The Packing and Covering Functions of Some Self-similar Fractals

STEVEN P. LALLEY

1. Introduction. *Self-similar sets* in $\mathbf{R}^d$ occur as the limit sets (equivalently, the minimal closed invariant sets) of certain semigroups of contractive Euclidean similarity transformations ([4], [7]). The purpose of this note is to describe the asymptotic behavior as $\varepsilon \to 0$ of the number $N(\varepsilon)$ of points in a maximal ε-separated subset and the number $M(\varepsilon)$ of ε-balls needed to cover a self-similar set, and to investigate the relationships between maximal packings, minimal coverings, and Hausdorff measure. The functions $N(\varepsilon)$ and $M(\varepsilon)$ are used to define the *packing* and *covering dimensions* (often called the *capacity* and *metric entropy*): see below.

A *similarity transformation* $S : \mathbf{R}^d \to \mathbf{R}^d$ has the form $S = rJ$, where $J : \mathbf{R}^d \to \mathbf{R}^d$ is an isometry and $r > 0$ is a scalar; if $0 < r < 1$ then S is called *contractive*. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$ be a finite set of contractive similarity transformations. Then for any sequence $i_1, i_2, \cdots$ of indices and any $x \in \mathbf{R}^d$

$$\lim_{n \to \infty} S_{i_1} S_{i_2} \ldots S_{i_n} x \overset{\triangle}{=} k_{i_1, i_2 \ldots}$$

exists, and the limit is independent of x ([4], Section 3; two different sequences $i_1, i_2, \ldots$ and $i_1', i_2', \ldots$ may yield the same limit). Let

$$K = \{k_{i_1 i_2 \ldots}\}$$

be the set of all possible limit points: this set will be the principal object of study in this paper.

Most of the fractals in [7], Section 6–8, 14 arise in this manner. Some examples:

(1) Let $S_1 x = rx$ and $S_2 x = rx + 1 - r$, where $0 < r \leq \frac{1}{2}$. If $r = \frac{1}{3}$ then K is the Cantor set; if $r = \frac{1}{2}$ then K is the unit interval ([7], plate 81).

(2) Let $S_i : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$S_1(x_1,x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2}\right)$$

$$S_2(x_1,x_2) = \left(\frac{1}{2} + \frac{x_1}{2}, \frac{x_2}{2}\right)$$

$$S_3(x_1,x_2) = \left(\frac{1}{4} + \frac{x_1}{2}, \frac{\sqrt{3}}{4} + \frac{x_2}{2}\right);$$

then K is the "Sierpinski gasket" ([7], p. 142).

(3) Let $a_1 = (0,0)$, $a_2 = \left(\frac{1}{3},0\right)$, $a_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$, $a_4 = \left(\frac{2}{3},0\right)$, and $a_5 = (1,0)$. Let S_i ($i = 1,2,3,4$) be the unique similarity transformation of $\mathbf{R}^2$ mapping $\overline{a_1 a_5}$ onto $\overline{a_i a_{i+1}}$ and having positive determinant. Then K is the "Koch snowflake" ([7], pp. 42–43).

The set K is always compact ([4], Section 3), as are the images

$$K_{i_1 i_2 \ldots i_n} \triangleq S_{i_1} S_{i_2} \ldots S_{i_n} K.$$

In the examples above the sets $K_1, K_2, \ldots, K_N$ are either pairwise disjoint or have "small" overlaps. In the former case the set K is totally disconnected and each point $x \in K$ has a unique representation $x = k_{i_1 i_2 \ldots}$; in the latter case, some points have multiple representations and K may be arcwise connected. It is always the case that $K = \bigcup_{i=1}^{N} K_i$.

Say that $\mathcal{S}$ satisfies the *open set condition* [4] if there exists a nonempty open subset U of $\mathbf{R}^d$ such that $S_i U \subset U$ for each i and $S_i U \cap S_j U = \emptyset$ if $i \neq j$. If U can be chosen so that $U \cap K \neq \emptyset$, say that $\mathcal{S}$ satisfies the *strong open set condition*. Notice that this holds in the examples above.

Write $S_i = r_i J_i$, where $0 < r_i < 1$ and J_i is an isometry. The *similarity dimension* of $\mathcal{S}$ ([4],[6]) is the unique $D > 0$ such that

$$\sum_{i=1}^{N} r_i^D = 1.$$

Let $H^D(\cdot)$ be the D-dimensional Hausdorff measure on $\mathbf{R}^d$ ([4]).

Theorem 0 ([4]). *If $\mathcal{S}$ satisfies the open set condition then $0 < H^D(K) < \infty$ and $H^D(K_i \cap K_j) = 0$ for $i \neq j$.*

Thus, D is the Hausdorff dimension of K. Since $H^D(K_i \cap K_j) = 0$ it follows that

$$H^D(K_{i_1 i_2 \ldots i_n}) = (r_{i_1} r_{i_2} \cdots r_{i_n})^D H^D(K).$$

Therefore, if one chooses indices $i, i_2, \ldots$ at random from the set $\{1, 2, \ldots, N\}$ according to the multinomial distribution $\{r_1^D, r_2^D, \ldots, r_N^D\}$, then the random point $k_{i_1 i_2 \ldots}$ will be "uniformly distributed" on K relative to D-dimensional Hausdorff measure.

Call a finite subset F of K ε-*separated* if dist $(x, x') \geq \varepsilon$ for all $x, x' \in F$ such that $x \neq x'$. Let $N(\varepsilon)$ be the maximum cardinality of an ε-separated subset of K; this will be called the *packing function*. Call a finite subset F of K an ε-*covering* if for every $y \in K$ there exists $x \in F$ such that dist $(x, y) < \varepsilon$. Let $M(\varepsilon)$ be the minimum cardinality of an ε-covering subset of K; this will be called the *covering function*. The *packing* and *covering dimensions* D_P and D_C are defined by

$$D_P = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon^{-1}},$$

$$D_C = \lim_{\varepsilon \to 0} \frac{\log M(\varepsilon)}{\log \varepsilon^{-1}},$$

provided these limits exist. (The covering dimension was introduced in [5], the packing dimension in [8]. They are usually called the *metric entropy* and *capacity*.) A simple argument shows that $N(3\varepsilon) \leq M(\varepsilon) \leq N(\varepsilon)$, so $D_P = D_C$ whenever either limit exists.

Theorem 1. *Assume that the strong open set condition holds.*

(a) *If the additive group generated by* $\log r_1, \log r_2, \ldots, \log r_N$ *is dense in* $\mathbf{R}$, *then there exist constants* $C, C' > 0$ *such that as* $\varepsilon \to 0$

(1.1) $$N(\varepsilon) \sim C \varepsilon^{-D}$$

and

(1.2) $$M(\varepsilon) \sim C' \varepsilon^{-D}.$$

(b) *If the additive group generated by* $\log r_1, \log r_2, \ldots, \log r_n$ *is* $h\mathbf{Z}(h > 0)$ *then for each* $\beta \in [0, h)$ *there exist constants* $C_\beta, C'_\beta > 0$, *uniformly bounded, such that as* $n \to \infty$

(1.3) $$N(e^{-nh+\beta}) \sim C_\beta \exp\{D(-nh + \beta)\}$$

and

(1.4) $$M(e^{-nh+\beta}) \sim C'_\beta \exp\{D(-nh + \beta)\}.$$

Observe that case (b) obtains for the Cantor set, the Koch snowflake, and the Sierpinski gasket.

Corollary.   *If the strong open set condition holds then*

$$D = D_P = D_C.$$

This answers a query in [1]. (After writing this note I learned that this relation is part of the folklore: see, for example, [9].)

Let F_ϵ be an ϵ-separated subset of K having maximum cardinality, and let G_ϵ be an ϵ-covering subset of K having minimum cardinality. Define Borel probability measures $\mu_\epsilon(\nu_\epsilon)$ on K by putting mass $\frac{1}{N(\epsilon)}$ $\left(\frac{1}{M(\epsilon)}\right)$ at each point of $F_\epsilon(G_\epsilon)$.

Theorem 2.   *If the strong open set condition holds then as $\epsilon \to 0$*

(1.5)
$$\mu_\epsilon \xrightarrow{\mathcal{D}} \frac{H^D}{H^D(K)}$$

and

(1.6)
$$\nu_\epsilon \xrightarrow{\mathcal{D}} \frac{H^D}{H^D(K)}.$$

Theorems 1 and 2 help clarify the relations between packings, coverings, and Hausdorff masures. Maximal ϵ-separated sets and minimal ϵ-separated sets are usually very difficult to find. In the totally disconnected case (i.e., $K_i \cap K_j = \emptyset$ for $i \neq j$) one may give an algorithm for obtaining an ϵ-separated set whose cardinality is within $0(1)$ of $N(\epsilon)$. In general one may produce an ϵ-separated set whose cardinality is within $0(\epsilon^{-D+\delta})$ of $N(\epsilon)$ for some $\delta > 0$. The proofs below should suggest how this may be done.

In proving Theorems 1–2, I shall consider only the packing function $N(\epsilon)$. The same arguments apply to the covering function $M(\epsilon)$.

2. Totally Disconnected K. This case is particularly simple. Assume that $K_1, K_2, \ldots, K_N$ are pairwise disjoint; since each K_i is compact there exists $\delta > 0$ such that if $x \in K_i$ and $x' \in K_j$, $i \neq j$, then $\text{dist}(x, x') > \delta$. Now if $\epsilon < \delta$ then one may obtain an ϵ-separated subset of maximum cardinality by finding maximal ϵ-separated subsets of $K_1, K_2, \ldots, K_N$ and taking their union. Since $K_i = S_i K$ is similar to K, a maximal ϵ- separated subset of K_i is similar to a maximal ϵr_i^{-1}-separated subset of K, and therefore its cardinality is $N(\epsilon r_i^{-1})$. Hence, if $\epsilon < \delta$ then $N(\epsilon) = \sum_{i=1}^{N} N(\epsilon r_i^{-1})$. It follows that

(2.1)
$$N(\epsilon) = \sum_{i=1}^{N} N(\epsilon r_i^{-1}) + L(\epsilon)$$

for all $\varepsilon > 0$. Since $N(\varepsilon)$ is a nonincreasing integer-valued function that equals 1 for all sufficiently large ε, $L(\varepsilon)$ is a piecewise continuous function with only finitely many discontinuities that vanishes for $0 < \varepsilon < \delta$.

Equation (2.1) may be rewritten as a renewal equation ([3], Chapter 11) in the following manner. Define

$$Z(a) = e^{-aD} N(e^{-a})$$

for $a > 0$; since $\sum_{i=1}^{N} r_i^D = 1$, it follows from (2.1) that

$$Z(a) = z(a) + \int_{(0,a]} Z(a-x) F(dx), \quad a > 0,$$

where $F(dx)$ is the probability measure that puts mass r_i^D at $-\log r_i$, $i = 1, 2, \ldots, N$. Because F has finite support and L is piecewise continuous with only finitely many discontinuities, z is also piecewise continuous with only finitely many discontinuities. Moreover, z has compact support in $[0,\infty)$ since L vanishes in $(0,\delta)$. Therefore, z is directly Riemann integrable ([3], Chapter 11).

There are now two cases, the nonlattice case and the lattice case, corresponding to (a) and (b) of Theorem 1. In the nonlattice case the renewal theorem ([3], Chapter 11) implies that

$$\lim_{a \to \infty} Z(a) = \frac{\displaystyle\int_0^\infty z(x)\, dx}{\displaystyle\sum_{i=1}^{N} r_i^D \log r_i^{-1}}.$$

This is equivalent to (1.1). In the lattice case the renewal theorem ([2], Chapter 13) implies that for $0 \le \beta < h$

$$\lim_{n \to \infty} Z(nh + \beta) = \frac{\displaystyle\sum_{n=1}^{\infty} z(nh + \beta)}{\displaystyle\sum_{i=1}^{N} r_i^D \log r_i^{-1}}.$$

This is equivalent to (1.3). Note that the constants C_β must be uniformly bounded because $N(\varepsilon)$ is nonincreasing.

3. The General Case. If $K_1, K_2, \ldots, K_N$ are not pairwise disjoint then the argument of the preceding section fails because the union of ε-separated subsets of $K_1, \ldots K_N$ will not generally be ε-separated. Nevertheless, since $K = \bigcup_{i=1}^{N} K_i$,

$$N(\varepsilon) \leq \sum_{i=1}^{N} N(\varepsilon r_i^{-1}).$$

Define

$$L(\varepsilon) = \sum_{i=1}^{N} N(\varepsilon r_i^{-1}) - N(\varepsilon).$$

Proposition 1: *Assume that the strong open set condition holds. Then there exist constants $\gamma > 0, \delta > 0$ such that*

$$L(\varepsilon) \leq \gamma \varepsilon^{\delta - D}.$$

The proof is deferred to Section 5.

Define, as in Section 2, $Z(a) = e^{-aD} N(e^{-a})$, and write

$$Z(a) = z(a) + \int_{(0,a]} Z(a - x) F(dx)$$

where $F(dx)$ puts mass r_i^D at $\log r_i^{-1}$, $i = 1, 2, \ldots, N$. Observe that for all sufficiently large a, $z(a) = -e^{-aD} L(e^{-a})$. Moreover, since $N(\varepsilon)$ is a nonincreasing, nonnegative integer valued function and $F(dx)$ has finite support, $z(a)$ is a piecewise continuous function with only finitely many discontinuities in any finite interval. Proposition 1 implies that

$$|z(a)| \leq \gamma e^{-a\delta}$$

for all sufficiently large a. It follows that $z(a)$ is directly Riemann integrable. Therefore, in the nonlattice case

$$\lim_{a \to \infty} Z(a) = \frac{\displaystyle\int_0^\infty z(x)\,dx}{\displaystyle\sum_{i=1}^{N} r_i^D \log r_i^{-1}},$$

and in the lattice case

$$\lim_{n \to \infty} Z(nh + \beta) = \frac{\displaystyle\sum_{n=1}^{\infty} z(nh + \beta)}{\displaystyle\sum_{i=1}^{N} r_i^D \log r_i^{-1}}$$

for every $\beta \in [0, h)$. This proves (1.1) and (1.3). As before, the constants C_β are uniformly bounded because $N(\varepsilon)$ is nonincreasing.

4. Maximal Packings and Hausdorff Measure. Recall that μ_ϵ is the probability measure that puts mass $\frac{1}{N(\epsilon)}$ at each point of a maximal ϵ-separated set.

Proposition 2. *Assume that the strong open set condition holds. For each pair of distinct sequences $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$,*

$$\lim_{\epsilon \to 0} \mu_\epsilon(K_{i_1 i_2 \ldots i_n} \cap K_{j_1 j_2 \ldots j_n}) = 0.$$

The proof will be given in Section 5.

Since the support of μ_ϵ is an ϵ-separated subset of K, and since K_i is similar to K, it follows that

$$(4.1) \qquad \mu_\epsilon(K_i) \le \frac{N(\epsilon r_i^{-1})}{N(\epsilon)}, \quad i = 1, 2, \ldots, N.$$

For small ϵ, $\sum_{i=1}^N \left(N(\epsilon r_i^{-1}) / N(\epsilon) \right) \sim 1$ by Theorem 1, and $\mu_\epsilon(K_i \cap K_j) = o(1)$ for $i \ne j$, by Proposition 2. Since $\mu_\epsilon(K) = 1$ and $K = \bigcup K_i$, (4.1) implies that

$$\mu_\epsilon(K_i) \sim \frac{N(\epsilon r_i^{-1})}{N(\epsilon)}$$

$$\sim r_i^D = \frac{H^D(K_i)}{H^D(K)}.$$

Now the sets $K_{i_1 i_2 \ldots i_n}$ are all similar to K, so by an easy induction argument

$$\lim_{\epsilon \to 0} \mu_\epsilon(K_{i_1 i_2 \ldots i_n}) = \frac{H^D(K_{i_1 i_2 \ldots i_n})}{H^D(K)}$$

for each sequence $i_1, i_2, \ldots, i_n$. Since

$$K = \bigcup K_{i_1 i_2 \ldots i_n} \qquad \text{and} \qquad \operatorname{diam} K_{i_1 i_2 \ldots i_n} \le \left(\max_{1 \le i \le N} r_i \right)^n \to 0,$$

it follows easily that for any continuous function $f : K \to \mathbf{R}$

$$\lim_{\epsilon \to 0} \int_K f \, d\mu_\epsilon = \frac{\int_K f(x) H^D(dx)}{H^D(K)}.$$

This proves (1.5).

5. The Key Estimate. Assume that the strong open set condition holds. Let $i,j \in \{1,2,\ldots,N\}$, $i \neq j$. Define $Q_{ij}(\varepsilon)$ to be the maximum cardinality of an ε-separated subset F of K_i such that for each $x \in F$, $\text{dist}(x,K_j) \leq \varepsilon$.

Proposition 3. *There exists $\delta > 0$ such that as $\varepsilon \to 0$,*

$$(5.1) \qquad\qquad\qquad Q_{ij}(\varepsilon) = O(\varepsilon^{\delta-D}).$$

Proposition 3 implies Proposition 1. To see this observe that one gets an ε-separated subset of K by taking maximal ε-separated subsets of K_i, $i = 1,2,\ldots,N$, deleting all points from K_i within ε of $\bigcup_{j:j \neq i} K_j$, then taking the union. Thus,

$$N(\varepsilon) \geq \sum_{i=1}^{N} N(\varepsilon r_i^{-1}) - \sum_{i \neq j}\sum Q_{ij}(\varepsilon),$$

and Proposition 1 follows.

Proposition 3 also implies Proposition 2. First notice that to prove Proposition 2 it suffices, since $K_{i_1} \supset K_{i_1 i_2} \supset \ldots$, to establish that if $i \neq j$ then

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(K_{i_1 i_2 \ldots i_n i} \cap K_{i_1 i_2 \ldots i_n j}) = 0.$$

Recall that $\mu_\varepsilon(G)$ is $N(\varepsilon)^{-1} \times$ the cardinality of $F_\varepsilon \cap G$, where F_ε is a maximal ε-separated subset of K. Since

$$K_{i_1 i_2 \ldots i_n i} \cap K_{i_1 i_2 \ldots i_n j} = S_{i_1} S_{i_2} \ldots S_{i_n}(K_i \cap K_j)$$

and since $S_{i_1} \ldots S_{i_n}$ is a similarity transformation that contracts distances by a factor of $r_{i_1} r_{i_2} \ldots r_{i_n} = \rho$,

$$\mu_\varepsilon(K_{i_1 i_2 \ldots i_n i} \cap K_{i_1 i_2 \ldots i_n j}) \leq \frac{Q_{ij}(\varepsilon\rho^{-1}) + Q_{ji}(\varepsilon\rho^{-1})}{N(\varepsilon)}.$$

Proposition 3 and Theorem 1 imply that this converges to 0 as $\varepsilon \to 0$.

6. Proof of the Key Estimate. Recall that the open set condition holds if there is an open set $U \subset \mathbf{R}^d$ such that $S_i U \subset U$ for each i and $S_i U \cap S_j U = \emptyset$ for $i \neq j$. Let $U_{i_1 i_2 \ldots i_n} = S_{i_1} S_{i_2} \ldots S_{i_n} U$. If the open set condition holds then

(a) $U \supset U_{i_1} \supset U_{i_1 i_2} \supset \cdots$;

(b) $K_{i_1 i_2 \ldots i_n} \subset \overline{U}_{i_1 i_2 \ldots i_n}$;

(c) $K_{j_1 j_2 \ldots j_n} \cap U_{i_1 i_2 \ldots i_n} = \emptyset$ unless $(i_1,\ldots,i_n) = (j_1,\ldots,j_n)$

$\left([4], \text{Section 5.2 (3)}\right).$

If the open set U can be chosen so that $U \cap K \neq \emptyset$ then the strong open set condition holds. Assume that this is the case. Then there exists a point $k_{j_1 j_2 \ldots} \in U$. Now the diameters of the sets $K_{j_1 j_2 \ldots j_n}$ converge to zero as $n \to \infty$, and $k_{j_1 j_2 \ldots}$ is an element of each; consequently, there exists a finite sequence $j_1, j_2, \ldots, j_p$ such that

$$K_{j_1 j_2 \ldots j_p} \subset U.$$

Since $K_{j_1 j_2 \ldots j_p}$ is compact there exists $\alpha > 0$ such that

$$\text{dist}(x, U^c) > \alpha \qquad \forall x \in K_{j_1 j_2 \ldots j_p}.$$

It follows upon applying the similarity transformation $S_{i_1} S_{i_2} \ldots S_{i_n}$ that for any sequence $i_1, i_2, \ldots, i_n$

$$K_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_p} \subset U_{i_1 i_2 \ldots i_n}$$

and that for each $x \in K_{i_1 i_2 \ldots i_n j_1 \ldots j_p}$

(6.1) $$\text{dist}(x, U^c_{i_1 i_2 \ldots i_n}) > \alpha r_{i_1} r_{i_2} \ldots r_{i_n}.$$

Let $j \in \{1, 2, \ldots, N\}$ and let $i_1, i_2, \ldots, i_n$ be a finite sequence such that $i_1 \neq j$ and $\alpha r_{i_1} r_{i_2} \ldots r_{i_n} > \varepsilon$. If $x \in K_{i_1 i_2 \ldots i_n}$ and $\text{dist}(x, K_j) \leq \varepsilon$ then the sequence $j_1, j_2, \ldots, j_p$ cannot occur in $i_1, i_2, \ldots, i_n$, because of (6.1) and the fact that $U_{i_1 i_2 \ldots i_n} \cap K_j = \emptyset$.

Now let F be an ε-separated subset of K_i such that for each $x \in F$, $\text{dist}(x, K_j) \leq \varepsilon$ (where $i \neq j$). Each $x \in F$ lies in a set $K_{i_1 i_2 \ldots i_m}$ such that $i_1 = i$ and

(6.2) $$r_{i_1} r_{i_2} \cdots r_{i_m} \text{ diam } K < \varepsilon \leq r_{i_1} r_{i_2} \cdots r_{i_{m-1}} \text{ diam } K;$$

since diam $K_{i_1 i_2 \ldots i_m} = r_{i_1} \ldots r_{i_m}$ diam $K < \varepsilon$ and F is ε-separated, each $x \in F$ has its own unique sequence $i_1, i_2, \ldots, i_m$ satisfying (6.2). Let $r_* = \max(r_1, r_2, \ldots, r_N) < 1$ and let $q \geq 1$ be an integer such that r_*^{q-1} diam $K < \alpha$; then (6.2) implies that $\alpha r_{i_1} r_{i_2} \cdots r_{i_{m-q}} > \varepsilon$. Consequently, if $x \in F \cap K_{i_1 i_2 \ldots i_m}$ and (6.2) holds then by the preceding paragraph the sequence $j_1, j_2, \ldots, j_p$ does not occur in $i_1, i_2, \ldots, i_{m-q}$. Therefore, the cardinality of F, and hence $Q_{ij}(\varepsilon)$, is bounded above by the number $A(\varepsilon)$ of distinct sequences $i_1, i_2, \ldots, i_m$ satisfying (6.2) such that the sequence $j_1, j_2, \ldots, j_p$ does not occur in $i_1, i_2, \ldots, i_{m-q}$. It remains to show that

(6.3) $$A(\varepsilon) = O(\varepsilon^{\delta - D})$$

as $\varepsilon \to 0$ for some $\delta > 0$.

Define $B(\varepsilon)$ to be the number of distinct sequences $i_1 i_2, \ldots, i_n$ such that the sequence $j_1, j_2, \ldots, j_p$ does not occur in $i_1, i_2, \ldots, i_n$ and $r_{i_1} r_{i_2} \cdots r_{i_n} \geq \varepsilon$. Then

$$A(\varepsilon) \leq N^q B\left(\frac{\varepsilon}{\operatorname{diam} K}\right);$$

consequently, to prove (6.3) it suffices to show that for some $D^* < D$

(6.4) $B(\varepsilon) = O(\varepsilon^{-D^*}).$

The function $B(\varepsilon)$ is a nonincreasing, nonnegative integer-valued function of $\varepsilon > 0$. Each sequence $i_1, i_2, \ldots, i_n$ counted in $B(\varepsilon)$ begins with some

$$(i_1, i_2, \ldots, i_p) \neq (j_1, j_2, \ldots, j_p),$$

provided $\varepsilon < (\min_{1 \leq i \leq N} r_i)^p$, so

(6.5) $B(\varepsilon) \leq \sum_{(i_1, \ldots, i_p) \neq (j_1, \ldots, j_p)} B\left(\frac{\varepsilon}{r_{i_1} r_{i_2} \cdots r_{i_p}}\right)$

for all $\varepsilon < (\min_{1 \leq i \leq N} r_i)^p$. Let D^* be the unique real number such that

(6.6) $\sum_{(i_1, \ldots, i_p) \neq (j_1, \ldots, j_p)} (r_{i_1} r_{i_2} \ldots r_{i_p})^{D*} = 1.$

Notice that $D^* < D$ because

$$\sum_{(i_1, \ldots, i_p)} (r_{i_1} r_{i_2} \ldots r_{i_p})^D = \left(\sum_i r_i^D\right)^p = 1.$$

Define $Z(x) = e^{-xD^*} B(e^{-x})$; then by (6.5)

(6.7) $Z(x) \leq \sum_{(i_1, \ldots, i_p) \neq (j_1, \ldots, j_p)} Z\big(x + \log(r_{i_1} r_{i_2} \ldots r_{i_p})\big)(r_{i_1} \ldots r_{i_p})^{D*}$

for all sufficiently large $x \in \mathbf{R}$. Moreover, for each $a \in \mathbf{R}$, $Z(x)$ is bounded on $(-\infty, a]$, because $B(\varepsilon) = 0$ for large ε. It now follows from (6.6) and (6.7) that for all sufficiently large $a \in \mathbf{R}$,

$$\sup\{Z(x) : x \leq a + \min_{(i_1, \ldots, i_p)} \log(r_{i_1} r_{i_2} \cdots r_{i_p})^{-1}\} \leq \sup\{Z(x) : x \leq a\}.$$

Therefore, $Z(x)$ is bounded on $\mathbf{R}$. This proves (6.4).

7. Concluding Remarks

(1) The methods used here may also be used to determine the asymptotic behavior of various other functions. For example, let $x \in \mathbf{R}^d \backslash K$ be a point in the complement of K whose orbit $\mathcal{O}(x) = \{S_{i_1} S_{i_2} \ldots S_{i_n} x\}$ is disjoint from K; define $Q(\varepsilon) = \#\{y \in \mathcal{O}(x)$: distance $(y,K) \geq \varepsilon\}$. Then $Q(\varepsilon)$ satisfies an asymptotic relation analogous to (1.1)-(1.4).

(2) The methods of this paper rely heavily on the *strict* self-similarity of K. For fractals with some *approximate* self-similarity, such as limit sets of Kleinian groups, the analogous problems are considerably harder, but similar results obtain (cf. [6]).

REFERENCES

[1] P. DIACONIS & M. SHAHSHAHANI, *Products of random matrices and computer image generation*, in: *Random Matrices and Their Applications*, ed. J. Cohen et. al., Amer. Math. Soc. (1986).

[2] W. FELLER, *An Introduction to Probability Theory and its Applications*, Vol. 1, 3rd ed., Wiley, New York (1986).

[3] W. FELLER, *An Introduction to Probability Theory and its Applications*, Vol. 2, 2nd ed., Wiley, New York (1971).

[4] J. HUTCHINSON, *Fractals and self-similarity*, Indiana Univ. J. Math. **30** (1981), 713–747.

[5] A.N. KOLMOGOROV & V.M. TIHOMIROV, *Epsilon-entropy and epsilon-capacity of sets in functional spaces*, Amer. Math. Soc. Transl. (Ser. 2) **17** (1959), 277–364.

[6] S. LALLEY, *Renewal theorems in symbolic dynamics, with applications to geodesic flows, noneuclidean tessellations, and their fractal limits*, Purdue Univ., Technical Report 87–40 (1987).

[7] B. MANDELBROT, *The Fractal Geometry of Nature*, Freeman, San Francisco (1983).

[8] L. PONTRJAGIN & L. SCHNIRELMAN, *Sur une propriété métrique de la dimension*, Ann. Math. **33** (1932), 156–162.

[9] L.S. YOUNG, *Dimension, entropy, and Lyapunov exponents*, Ergodic Th Dynamical Systems **2** (1982), 109–124.

Department of Mathematics
Purdue University
West Lafayette, Indiana 47907