ are quasi-isometric, that is, there are constants $0 < C_1 < C_2 < \infty$ such that for all $x, y \in \Gamma$,

$$C_1 d(x, y) \leq d_{RW}(x, y) \leq C_2 d(x, y).$$

(27)

Proof. If inequality (6) holds for all $x, y \in \Gamma$, then the first inequality in (27) will hold with $C_1 = -\log \rho$. □

Remark 9. Except in the simplest cases — when the Cayley graph is a tree, as for free groups and free products of cyclic groups — the random walk metric $d_{RW}$ does not extend from $\Gamma$ to a metric on the full Cayley graph $G^\Gamma$. To see this, observe that if it did extend, then the resulting metric space $(G^\Gamma, d_{RW})$ would be path-connected, and therefore would have the Hopf-Rinow property: any two points would be connected by a geodesic segment. But this would imply that for vertices $x, z \in \Gamma$ such that $d(x, z) \geq 2$ there would be a $d_{RW}$-geodesic segment from $x$ to $z$, and such a geodesic would necessarily pass through a point $y$ such that $d(x, y) = 1$. For any such triple $x, y, z \in \Gamma$ it would then necessarily be the case that

$$F_R(x, z) = F_R(x, y)F_R(y, z).$$

This is possible only when every random walk path from $x$ to $z$ must pass through $y$ — in particular, when the Cayley graph of $\Gamma$ is a tree.

2.8. Green’s function and branching random walks. There is a simple interpretation of the Green’s function $G_r(x, y)$ in terms of the occupation statistics of branching random walks. A branching random walk is built using a probability distribution $Q = \{q_k\}_{k \geq 0}$ on the nonnegative integers, called the offspring distribution, together with the step distribution $P := \{p(x, y) = p(x^{-1}y)\}_{x, y \in \Gamma}$ of the underlying random walk, according to the following rules: At each time $n \geq 0$, each particle fissions and then dies, creating a random number of offspring with distribution $Q$; the offspring counts for different particles are mutually independent. Each offspring particle then moves from the location of its parent by making a random jump according to the step distribution $p(x, y)$; the jumps are once again mutually independent. Consider the initial condition which places a single particle at site $x \in \Gamma$, and denote the corresponding probability measure on population evolutions by $Q^x$. 

□
Proposition 10. Under $Q^x$, the total number of particles in generation $n$ evolves as a Galton-Watson process with offspring distribution $Q$. If the offspring distribution has mean $r \leq R$, then under $Q^x$ the expected number of particles at location $y$ at time $n$ is $r^n P^x \{X_n = y\}$, where under $P^x$ the process $X_n$ is an ordinary random walk with step distribution $P$. Therefore, $G_r(x, y)$ is the mean total number of particle visits to location $y$.

Proof. The first assertion follows easily from the definition of a Galton-Watson process – see [4] for the definition and basic theory. The second one shows by induction on $n$. The third then follows from the formula (1) for the Green’s function. □

There are similar interpretations of the restricted Green’s function $G_r(x, y; \Omega)$ and the first-passage generating function $F_r(x, y)$. Suppose that particles of the branching random walk are allowed to reproduce only in the region $\Omega$; then $G_r(x, y; \Omega)$ is the mean number of particle visits to $y$ in this modified branching random walk.

3. A PRIORI ESTIMATES FOR THE SURFACE GROUP

3.1. Symmetries of simple random walk on $\Gamma_g$. Recall that the generating set $A_g$ of the surface group $\Gamma_g$ consists of $2g$ letters $a_i, b_i$ and their inverses, which are subject to the relation $\prod [a_i, b_i] = 1$. This fundamental relation implies others, including

\begin{align}
\prod_{i=0}^{g-1} [b_{g-i}, a_{g-i}] &= 1 \quad \text{and} \\
\prod_{i=k+1}^{g} [a_i, b_i] \prod_{i=1}^{k} [a_i^{-1}, b_i^{-1}] &= 1.
\end{align}

Since each of these has the same form as the fundamental relation, each leads to an automorphism of the group $\Gamma_g$; relation (28) implies that the bijection

\begin{align*}
& a_i^{\pm 1} \mapsto b_{g-i}^{\pm 1}, \\
& b_i^{\pm 1} \mapsto a_{g-i}^{\pm 1},
\end{align*}

extends to an automorphism, and similarly relation (29) implies that the mapping

\begin{align*}
& a_i^{\pm 1} \mapsto a_{i+1}^{\pm 1}, \quad 1 \leq i \leq g-1, \\
& b_i^{\pm 1} \mapsto b_{i+1}^{\pm 1}, \quad 1 \leq i \leq g-1, \\
& a_g^{\pm 1} \mapsto a_1^{\mp 1}, \\
& b_g^{\pm 1} \mapsto b_1^{\mp 1}
\end{align*}

extends to an automorphism. Clearly, each of these automorphisms preserves the uniform distribution on $A_g$, and so it must also fix each of the generating functions $G_r(1, x), F_r(1, x),$ and $A_r(1; x)$. This implies
Corollary 11. For simple random walk on $\Gamma_g$,
\begin{align}
A_r(1; x) &= A_r(1; y) := A_r, \quad \forall \ x, y \in A_g, \\
F_r(1, x) &= F_r(1, y) := F_r, \quad \forall \ x, y \in A_g, \quad \text{and} \\
G_r(1, x) &= G_r(1, y) := G_r, \quad \forall \ x, y \in A_g.
\end{align}

Consequently, by the renewal equation, $G_r = 1/(1 - rF_r)$. Since $G_R < \infty$, this implies that for the simple random walk on $\Gamma_g$ the first-passage generating functions $F_r(1, x)$ for $x \in A_g$ are bounded by $1/R$:
\begin{align}
F_r(1, x) &= F_r(1, y) := F_r, \quad \forall \ x, y \in A_g.
\end{align}

3.2. Large genus asymptotics for $G_R(1, 1)$. For simple random walk on $\Gamma_g$, the symmetry relations $(30)$ and the inequalities $(21)$, $(22)$, and $(8)$ can be combined to give upper bounds for the Green’s function. Asymptotically, these take the following form:

Proposition 12. $\lim_{g \to \infty} G_R(1, 1) = 2$.

Proof. Proposition 13 below implies that $\lim \inf_{g \to \infty} \geq 2$, so it suffices to prove the reverse inequality $\lim \sup_{g \to \infty} \leq 2$. For notational convenience, write $G = G_R(1, 1)$, $F = F_R(1, x)$, and $A = A_R(1, x)$; by Corollary 11 the latter two quantities do not depend on the generator $x$. As noted above, the renewal equation implies that $RF = 1 - 1/G$, and by $(31)$ above, $F < 1/R$. By Zuk’s inequality (8), the spectral radius $R = R_g$ is at least $\sqrt{g}$, so it follows that $F < 1/\sqrt{g}$, which is asymptotically negligible as the genus $g \to \infty$. On the other hand, the retracing inequality (21), together with Zuk’s inequality, gives
\begin{align}
1 - 1/G = RF > R^2/(4g(1 - A)) > 1/(4(1 - A)).
\end{align}

This implies that $A < 1$. In the other direction, the renewal inequality (22) implies that
\begin{align}
1/G \geq (1 - A - F^2).
\end{align}

Combining the last two inequalities yields
\begin{align}
1/4(1 - A) < A + F^2
\end{align}

Since $F \to 0$ as $g \to \infty$, it follows that $\lim \inf_{g \to \infty} A \geq 1/2$. Finally, using once again the renewal inequality (22) and the fact that $F$ is asymptotically negligible as $g \to \infty$,
\begin{align}
\lim \sup_{g \to \infty} G_R(1, 1) \leq 2.
\end{align}

3.3. The covering random walk. The simple random walk $X_n$ on the surface group $\Gamma_g$ can be lifted in an obvious way to a simple random walk $\tilde{X}_n$, called the covering random walk, on the free group $F_{2g}$ on $2g$ generators. Clearly, on the event $\tilde{X}_{2n} = 1$ that the lifted walk returns to the root at time $2n$, it must be the case that the projection $X_{2n} = 1$ in $\Gamma_g$. Thus, the return probabilities for $X_{2n}$ are bounded below by those of $\tilde{X}_{2n}$, and so the spectral radius $R$ is bounded above by the spectral radius $\tilde{R}$ of the covering random walk. The return probabilities of the covering random walk are easy to estimate. Each step of $\tilde{X}_n$ either increases or decreases the distance from the root 1 by 1; the probability that the distance increases is $(4g - 1)/4g$, unless the walker is at the root, in which
case the probability that the distance increases is 1. Consequently, the probability that $X_{2n} = 1$ can be estimated from below by counting up/down paths of length $2n$ in the nonnegative integers that begin and end at 0. It is well known that the number of such paths is the $n$th Catalan number $\kappa_n$. Thus,

$$P^1\{X_{2n} = 1\} \geq \kappa_n \left(\frac{4g - 1}{4g}\right)^n \left(\frac{1}{4g}\right)^n. \quad (33)$$

The generating function of the Catalan numbers is (see )

$$\sum_{n=0}^{\infty} \kappa_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}.$$ 

The smallest positive singularity is at $z = 1/4$, and the value of the sum at this argument is 2. Since the spectral radius satisfies $R^2_g/4g \to 1/4$ as $g \to \infty$, by Zuk’s inequality and results of Kesten [13], the inequality (33) has the following consequence.

**Proposition 13.** For every $\varepsilon > 0$ there exist $g(\varepsilon) < \infty$ and $m(\varepsilon) < \infty$ such that if $g \geq g(\varepsilon)$ then

$$\sum_{n=0}^{m(\varepsilon)} P^1\{X_{2n} = 1\} R^{2n} \geq 2 - \varepsilon. \quad (34)$$

The Green’s function and first-passage generating functions for the covering random walk can be exhibited in closed form, using the renewal equation and a retracing identity. The key is that the Cayley graph of the free group is the infinite homogeneous tree $T_{4g}$ of degree $4g$. Since there are no cycles, for any two distinct vertices $x, y \in \Gamma_g$ there is only one self-avoiding path (and hence only one geodesic segment) from $x$ to $y$; therefore, if $x = x_1x_2 \cdots x_m$ is the word representation of $x$ then

$$\tilde{F}_r(1, x) = \prod_{i=1}^{m} \tilde{F}_r(1, x_i) = \tilde{F}_r^{|x|}, \quad (35)$$

the last because symmetry forces $\tilde{F}_r(1, y) = \tilde{F}_r$ to have a common value for all generators $y$. Now fix a generator $x \in A_g$, and consider the first-passage generating function $\tilde{F}_r(1, x) = \tilde{F}_r$. Since $T_{4g}$ has no cycles, any random walk path from 1 to $x$ must either jump directly from 1 to $x$, or must first jump to a generator $y \neq x$, then return to 1, and then eventually find its way to $x$. Consequently, with $p = p_g = 1 - q = 1/4g$,

$$\tilde{F}_r = pr + qr \tilde{F}_r^2,$$

from which it follows that

$$\tilde{F}_r = \frac{1 - \sqrt{1 - 4pqr^2}}{2qr}, \quad (36)$$

$$\tilde{G}_r = \frac{2q}{2q - 1 - \sqrt{1 - 4pqr^2}},$$

and

$$\tilde{R}^2 = \frac{1}{4pq} = \frac{4g^2}{4g - 1}.$$
3.4. Uniform bounds on the Green’s function. Recall from section 2.6 that the first-passage generating function $F_R(1, x)$ is bounded by the tail-sums of the Green’s function $G_R(1, 1)$: in particular,

$$F_R(1, x)^2 \leq F_R(1, x)^2 G_R(1, 1) \leq \sum_{n=2|x|}^{\infty} P\{X_n = 1\} R^n. \quad (37)$$

Propositions 12, 13 imply that, for large genus $g$, these tail-sums can be made uniformly small by taking $|x|$ sufficiently large. In fact, the first-passage generating functions can be bounded away from 1 uniformly in $x \in \Gamma_g \setminus \{1\}$ provided the genus is sufficiently large:

**Proposition 14.** For any $\alpha > 3/4$ there exists $g_\alpha < \infty$ so that

$$\sup_{g \geq g_\alpha} \sup_{x \neq 1} F_R(1, x) < \sqrt{\alpha}. \quad (38)$$

**Proof.** By Proposition 12, $G_R(1, 1)$ is close to 2 for large genus $g$. On the other hand, by inequality (33), $P\{X_2 = 1\} R^2 \geq (1 - 1/4g)(R^2/4g)$, and by taking $g$ large this can be made arbitrarily close to $1/4$. Hence, by taking $g \geq g_\ast$ with $g_\ast$ large,

$$\sum_{n=3}^{\infty} P\{X_n = 1\} R^n \leq \alpha$$

where $\alpha$ can be taken arbitrarily close to $3/4$ by letting $g_\ast \to \infty$. Inequality (37) now implies that $F_R(1, x)^2 \leq \alpha$ for all $|x| \geq 2$ and all $g \geq g_\ast$. But for $|x| = 1$, the symmetry relations of Corollary 11 and the renewal equation (18) imply that $F_R(1, x) < 1/R$, which tends to 0 as $g \to \infty$. \hfill $\Box$

A more sophisticated version of this argument shows

**Proposition 15.**

$$\lim_{g \to \infty} \sup_{x \neq 1} F_R(1, x) = 0. \quad (39)$$

**Proof.** First, consider a vertex $x \in \Gamma_g$ at distance $\geq 2g$ from the root 1: By inequality (37), $F_R(1, x)^2$ is bounded by the tail-sum $\sum_{n \geq 4g} P\{X_n = 1\} R^n$, which by Propositions 12, 13 converges to zero as $g \to \infty$. Thus, it remains only to show that $F_R(1; x) \to 0$ as $g \to \infty$ uniformly for vertices $x$ at distance $< 2g$ from the root.

Fix a vertex $x \in \Gamma_g$ such that $|x| < 2g$, and consider a path $\gamma$ from 1 to $x$. If $\gamma$ is of length $< 2g$ then it has no nontrivial cycles, because the fundamental relation (5) has length $4g$. Consequently, it lifts to a path $\tilde{\gamma}$ in the free group $F_{2g}$ from 1 to the unique covering point $\tilde{x}$ of $x$ at distance $< 2g$ from the group identity 1 in $F_{2g}$. Since $R \leq \tilde{R}$, it follows that

$$\sum_{\gamma: |\gamma| < 2g} w_R(\gamma) \leq \tilde{F}_R(1, \tilde{x}) \leq \tilde{F}_R(1, x) = \tilde{F}_R^{[x]}$$

where the sum is over all paths from 1 to $x$ of length $< 2g$. By (36), this converges to 0 as $g \to \infty$. On the other hand, since the concatenation of a path $\gamma$ from 1 to $x$ of length $\geq 2g$ with a path $\gamma'$ of
length $\geq 2g$ from $x$ to $1$ is a path from $1$ to $1$ of length $\geq 4g$,
\[
\left( \sum_{\gamma:|\gamma|\geq 2g} w_R(\gamma) \right)^2 \leq \sum_{n=4g}^{\infty} P^1\{X_n = 1\} R^n;
\]
here the sum is over all paths from $1$ to $x$ of length $\geq 2g$. By Propositions [13,14] the tail-sum on the right side converges to zero as $g \to \infty$. \hfill \square

4. The Walkabout Argument

According to Australian tradition (see the film Walkabout directed by Nicolas Roeg), at adolescence an Aborigine male embarks on a “walkabout” for six months in the outback, tracing the path of his tribal ancestors, surviving by hunting and trapping. From the viewpoint of a random walker, the hyperbolic plane is a vast outback; failure to follow, or nearly follow, the geodesic path from one point to another necessitates walkabouts whose extents grow exponentially with the deviation from the geodesic path. It is this that accounts for Ancona’s inequality. The following arguments make this precise.

4.1. Free subgroups and embedded trees. Recall that the surface group $\Gamma_g$ in its standard presentation has $2g$ generators which, together with their inverses, satisfy the relation (5). Denote by $F_A^+$ and $F_A^-$ the sub-semigroups of $\Gamma_g$ generated by \{a_i\}_{i \leq g}$ and \{a_i^{-1}\}_{i \leq g}$, respectively, and define $F_B^+$ similarly.

**Proposition 16.** The image of each of the semigroups $F_A^+$ and $F_B^+$ in the Cayley graph is a rooted tree of outdegree $g$. Every self-avoiding path in (the image of) any one of these semigroups is a geodesic in the Cayley graph.

**Proof.** This is an elementary consequence of Dehn’s algorithm (cf. [16]). Consider a self-avoiding path $\alpha = a_{(1)}a_{(2)}\cdots a_{(m)}$ in $F_A^+$. If this were not a geodesic segment, then there would exist a geodesic path $\beta = x_{1}^{-1}x_{n-1}\cdots x_{n}^{-1}$, with $n < m$, such that
\[
a_{(1)}a_{(2)}\cdots a_{(m)}x_{1}x_{2}\cdots x_{n} = 1.
\]
According to Dehn’s algorithm, either $a_{(m)} = x_{1}^{-1}$, or there must exist a block of between $2g + 1$ and $4g$ consecutive letters that can be shortened by using (a cyclic rewriting of) the fundamental relation (5). Since $\beta$ is geodesic, this block must include the last letter of $\alpha$; and because $a$’s and $b$’s alternate in the fundamental relation, it must actually begin with the last letter $a_{i(m)}$, and therefore include at least the first $2g$ letters $x_1, \ldots, x_{2g}$. Hence, the Dehn shortening results in
\[
a_{(1)}a_{(2)}\cdots a_{(m-1)}y_1y_2\cdots y_k = 1,
\]
where $k < m - 1$. Therefore, by induction on $m$, there exists $r \geq m - n \geq 1$ such that
\[
a_{(1)}a_{(2)}\cdots a_{(r)} = 1.
\]
But this is impossible, by Dehn. This proves that every self-avoiding path in $F_A^+$ beginning at the root $1$ is geodesic, and it follows by homogeneity that every self-avoiding path in $F_A^+$ is also geodesic. Finally, this implies that the image of $F_A^+$ in the Cayley graph is a tree. \hfill \square
Note 17. The presence of large free semigroups is a general property of word-hyperbolic groups (see for example [7], Th. 5.3.E), and for this reason it may well be possible to generalize the arguments below. The primary obstacle to generalization seems to be in obtaining suitable \textit{a priori} estimates on the first-passage generating functions to use in conjunction with Lemma [13] below.

4.2. Crossing a tree. Say that a path $\gamma$ in the Cayley graph $G^\Gamma$ \textit{crosses} a rooted subtree $T$ of degree $d$ if either it visits the root of the tree, at which time it terminates, or if it crosses each of the $d$ subtrees $T_i$ of $T$ attached to the root. (In the latter case, the path must terminate at the root of the last subtree it crosses.) Observe that if $G^\Gamma$ is planar, as when $\Gamma = \Gamma_g$ is a surface group, this definition of crossing accords with the usual topological notion of a crossing. For a vertex $x \in \Gamma$, let $P(x; T)$ be the set of all paths starting at $x$ that cross $T$. Let $P^m(x; T)$ be the set of all paths in $P(x; T)$ of length $\leq m$. Define

\[
H_r(x; T) := \sum_{\gamma \in P(x; T)} w_r(\gamma) \quad \text{and} \quad H_r^m(x; T) := \sum_{\gamma \in P^m(x; T)} w_r(\gamma).
\]

(Recall that $w_r(\gamma)$ is the $r-$weight of the path $\gamma$, defined by (11)). The following result is the essence of the walkabout argument.

Lemma 18. Suppose that $F_r(1, x) \leq \beta$ for every $x \neq 1$ and some constant $\beta < 1$. Then for every rooted subtree $T \subset G^\Gamma$ of degree $d \geq 2$ and every vertex $x \not\in T$,

\[
H_r(x; T) \leq \beta + \beta^d/(1 - \beta^d).
\]

Proof. It suffices to show that the inequality holds with $H_r(x; T)$ replaced by $H_r^m(x; T)$ for any $m \geq 1$. Now the generating function $H^m_r(x; T)$ is a sum over paths of length $\leq m$. In order that such a path $\gamma$ starting at $x$ crosses $T$, it must either visit the root of $T$, or it must cross each of the $d$ offshoot tree $T_i$. Since these are pairwise disjoint, a path $\gamma$ that crosses every $T_i$ can be decomposed as $\gamma = \gamma_1 \gamma_2 \cdots \gamma_d$, where $\gamma_i$ starts at $x$ and crosses $T_i$, and $\gamma_{i+1}$ starts at the endpoint of $\gamma_i$, in $T_i$, and crosses $T_{i+1}$. Each $\gamma_i$ must have length at least $1$; hence, since their concatenation has length $\leq m$, each $\gamma_i$ must have length $\leq m - d + 1 \leq m - 1$. Since the sum of $w_r(\gamma)$ over all paths $\gamma$ from $x$ to the root of $T$ is no larger than $\beta$, by hypothesis, it follows that

\[
\sup_{T} \sup_{x \not\in T} H_r^m(x; T) \leq \beta + \left( \sup_{T} \sup_{x \not\in T} H_r^{m-1}(x; T) \right)^d.
\]

Therefore, since $H^1(r; x) \leq \beta$,

\[
\sup_{T} \sup_{x \not\in T} H_r^m(x; T) \leq \beta + \beta^d + \beta^{2d} + \cdots.
\]

4.3. Exponential decay of the Green’s function. Assume now that $\Gamma$ has a planar Cayley graph $G^\Gamma$, and that this is embedded quasi-isometrically in the hyperbolic plane. Let $\gamma = x_0x_1 \cdots x_m$ be a geodesic segment in the Cayley graph $G^\Gamma$. Say that a vertex $x_k$ on $\gamma$ is a \textit{barrier point} if there are disjoint rooted subtrees $T_k, T'_k$ in the Cayley graph, both of of outdegree $d \geq 2$, and neither intersecting $\gamma$, whose roots $y_k$ and $z_k$ are vertices neighboring $x_k$ on opposite sides of $\gamma$. 

Call $T_k \cup T'_k \cup \{x_k\}$ a barrier, and the common outdegree $d$ the order of the barrier. Observe that a barrier must disconnect the hyperbolic plane in such a way that the initial and final segments $x_0x_1 \cdots x_{k-1}$ and $x_{k+1}x_{k+2} \cdots x_m$ of $\gamma$ lie in opposite components.

**Remark 19.** In the arguments of [3] and [1], the region of hyperbolic space separating two cones plays the role of a barrier.

**Proposition 20.** Let $\Gamma = \Gamma_g$ be the surface group of genus $g \geq 2$. There is a constant $\kappa = \kappa_g < \infty$ such that along every geodesic segment $\gamma$ of length $\kappa n$ there are $n$ disjoint barriers $B_i$, each of order $g$.

**Note 21.** The value of the constant $\kappa$ is not important in the arguments to follow. The argument below shows that $\kappa_g = 8g$ will work.

The proof of Proposition 20 is deferred to section 4.5 below. Given the existence of barriers, the tree-crossing Lemma 18 and the a priori estimate on the Green’s function provided by Proposition 15 the exponential decay of the Green’s function at the spectral radius follows routinely:

**Theorem 22.** If the genus $g$ is sufficiently large, then the Green’s function $G_R(x, y)$ of simple random walk on the surface group $\Gamma_g$, evaluated at the spectral radius $R = R_g$, decays exponentially in the distance $d(x, y)$, that is, there exist constants $C = C_g < \infty$ and $\varrho = \varrho_g < 1$ such that for every $x \in \Gamma_g$,

$$G_R(1, x) \leq C \varrho^{|x|}. \quad (41)$$

**Proof.** By Proposition 15 for any $\beta > 0$ there exists $g_{\beta} < \infty$ so large that if $g \geq g_{\beta}$ then the first-passage generating functions of the simple random walk on $\Gamma_g$ satisfy $F_R(1, x) < \beta$ for all $x \neq 1$. By Lemma 18 the tree-crossing generating functions $H_R(x; T)$ for trees of outdegree $g$ satisfy $H_R(x; T) \leq \beta + \beta^g/(1 - \beta^g)$. If $\beta$ is sufficiently small then it follows that for any barrier $B = T \cup T' \cup \{y\}$ of order $g$ and any vertex $x \not\in B$,

$$H_R(x; T) + H_R(x; T') + F_R(x, y) < 1/2.$$  

By Proposition 20 a path $\gamma$ from 1 to $x$ must cross $|x|/\kappa_g$ distinct barriers. Therefore,

$$F_R(1, x) \leq 2^{-|x|/\kappa_g}.$$

\[\square\]

4.4. **Action of $\Gamma_g$ on the hyperbolic plane.** The surface group $\Gamma = \Gamma_{2g}$ acts by hyperbolic isometries of the hyperbolic plane $\mathbb{H}$. This action provides a useful description of the Cayley graph $G^\Gamma$, using the tessellation $T = \{xP\}_{x \in \Gamma}$ of the hyperbolic plane $\mathbb{H}$ by fundamental polygons (“tiles”) $xP$ (see, e.g., [11], chs. 3–4); for the surface group $\Gamma_g$ the polygon $P$ can be chosen to be a regular $4g-$sided polygon (cf. [11], sec. 4.3, Ex. C). The tiles serve as the vertices of the Cayley graph; two tiles are adjacent if they share a side. Thus, each group generator $a_i^\pm, b_i^\pm$ maps $P$ onto one of the $4g$ tiles that share sides with $P$. The sides of $P$ (more precisely, the geodesics gotten by extending the sides) can be labeled clockwise, in sequence, as

$$A_1, B_1, \bar{A}_1, \bar{B}_1, \ldots, \bar{B}_g$$

in such a way that each generator $a_i$ maps the exterior of the geodesic $A_i$ onto the interior of $\bar{A}_i$, and similarly $b_i$ maps the exterior of the geodesic $B_i$ onto the interior of $\bar{B}_i$. Observe that $4g$ tiles meet
at every vertex of \( \mathcal{P} \); for each such vertex, the successive group elements in some cyclic rewriting of the fundamental relation (5), e.g.,

\[ a_1, a_1b_1, a_1b_1a_1^{-1} \cdots , \]

map the polygon \( \mathcal{P} \) in sequence to the tiles arranged around the vertex. Also, the full tessellation is obtained by drawing all of the geodesics \( xA_1, xB_1, xA_1, xB_1 \), where \( x \in \Gamma \); these partition \( \mathbb{H} \) into the congruent polygons \( xP \). Call the geodesics \( A_1, B_1, \ldots , B_g \) bounding geodesics of \( \mathcal{P} \), and their images by isometries \( x \in \Gamma \) bounding geodesics of the tessellation.

That the semigroups \( \mathcal{F}_A^{\pm 1}, \mathcal{F}_B^{\pm 1} \) defined in section 4.1 are free corresponds geometrically to the following important property of the tessellation \( \mathcal{T} \): The exteriors of two bounding geodesics of \( \mathcal{P} \) do not intersect unless the corresponding symbols are adjacent in the fundamental relation (5), e.g., the exteriors of \( A_1 \) and \( B_1 \) intersect, but the exteriors of \( A_1 \) and \( B_2 \) do not.

**Lemma 23.** Let \( \gamma \) be a geodesic segment in the Cayley graph that begins at \( \mathcal{P} \) and on its first step jumps from \( \mathcal{P} \) to the tile \( a_i \mathcal{P} \) (respectively, \( b_i \mathcal{P} \), or \( a_i^{-1} \mathcal{P} \), or \( b_i^{-1} \mathcal{P} \)). Then \( \gamma \) must remain in the halfplane exterior to \( A_i \) (respectively, \( B_i, \bar{A}_i \), or \( \bar{B}_i \)) on all subsequent jumps.

**Proof.** By induction on the length of \( \gamma \). First, \( \gamma \) cannot recross the geodesic line \( A_i \) in \( \mathbb{H} \) in its first \( 2g + 1 \) steps, because to do so would require that \( \gamma \) cycle through at least \( 2g + 1 \) tiles that meet at one of the vertices of \( \mathcal{P} \) on \( A_i \). This would entail completing more than half of a cyclic rewriting of the fundamental relation (5), and so \( \gamma \) would not be a geodesic segment in the Cayley graph.

Now suppose that \( |\gamma| \geq 2g + 1 \). Since \( \gamma \) cannot complete more than \( 2g \) steps of a fundamental relation, it must on some step \( j \leq 2g \) jump to a tile that does not meet \( \mathcal{P} \) at a vertex. This tile must be on the other side of a bounding geodesic \( C \) that does not intersect \( A_i \) (by the observation preceding the lemma). The induction hypothesis implies that \( \gamma \) must remain thereafter in the halfplane exterior to this bounding geodesic, and therefore in the halfplane exterior to \( A_i \).

4.5. **Existence of barriers: Proof of Proposition 20** Let \( \gamma \) be a geodesic segment in the Cayley graph. Since \( \gamma \) cannot make more than \( 2g \) consecutive steps in a relator sequence (a cyclic rewriting of the fundamental relation), at least once in every \( 4g \) steps it must jump across a bounding geodesic \( xL \) into a tile \( xP \), and then on the next step jump across a bounding geodesic \( xL' \) that does not meet \( xL \). By Lemma 23, \( \gamma \) must remain in the halfplane exterior to \( xL' \) afterwards. Similarly, by time-reversal, \( \gamma \) must stay in the halfplane exterior to \( xL \) up to the time it enters \( xP \). Thus, the tile \( xP \) segments \( \gamma \) into two parts, past and future, that live in nonoverlapping halfplanes.

**Definition 24.** If a geodesic segment \( \gamma \) in the Cayley graph \( G^\Gamma \) enters a tile \( xP \) by crossing a bounding geodesic \( xL \) and exits by crossing a bounding geodesic \( xL' \) that does not intersect \( xL \), then the tile \( xP \) — or the vertex \( x \in \Gamma \) — is called a cut point for \( \gamma \).

**Lemma 25.** Let \( \gamma \) be a geodesic segment in \( G^\Gamma \) from \( u \) to \( v \). If \( x \) is a cut point for \( \gamma \), then it is also a barrier point. Moreover, every geodesic segment from \( u \) to \( v \) passes through \( x \).

**Proof.** Since \( \gamma \) jumps into, and then out of \( xP \) across bounding geodesics \( xL \) and \( xL' \) that do not meet, the sides \( xL'' \) and \( xL''' \) of \( xP \) adjacent to the side \( xL' \) are distinct from \( xL \). Denote by \( yP \) and \( zP \) the tiles adjacent to \( xP \) across these bounding geodesic lines \( xL'' \) and \( xL''' \). For each of these tiles \( \tau \), at least one of the four trees rooted at \( \tau \) obtained by translation of the four semigroups of Proposition 16 will lie entirely in the intersections of the halfplanes interior to \( xL \) and \( xL' \), by
Therefore, each of the tiles \( y \mathcal{P} \) and \( z \mathcal{P} \) is the root of a tree that does not intersect \( \gamma \). These trees, by construction, lie on opposite sides of \( \gamma \). This proves that \( x \) is a barrier point.

Suppose now that \( \gamma' \) is another geodesic segment from \( u \) to \( v \). If \( \gamma' \) did not pass through the tile \( x \mathcal{P} \), then it would have to circumvent it by passing through one of the tiles \( y \mathcal{P} \) or \( z \mathcal{P} \). To do this would require either that it complete a relation or pass through \( g \) trees. In either case, the path \( \gamma' \) could be shortened by going through \( x \mathcal{P} \).

To complete the proof of Proposition 20 it remains to show that the successive barriers along \( \gamma \) constructed above are pairwise disjoint. But the attached trees at the tiles intersect the barrier at \( x \). The past and future segments of \( \gamma \) lie in the exteriors. Hence, at each new barrier along (say) the future segment, the attached trees will lie in halfplanes contained in these exteriors, and so will not intersect the barrier at \( x \mathcal{P} \).

4.6. Ancona’s inequality. The Ancona inequalities (3) state that the major contribution to the Green’s function \( G_R(x_0, x_m) \) comes from random walk paths that pass within a bounded distance of \( x_n \). To prove this it suffices, by Lemma 25 to show that (44) holds for cut points \( x_m \). The key to this is that a path from \( x_0 \) to \( x_m \) that does not pass within distance \( n \) of \( x_m \) must cross \( g^{n-1} \) trees of outdegree \( g \).

**Lemma 26.** Let \( \gamma \) be a geodesic segment from \( u \) to \( v \) that passes through the root vertex \( 1 \), and suppose that vertex \( 1 \) is a cut point for \( \gamma \). Assume that both \( u, v \) are exterior to the sphere \( S_n := \{ x \in \Gamma : |x| = n \} \) of radius \( n \) in the Cayley graph \( G^\Gamma \) centered at 1. If \( F_R(1, x) \leq \beta \) for all vertices \( x \neq 1 \), then

\[
G_R(u; v; G^\Gamma \setminus S_n) \leq 2 \left( \beta + \beta^9/(1-\beta^9) \right) g^{n-1}.
\]

**Proof.** Since both \( u, v \) are exterior to \( S_n \), the restricted Green’s function is the sum over all paths from \( u \) to \( v \) that do not enter the sphere \( S_n \) (recall definition (15)). Since 1 is a barrier point for \( \gamma \), there are trees \( T, T' \) of outdegree \( g \) with roots adjacent to 1 on either side of \( \gamma \). A path from \( u \) to \( v \) that does not enter \( S_n \) must cross either \( T \) or \( T' \), and it must do so without passing within distance \( n-1 \) of the root. Thus, it must cross \( g^{n-1} \) disjoint subtrees of either \( T \) or \( T' \). Consequently, the result follows from Lemma 26.

**Proposition 27.** For all sufficiently large \( g \), there exists \( C = C_g < \infty \) such that the Green’s function of the simple random walk on the surface group \( \Gamma_g \) satisfy the Ancona inequalities: In particular, for every geodesic segment \( x_0 x_1 x_2 \cdots x_m \), every \( 1 < n < m \), and every \( 1 \leq r \leq R \),

\[
G_r(x_0, x_m) \leq C G_r(x_0, x_n) G_r(x_n, x_m).
\]

**Proof.** It is certainly true that for each distance \( m < \infty \) there is a constant \( \infty > C_m \geq 1 \) so that (44) holds for all geodesic segments of length \( m \), because (by homogeneity of the Cayley graph) there are only finitely many possibilities. The problem is to show that the constants \( C_m \) remain bounded as \( m \to \infty \).

As noted above, it suffices to consider only cut points \( x_n \) along the geodesic segment \( \gamma \). For ease of notation, assume that \( \gamma \) has been translated so that the cut point \( x_n = 1 \) is the root vertex of the Cayley graph, and write \( u = x_0 \) and \( v = x_m \) for the initial and terminal points. Assume
also that \( d(u, 1) \leq m/2 \); this can be arranged by switching the endpoints \( u, v \), if necessary. Thus, there is a cut point \( w \) on the geodesic segment between 1 and \( v \) so that \( .7m \leq d(u, w) \leq .8 \). Let \( S = S_{\sqrt{m}}(w) \) and \( B = B_{\sqrt{m}}(w) \) be the sphere and ball, respectively, of common radius \( \sqrt{m} \), centered at \( w \). Any path from \( u \) to \( v \) (or any path from 1 to \( v \)) must either pass through the ball \( B \) or not; hence

\[
G_r(u, v) = G_r(u, v; B^c) + \sum_{z \in S} G_r(u, z)G_r(z, v; B^c).
\]

If \( m \) is sufficiently large that \( \sqrt{m} < .1m \), then any point \( z \in S \) must be at distance

\[.6m \leq d(u, z) \leq .9m\]

from \( u \). Moreover, by Lemma 25, every geodesic segment from \( u \) to \( z \) must pass through 1. (This follows because 1 is a cut point for \( \gamma \).) Similarly, since \( w \) is also a cut point, every geodesic segment from 1 to \( v \) passes through \( w \). Consequently, for every \( z \in S \),

\[G_r(u, z) \leq C_{[.9m]}G_r(u, 1)G_r(1, z).
\]

By Lemma 26,

\[G_r(u, v; B^c) \leq G_R(u, v; B^c) \leq 2\alpha g^{\sqrt{m}}\]

where \( \alpha = \beta + \beta g / (1 - \beta g) < 1/2 \), provided the genus \( g \) is sufficiently large. On the other hand, the Harnack inequalities ensure that for some \( g > 0 \) and all \( r \geq 1 \)

\[
G_r(u, 1) \geq g^m \quad \text{and} \quad G_r(1, v) \geq g^m
\]

Therefore,

\[
G_r(u, v) = G_r(u, v; B^c) + \sum_{z \in S} G_r(u, z)G_r(z, v; B^c)
\leq 2\alpha g^{\sqrt{m}} + C_{[.9m]} \sum_{z \in S} G_r(u, 1)G_r(1, z)G_r(z, v; B^c)
\leq 2\alpha g^{\sqrt{m}} + C_{[.9m]}G_r(u, 1)G_r(1, v)
\leq (1 + 2\alpha g^{\sqrt{m}}/g^{2m})C_{[.9m]}G_r(u, 1)G_r(1, v),
\]

This shows that

\[C_m \leq (1 + 2\alpha g^{\sqrt{m}}/g^{2m})C_{[.9m]},\]

and it now follows routinely that the constants \( C_{m} \) remain bounded as \( m \to \infty \). \( \square \)

REFERENCES