ADVANCES IN APPLIED MATHEMATICS 8, 154–193 (1987)

Distribution of Periodic Orbits of Symbolic and Axiom *A* Flows

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0. INTRODUCTION

A recent paper by Parry [11] begins:

> For some time now, in fact probably since Selberg's paper [16], there has been a growing awareness of affinities between the distribution problems of number theory and those of dynamical systems.

Indeed, Margulis [10] announced that for the geodesic flow on a d-dimensional compact manifold of curvature -1 the number of periodic orbits τ with (minimal) period $\tau(1) \leq x$ is asymptotic to $e^{(d-1)x}/(d-1)x$. This result bears a striking resemblance to the prime number theorem. Parry and Pollicott [12], following earlier work by Bowen [2, 4], generalized Margulis' theorem to weakly mixing Axiom A flows, proving that $\#\{\tau\colon \tau(1) \leq x\} \sim e^{hx}/hx$, where h is the topological entropy of the flow. Sarnak [15] has related results for the horocycle flow. Bowen [3] and Parry [11] proved analogues of the Dirichlet density theorem for mixing Axiom A flows, e.g., if $\tau(G)$ represents the integral of the continuous function G over one period of τ, then $\sum_{\tau(1) \leq x} \tau(G)/\tau(1) \sim (e^{hx}/hx)\int G\, d\bar{\mu}$, where $\bar{\mu}$ is the invariant probability measure of maximum entropy.

This paper pursues an altogether different analogy, this between the distribution problems for periodic orbits of Axiom A and symbolic flows and those of classical probability theory. This analogy leads to theorems which apparently have no counterparts in number theory. Moreover, it leads to techniques quite different from those commonly used in studying periodic orbits: in particular, there is no use of zeta functions or any of the attendant Tauberian theorems. We do not believe that the main results of this paper can be obtained by analyzing zeta functions. These results do, however, make use of the groundwork done by Bowen in [4], which reduces

0196-8858/87 $7.50

the study of Axiom A flows to the study of (hyperbolic) *symbolic flows* (called *suspensions* in [11] and [12]).

The main result is an analogue of the local limit theorem for large deviations in classical probability. Let $G_1, G_2, \ldots, G_d$ be real-valued continuous functions; for $x_1, x_2, \ldots, x_d \in \mathscr{R}$ let $-\bar{\gamma}(\mathbf{x}) = \sup\{\bar{H}(\bar{\mu})\colon \bar{\mu}$ invariant; $\int G_i\, d\bar{\mu} = x_i,\ i = 1, 2, \ldots, d\}$, where $\bar{H}$ denotes entropy.

THEOREM I. *Under certain conditions*

$$\#\{\tau\colon 0 \le \tau(1) - a \le \delta;\ 0 \le \tau(G_i) - ax_i \le \delta,\ i = 1, \ldots, d\}$$
$$\sim e^{-a\bar{\gamma}(\mathbf{x})} a^{-(d+2)/2} (2\pi)^{-d/2} \left(\det \nabla^2 \bar{\gamma}(\mathbf{x})\right) C(\mathbf{x}, \delta) \tag{0.1}$$

as $a \to \infty$. This holds uniformly in $\mathbf{x}$ *locally.*

The entropy $-\bar{\gamma}(\mathbf{x})$ plays the same role as the Kullback–Leibler information function for analogous results in probability; $\nabla^2\bar{\gamma}(\mathbf{x})$ may be interpreted as a Fisher information matrix.

It is obvious that some hypotheses on $\mathbf{x}$ and $G_1, \ldots, G_d$ are needed. If, for example, $G_1 = G_2 = \cdots = G_d = 1$, then (0.1) is false; if $x_1, \ldots, x_d$ are sufficiently large or small then $0 \le \tau(1) - a \le \delta$ may be incompatible with $0 \le \tau(G_i) - ax_i \le \delta$, since $G_1, \ldots, G_d$ are bounded.

Setting $d = 0$ in (0.1) leads to the "prime number theorem" for periodic orbits of [10] and [12].

The next result is an analogue of the weak law of large numbers. It improves the equidistribution theorems of [3] and [11].

THEOREM II. *Under certain conditions, if you choose τ at random from $\{\tau\colon 0 \le \tau(1) - a \le \delta;\ 0 \le \tau(G_i) - ax_i \le \delta,\ i = 1, \ldots, d\}$, then*

$$\operatorname{Prob}\left\{\left|\frac{\tau(G)}{\tau(1)} - \int G\, d\bar{\mu}_{\mathbf{x}}\right| < \varepsilon\right\} \to 1$$

as $a \to \infty$, for every $\varepsilon > 0$ and every continuous function G, where $\bar{\mu}_{\mathbf{x}}$ is the invariant probability measure maximizing entropy subject to the constraints $\int G_i\, d\bar{\mu}_{\mathbf{x}} = x_i,\ i = 1, \ldots, d$.

Setting $d = 0$, one sees that almost every periodic orbit is nearly uniformly distributed according to the maximum entropy measure.

There is also a central limit theorem.

THEOREM III. *Under certain conditions, if you choose τ at random from $\{\tau\colon 0 \le \tau(1) - a \le \delta;\ 0 \le \tau(G_1) - ax_i \le \delta,\ i = 1, 2, \ldots, d\}$ then for some*

$c_x > 0$,

$$\text{Prob}\left\{ a^{1/2} c_x \left(\frac{\tau(G)}{\tau(1)} - \int G \, d\bar{\mu}_x \right) \geq y \right\} \to \int_y^\infty e^{-t^2/2} \, dt/(2\pi)^{1/2}$$

as $a \to \infty$, *for all* $y \in \mathscr{R}$.

Theorems I–III are stated more precisely for symbolic flows in Section 6, and for Axiom A flows in Section 9. The important counting arguments are in Sections 7 and 8: these rely on certain facets of the "thermodynamic formalism" developed by Ruelle [14] (cf. also [15]). A brief resume of "thermodynamic" results is given in Sections 1–3, and properties of the thermodynamic functions are presented in Sections 4, 5. Nearly the entire paper is concerned with the study of symbolic flows: the results for Axiom A flows follow directly from the corresponding results for symbolic flows, in view of the construction in [4].

Theorems II, III are familiar in the context of a much simpler dynamical system than an Axiom A flow, to wit, the shift σ on the space of sequences $\prod_0^\infty \{0,1\}$. "Periodic orbits" for this system come from sequences ξ satisfying $\sigma^n \xi = \xi$ for some n. The number of periodic *orbits* with minimal period n is $n^{-1} \sum_{d|n} \mu(n/d) 2^d$, where μ is the Moebius function; this is obviously asymptotic to $n^{-1} 2^n$ for large n. For an orbit of period n chosen at random the fraction of 1's is near $\frac{1}{2}$ with high probability, by the law of large numbers, and the probability that the fraction of 1's is greater than $\frac{1}{2} + \frac{1}{2} y n^{-1/2}$ is about $\int_y^\infty e^{-t^2/2} \, dt/(2\pi)^{1/2}$, by the Central Limit Theorem. Thus, for this system some features of the distribution theory for periodic orbits exactly coincide with the classical limit laws for Bernoulli sequences.

1. Preliminaries: Shifts and Symbolic Flows

Forward Shift

Let A be an irreducible, aperiodic, $l \times l$ matrix of zeros and ones ($l > 1$), and let

$$\Sigma_A = \left\{ \xi \in \prod_{n=-\infty}^{\infty} \{1, 2, \dots, l\} : A(\xi_n, \xi_{n+1}) = 1, \forall n \in \mathscr{Z} \right\},$$

$$\Sigma_A^+ = \left\{ \xi \in \prod_{n=0}^{\infty} \{1, 2, \dots, l\} : A(\xi_n, \xi_{n+1}) = 1, \forall n \in \mathscr{N} \right\}.$$

The spaces Σ_A, Σ_A^+ are compact and metrizable in the product topology.

Define the forward shift operators

$$\sigma\colon\ \Sigma_A \to \Sigma_A \qquad \text{and} \qquad \sigma\colon\ \Sigma_A^+ \to \Sigma_A^+$$

by $(\sigma\xi)_n = \xi_{n+1}$ for all $n \in \mathscr{Z}$ $(n \in \mathscr{N})$. Observe that $\sigma\colon \Sigma_A \to \Sigma_A$ is a homeomorphism, whereas $\sigma\colon \Sigma_A^+ \to \sigma_A^+$, although continuous and surjective, is not generally 1-to-1.

Hölder Continuity

Let $C(\Sigma_A), C(\Sigma_A^+)$ denote the spaces of continuous, complex-valued functions on Σ_A, Σ_A^+. Define

$$\mathrm{var}_n(f) = \sup\{|f(\xi) - f(\zeta)|\colon \xi_j = \zeta_j, \forall |j| \le n\}, \qquad f \in C(\Sigma_A),$$
$$\mathrm{var}_n(f) = \sup\{|f(\xi) - f(\zeta)|\colon \xi_j = \zeta_j, \forall 0 \le j \le n\}, \qquad f \in C(\Sigma_A^+).$$

For $0 < \rho < 1$, let

$$|f|_\rho = \sup_{n \in \mathscr{N}} (\mathrm{var}_n f)/\rho^{2n+1}, \qquad f \in C(\Sigma_A),$$
$$|f|_\rho = \sup_{n \in \mathscr{N}} (\mathrm{var}_n f)/\rho^{n}, \qquad f \in C(\Sigma_A^+),$$
$$\mathscr{F}_\rho = \{f \in C(\Sigma_A)\colon |f|_\rho < \infty\},$$
$$\mathscr{F}_\rho^+ = \{f \in C(\Sigma_A^+)\colon |f|_\rho < \infty\}.$$

Elements of $\mathscr{F}_\rho, \mathscr{F}_\rho^+$ are referred to as Hölder continuous functions. The spaces $\mathscr{F}_\rho, \mathscr{F}_\rho^+$ are Banach algebras when endowed with the norm(s) $\|\cdot\|_\rho = |\cdot|_\rho + \|\cdot\|_\infty$. Note that $\mathscr{F}_\rho^+$ is naturally embedded in $\mathscr{F}_\rho$.

Symbolic Flow

For $f \in \mathscr{F}_\rho$ a strictly positive function, define the suspension space

$$\Sigma_A^f = \{(\xi, t)\colon \xi \in \Sigma_A \text{ and } 0 \le t \le f(\xi)\},$$

with $(\xi, f(\xi))$ and $(\sigma\xi, 0)$ identified for all $\xi \in \Sigma_A$. The symbolic flow under f (sometimes called the *f-suspension*), written σ_t^f, is defined by

$$\sigma_t^f(\xi, s) = (\xi, s + t), \qquad 0 \le s + t \le f(\xi)\colon$$

note that when $s + t = f(\xi)$, the point $\sigma_t^f(\xi, s) = (\xi, f(\xi)) = (\sigma\xi, 0)$, so σ_t^f is well defined for all $t \in \mathscr{R}$. Moreover, $\sigma_t^f\colon \Sigma_A^f \to \Sigma_A^f$ is a homeomorphism for every $t \in \mathscr{R}$.

Homology

Two functions $f, g \in C(\Sigma_A)$ (or $C(\Sigma_A^+)$) are said to be homologous, written $f \cong g$, if there exists $h \in C(\Sigma_A)$ (or $h \in C(\Sigma_A^+)$) such that

$$f - g = h \circ \sigma - h. \tag{1.1}$$

Homology is clearly an equivalence relation.

For $f \in C(\Sigma_A)$ (or $C(\Sigma_A^+)$) and $n \geq 0$, define

$$S_n f = \sum_{i=0}^{n-1} f \circ \sigma^i, \qquad n \geq 1,$$
$$S_0 f \equiv 0. \tag{1.2}$$

Observe that if (1.1), then $S_n f - S_n g = h \circ \sigma^n - h$; and if $\sigma^n \xi = \xi$, then $S_n f(\xi) = S_n g(\xi)$.

Lemma A. *If $f \in \mathscr{F}_\rho$ then $f \cong f^*$ for some $f^* \in \mathscr{F}_{\rho^{1/2}}$.*

For the proof, see [5, Lemma 1.6].

σ^f-Homology

Let $\mathscr{G}$ be the infinitesimal generator of the semigroup on $C(\Sigma_A^f)$ induced by the symbolic flow σ^f. Thus $F \in \mathscr{D}(\mathscr{G})$ iff for every $(\xi, t) \in \Sigma_A^f$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\big(F \circ \sigma_\varepsilon^f(\xi, t) - F(\xi, t)\big) = \mathscr{G}F(\xi, t)$$

exists, and $\mathscr{G}F \in C(\Sigma_A^f)$. Two functions $G_1, G_2 \in C(\Sigma_A^f)$ are said to be σ^f-homologous, written $G_1 \cong G_2$, if there exists $F \in \mathscr{D}(\mathscr{G})$ such that

$$G_1 - G_2 = \mathscr{G}F. \tag{1.3}$$

Observe that σ^f-homology is an equivalence relation; also, if (1.3) holds then

$$\int_{s=0}^{t} (G_1 - G_2) \circ \sigma_s^f \, ds = F \circ \sigma_t^f - F.$$

In particular, if $G_1 \cong G_2$ then $G_1 - G_2$ integrates to zero over any periodic orbit of σ^f.

Lemma B. *Let $G_1, G_2 \in C(\Sigma_A^f)$, and let $g_i(\xi) = \int_0^{f(\xi)} G_i(\xi, t)\, dt$, $i = 1, 2$. Then $g_1 \cong g_2$ iff $G_1 \cong G_2$.*

The proof is easy, and therefore omitted.

Notational Conventions

Elements of $\mathscr{R}^{d+1}$ (or $\mathscr{C}^{d+1}$) will be denoted by boldface letters $\mathbf{x}, \mathbf{z}, \boldsymbol{\Theta}$, etc. Functions on Σ_A (or Σ_A^+) will always be denoted by lower case letters f, g, h, φ, ψ, etc.; functions on Σ_A^f will be denoted by upper case letters F, G, Φ, etc. For functions $f_0, f_1, \ldots, f_d$ on Σ_A (or Σ_A^+) $\mathbf{f} = (f_0, f_1, \ldots, f_d)$ will denote the corresponding vector-valued function; for functions $G_1, \ldots, G_d$ on Σ_A^f, $\mathbf{G} = (G_1, G_2, \ldots, G_d)$. The inner product on $\mathscr{C}^d$ or $\mathscr{C}^{d+1}$ is denoted by $\langle \mathbf{x} | \mathbf{z} \rangle$. If B is a Hermitian matrix, then $\langle \mathbf{x} | B | \mathbf{z} \rangle = \langle \mathbf{x} | B\mathbf{z} \rangle$.

The set of bounded, Borel measurable, real-valued functions on Σ_A^f will be denoted by $B(\Sigma_A^f)$. Measures on Σ_A^f will be denoted by $\bar{\mu}$, $\bar{\nu}$, etc., whereas measures on Σ_A, Σ_A^+ will be denoted by μ, ν, etc.

2. Ruelle's Perron–Frobenius Operators

For f, $g \in C(\Sigma_A^+)$ define $\mathscr{L}_f g \in C(\Sigma_A^+)$ by

$$\mathscr{L}_f g(\xi) = \sum_{\zeta:\, \sigma\zeta = \xi} e^{f(\zeta)} g(\zeta).$$

For each $f \in \mathscr{F}_\rho^+$, the operator $\mathscr{L}_f$: $\mathscr{F}_\rho^+ \to \mathscr{F}_\rho^+$ is a continuous linear operator; if f is real-valued, then $\mathscr{L}_f$ is a positive operator. Observe that for any f, g, $\varphi \in C(\Sigma_A^+)$ and any $n \in \mathscr{N}$,

$$\begin{aligned} \mathscr{L}_f^n g(\xi) &= \mathscr{L}_\varphi^n e^{S_n(f-\varphi)} g(\xi) \\ &= \sum_{\zeta:\, \sigma^n\zeta = \xi} \exp\{S_n f(\zeta)\} g(\zeta). \end{aligned}$$

Theorem A (Ruelle). *For each real-valued $f \in \mathscr{F}_\rho^+$ there exists a real number $\lambda_f \in (0, \infty)$ which is a simple eigenvalue of $\mathscr{L}_f$: $\mathscr{F}_\rho^+ \to \mathscr{F}_\rho^+$, with strictly positive eigenfunction h_f. Moreover, spectrum $(\mathscr{L}_f) \setminus \{\lambda_f\}$ is contained in a disc of radius strictly less than λ_f.*

Proof may be found in [5, 13, 14]. Bowen and Ruelle also prove

Theorem B. *For each real-valued $f \in \mathscr{F}_\rho^+$ there is a positive Borel measure ν_f on Σ_A^+ such that $\mathscr{L}_f^* \nu_f = \lambda_f \nu_f$, and such that for all $g \in C(\Sigma_A^+)$,*

$$\lim_{n \to \infty} \left\| \mathscr{L}_f^n g / \lambda_f^n - \left(\int_{\Sigma_A^+} g \, d\nu_f \right) h_f \right\|_\infty = 0. \tag{2.1}$$

Theorem C (Pollicott). *Suppose $f = u + iv$, where $u, v \in \mathscr{F}_\rho^+$ are real-valued.*

(a) *If for some constant $a \in [-\pi, \pi]$ the function $(v - a)/2\pi$ is homologous to an integer-valued function, then $e^{ia}\lambda_u$ is a simple eigenvalue of*

$\mathscr{L}_f$, and the rest of the spectrum is contained in a disc of radius strictly less than λ_u.

(b) *Otherwise, the entire spectrum of $\mathscr{L}_f$ is contained in a disc of radius strictly less than λ_u.*

The proof is given in [13].

Standard arguments in regular perturbation theory show that λ_f, h_f, ν_f are "analytic" in f, and that the convergence in (2.1) is locally uniform in f, g. These arguments are carried out in Appendix 1. Assume that for all real-valued $f \in \mathscr{F}_\rho^+$

$$\begin{aligned} 1 &= \int_{\Sigma_A^+} 1\, d\nu_0 \\ &= \int_{\Sigma_A^+} h_f\, d\nu_0 \\ &= \int_{\Sigma_A^+} h_f\, d\nu_f. \end{aligned}$$

This normalization will be used implicitly in the perturbation arguments of Appendix 1.

Observe that if $f - g = \varphi - \varphi \circ \sigma$, then

$$\begin{aligned} \mathscr{L}_f \psi &= e^{-\varphi} \mathscr{L}_g (e^{\varphi} \psi), \\ h_f &= e^{-\varphi} h_g, \\ d\nu_f &= e^{\varphi}\, d\nu_g, \\ \lambda_f &= \lambda_g. \end{aligned}$$

3. Gibbs States, Pressure, and the Variational Principle

For each real-valued $f \in \mathscr{F}_\rho^+$ the measure μ_f defined by $d\mu_f/d\nu_f = h_f$ is a σ-invariant probability measure on Σ_A^+ which, because of the σ-invariance, extends to a σ-invariant probability measure on Σ_A, also denoted by μ_f. If $f, g \in \mathscr{F}_\rho^+$ then $\mu_f = \mu_g$ iff $f - g \cong$ constant (cf. [5, Theorem 1.28]); otherwise $\mu_f \perp \mu_g$ (because μ_f, μ_g are ergodic). The measure μ_f is called the *Gibbs measure* for f.

If $f \in \mathscr{F}_\rho$ then by Lemma A there exists $f^* \in \mathscr{F}_{\rho^{1/2}}^+$ such that $f \cong f^*$. For any real-valued $f \in \mathscr{F}_\rho$, define

$$\mu_f = \mu_{f^*}, \tag{3.1}$$

$$P(f) = P(f^*) = \log \lambda_{f^*}. \tag{3.2}$$

These are valid definitions because if $f \cong f_1^*$ and $f \cong f_2^*$ as functions in $C(\Sigma_A)$, then $f_1^* \cong f_2^*$ as functions in $C(\Sigma_A^+)$, whence $\mu_{f_1^*} = \mu_{f_2^*}$ and $\lambda_{f_1^*} = \lambda_{f_2^*}$.

Let $\mathscr{I}$ denote the set of σ-invariant probability measures on Σ_A, and for $\mu \in \mathscr{I}$ let $H(\mu)$ be the entropy of the dynamical system (Σ_A, σ, μ).

THEOREM D (Gibbs Variational Principle). *For any real-valued $f \in \mathscr{F}_\rho$ and any $\mu \in \mathscr{I}$,*

$$P(f) - H(\mu) \geq \int f \, d\mu. \tag{3.3}$$

Equality holds iff $\mu = \mu_f$.

See [5, Proposition 1.21 and Theorem 1.22].

THEOREM E. *The pressure functional is convex, i.e., if $f, g \in \mathscr{F}_\rho$ are real-valued and $0 < \alpha < 1$ then*

$$P(\alpha f + (1 - \alpha) g) \leq \alpha P(f) + (1 - \alpha) P(g).$$

Equality holds iff $f - g$ is homologous to a constant.

This follows directly from Theorem D (recall that $\mu_f = \mu_g$ iff $f - g \cong$ constant).

4. THERMODYNAMIC FUNCTIONS FOR THE SHIFT

Let $f_0, f_1, \ldots, f_d \in \mathscr{F}_\rho$ be real-valued, and let $\mathbf{f} = (f_0, f_1, \ldots, f_d)$. For $\mathbf{z} = (z_0, z_1, \ldots, z_d) \in \mathscr{R}^{d+1}$, define

$$\beta(\mathbf{z}) = \beta_{\mathbf{f}}(\mathbf{z}) = \mathbf{P}(\langle \mathbf{z} | \mathbf{f} \rangle). \tag{4.1}$$

In Appendix 1 we will show that $\beta(\mathbf{z})$ is a real analytic function.

HYPOTHESIS A. *If $\sum_{i=0}^d a_i f_i$ is homologous to a constant for some $a_0, a_1, \ldots, a_d \in \mathscr{R}$, then $a_0 = a_1 = \cdots = a_d = 0$.*

Under Hypothesis A, $\beta(\mathbf{z})$ enjoys the following properties:

(a) $f_i > 0 \Rightarrow \beta(\mathbf{z})$ *is strictly increasing in* z_i;

(b) $\beta(\mathbf{z})$ *is strictly convex on* $\mathscr{R}^{d+1}$;

(c) $\nabla^2 \beta(\mathbf{z})$ *is strictly positive definite*, $\forall \mathbf{z} \in \mathscr{R}^{d+1}$;

(d) $\nabla \beta$: $\mathscr{R}^{d+1} \to$ *range* $(\nabla \beta)$ *is a diffeomorphism*;

(e) $\nabla \beta(\mathbf{z}) = \int \mathbf{f} \, d\mu_{\langle \mathbf{z} | \mathbf{f} \rangle}, \forall \mathbf{z} \in \mathscr{R}^{d+1}$.

Property (a) is apparent from the variational principle; property (b) follows immediately from Hypothesis A and Theorem E; and property (d) follows from (c). Properties (c) and (e) are known (cf. [14, Chap. 5, Ex. 5]). Proofs may be found in [9].

The *Legendre transform* $\gamma = \gamma_{\mathbf{f}}$ of the convex function $\beta_{\mathbf{f}}$ is defined by

$$\gamma(\mathbf{x}) = \sup_{\mathbf{z} \in \mathscr{R}^{d+1}} (\langle \mathbf{x} | \mathbf{z} \rangle - \beta(\mathbf{z})). \tag{4.2}$$

The function $\gamma(\mathbf{x})$ is convex on $\mathscr{R}^d$. For $\mathbf{x} \in \mathscr{B} = \nabla\beta(\mathscr{R}^{d+1})$, the supremum is *uniquely* attained at the value of $\mathbf{z}$ for which $\nabla\beta(\mathbf{z}) = \mathbf{x}$, since $\langle \mathbf{x} | \mathbf{z} \rangle - \beta(\mathbf{z})$ is a strictly concave function of z. The inverse function theorem now implies that $\gamma(\mathbf{x})$ is smooth on $\mathscr{B}$ and differential calculus therefore applies. Thus,

(f) $\nabla\gamma \circ \nabla\beta =$ *identity on* $\mathscr{R}^{d+1}$;

(g) $\nabla\beta \circ \nabla\gamma =$ *identity on* $\mathscr{B} = \nabla\beta(\mathscr{R}^{d+1})$;

(h) $\nabla\beta(\mathbf{z}) = \mathbf{x} \Leftrightarrow \nabla\gamma(\mathbf{x}) = \mathbf{z} \Rightarrow \nabla^2\gamma(\mathbf{x}) = (\nabla^2\beta(\mathbf{z}))^{-1}$;

(i) $\nabla\beta(\mathbf{z}) = \mathbf{x} \Rightarrow \gamma(\mathbf{x}) + \beta(\mathbf{z}) = \langle \mathbf{x} | \mathbf{z} \rangle$;

(j) $\nabla\beta(\mathbf{z}) \neq \mathbf{x} \Rightarrow \gamma(\mathbf{x}) + \beta(\mathbf{z}) > \langle \mathbf{x} | \mathbf{z} \rangle$;

(k) $\nabla\beta(\mathbf{z}) = \mathbf{x} \Rightarrow \gamma(\mathbf{x}) = -H(\mu_{\langle \mathbf{z} | \mathbf{f} \rangle}) < 0$.

Property (k) follows from (e) and (i), together with Theorem D.

Unfortunately, it is difficult to describe the region $\mathscr{B}$. However, it should be noted that by (e)

$$\mathscr{B} \subset \prod_{i=0}^{d} \left[\min_{\Sigma_A} f_i, \max_{\Sigma_A} f_i \right].$$

Define $\mathscr{B}^* = \{\mathbf{x} \in \mathscr{R}^{d+1} : \mathbf{x} = \int \mathbf{f}\, d\mu, \text{ some } \mu \in \mathscr{I}\}$ where $\mathscr{I}$ is the set of σ-invariant probability measures on Σ_A. Since $\mathscr{I}$ is convex and closed in the weak-$*$ topology, $\mathscr{B}^*$ is closed and convex in $\mathscr{R}^{d+1}$: in fact, it can be shown (using Theorem E) that $\mathscr{B}^*$ is the closed convex hull of $\mathscr{B}$. Define

$$\alpha(\mathbf{x}) = \sup\left\{ H(\mu) : \mu \in \mathscr{I}, \int \mathbf{f}\, d\mu = \mathbf{x} \right\}.$$

Then α is concave on $\mathscr{B}^*$ (since $H(\mu)$ is concave on $\mathscr{I}$), and by (k)

$$\alpha(\mathbf{x}) = -\gamma(\mathbf{x}), \qquad \mathbf{x} \in \mathscr{B}.$$

Fix $\mathbf{x} \in \mathscr{R}^d$, and consider the function $t \to t\alpha(\mathbf{x}/t)$, defined for all $t \in \mathscr{R}^+$ such that $\mathbf{x}/t \in \mathscr{B}^*$. This set of t is an interval, since $\mathscr{B}^*$ is convex. It is easily shown that $t\alpha(\mathbf{x}/t)$ is concave in t (if $F(t)$ is concave, then $tF(a/t)$ is concave $\forall a \in \mathscr{R}^+$). Moreover, if $\mathbf{x}/t \in \mathscr{B}$, then

$$(d/dt)(-t\gamma(\mathbf{x}/t)) = \beta(\nabla\gamma(\mathbf{x}/t)), \tag{4.3}$$

$$(d^2/dt^2)(-t\gamma(\mathbf{x}/t)) = -t^{-3}\langle \mathbf{x}|\nabla^2\gamma(\mathbf{x}/t)|\mathbf{x}\rangle < 0. \tag{4.4}$$

If there is a $t > 0$ such that $\beta(\nabla\gamma(\mathbf{x}/t)) = 0$, then this t is unique, and will be denoted by $t_{\mathbf{x}}$.

Observe that if $t_{\mathbf{x}}$ exists at some $\mathbf{x} = \mathbf{x}^*$, then $t_{\mathbf{x}}$ exists and is smooth for $\mathbf{x}$ in a neighborhood of $\mathbf{x}^*$. For if $\beta(\nabla\gamma(\mathbf{x}^*/t_{\mathbf{x}^*})) = 0$ then by (4.3), (4.4) $t\alpha(\mathbf{x}^*/t)$ has a local maximum at $t = t^*_{\mathbf{x}}$, and hence by the smoothness in $(\mathbf{x}, t)$ of $t\alpha(\mathbf{x}/t)$, there must be a local maximum $t_{\mathbf{x}}$ of $t \to t\alpha(\mathbf{x}/t)$ for all $\mathbf{x}$ near $\mathbf{x}^*$.

5. Thermodynamic Functions for the Flow

Consider the symbolic flow (Σ_A^f, σ^f), where $f \in \mathscr{F}_\rho$ is strictly positive. For $G, G_i \in B(\Sigma_A^f)$, let

$$\begin{aligned} g(\xi) &= \int_0^{f(\xi)} G(\xi, t)\, dt, \\ g_i(\xi) &= \int_0^{f(\xi)} G_i(\xi, t)\, dt. \end{aligned} \tag{5.1}$$

Let $\mathscr{F}_\rho(\Sigma_A^f)$ be the set of $G \in B(\Sigma_A^f)$ such that $g \in \mathscr{F}_\rho$, where g is defined by (5.1).

For $G \in \mathscr{F}_\rho(\Sigma_A^f)$ real-valued, define the pressure $P_f(G)$ to be the unique real number such that

$$P\big(g - P_f(G)f\big) = 0, \tag{5.2}$$

where P is defined by (3.2). Such a real number always exists, because $P(g - zf) \to -\infty$ as $z \to \infty$ and $P(g - zf) \to \infty$ as $z \to -\infty$, since $f > 0$.

Let $\mathscr{I}_f$ be the set of σ_f-invariant probability measures on Σ_A^f. There is a 1-to-1 correspondence between $\mathscr{I}$ and $\mathscr{I}_f$, given by $\mu \leftrightarrow \bar{\mu}$ iff

$$\int_{\Sigma_A^f} G\, d\bar{\mu} = \int_{\Sigma_A} g\, d\mu \Big/ \int_{\Sigma_A} f\, d\mu, \tag{5.3}$$

where g, G satisfy (5.1). Moreover, if $H(\bar{\mu}, \sigma^f)$ is the entropy of the flow $(\Sigma_A^f, \sigma^f, \bar{\mu})$ for $\bar{\mu} \in \mathscr{I}_f$, then

$$H(\bar{\mu}, \sigma^f) = H(\mu) \Big/ \int f\, d\mu \tag{5.4}$$

(cf. [1], also [7]).

THEOREM F. *For any real-valued* $G \in \mathscr{F}_\rho(\Sigma_A^f)$ *and* $\bar{\mu} \in \mathscr{I}_f$,

$$P_f(G) - H(\bar{\mu}, \sigma^f) \geq \int G \, d\bar{\mu}. \tag{5.5}$$

Equality holds iff $\bar{\mu} = \bar{\mu}_G$, *where* $\bar{\mu}_G$ *is defined by*

$$\int G_1 \, d\bar{\mu}_G = \int g_1 \, d\mu \Big/ \int f \, d\mu, \qquad \forall G_1 \in B(\Sigma_A^f), \tag{5.6}$$

$\mu = \mu_{g - P_f(G)f}$ *being the Gibbs measure for the function* $g - P_f(G)f$, *and* g_1, G_1 *satisfy* (5.1).

The proof of Theorem F will be given at the end of the section. The measure $\bar{\mu}_G$ will be referred to as the Gibbs measure for G.

Note that in the special case $G \equiv 0$, $\bar{\mu}_G = \bar{\mu}_0$ is the unique invariant measure which maximizes entropy for the flow, and $H(\bar{\mu}_0, \sigma^f)$ equals the topological entropy $H^*(\sigma^f)$ of the flow. Consequently, by (5.2)

$$P(-H^*(\sigma^f)f) = 0. \tag{5.7}$$

Next, let $G_1, G_2, \ldots, G_d \in \mathscr{F}_\rho(\Sigma_A^f)$ be real-valued, let $\mathbf{G} = (G_1, G_2, \ldots, G_d)$, and let $g_1, g_2, \ldots, g_d \in \mathscr{F}_\rho$ be related to $G_1, \ldots, G_d$ as in (5.1). Define the thermodynamic functions

$$\bar{\beta}(\mathbf{z}) = \bar{\beta}_{\mathbf{G}}(\mathbf{z}) = P_f(\langle \mathbf{z} | \mathbf{G} \rangle), \qquad \mathbf{z} \in \mathscr{R}^d, \tag{5.8}$$

$$\bar{\gamma}(\mathbf{x}) = \bar{\gamma}_{\mathbf{G}}(\mathbf{x}) = \sup_{\mathbf{z} \in \mathscr{R}^d} \left(\langle \mathbf{x} | \mathbf{z} \rangle - \bar{\beta}(\mathbf{z}) \right), \qquad \mathbf{x} \in \mathscr{R}^d. \tag{5.9}$$

HYPOTHESIS $\bar{\text{A}}$. *If* $a_1, a_2, \ldots, a_d \in \mathscr{R}$ *are constants such that* $\sum_{i=1}^d a_i G_i \cong$ *constant, then* $a_1 = a_2 = \cdots = a_d = 0$.

By Lemma B, $\mathbf{G}$ satisfies Hypothesis $\bar{\text{A}}$ iff $(f, g_1, g_2, \ldots, g_d)$ satisfies Hypothesis A. Under Hypothesis $\bar{\text{A}}$, $\bar{\beta}_{\mathbf{G}}$ has the following properties:

($\bar{\text{a}}$) $G_i > 0 \Rightarrow \bar{\beta}_{\mathbf{G}}$ *is strictly increasing in* z_i;

($\bar{\text{b}}$) $\bar{\beta}(\mathbf{z})$ *is strictly convex on* $\mathscr{R}^d$;

($\bar{\text{c}}$) $\nabla^2 \bar{\beta}(\mathbf{z})$ *is strictly positive definite*;

($\bar{\text{d}}$) $\nabla \bar{\beta}$: $\mathscr{R}^d \to$ *range* $(\nabla \bar{\beta})$ *is a diffeomorphism*;

($\bar{\text{e}}$) $\nabla \bar{\beta}(\mathbf{z}) = \int \mathbf{G} \, d\bar{\mu}_{\langle \mathbf{z} | \mathbf{G} \rangle}$, $\forall \mathbf{z} \in \mathscr{R}^d$.

Property ($\bar{\text{a}}$) is an immediate consequence of Theorem F. Properties ($\bar{\text{b}}$) and ($\bar{\text{d}}$) follow immediatly from ($\bar{\text{c}}$). Property ($\bar{\text{c}}$) follows from Lemma 1 below.

Let $\beta = \beta_{(f, \mathbf{g})}$ be the pressure function for $(f, g_1, \ldots, g_d)$. Then $\bar{\beta}$ and β are related by

$$\beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) = 0, \qquad \forall \mathbf{z} \in \mathscr{R}^d. \tag{5.10}$$

Proof of (e). Taking the partial derivative with respect to z_i, $i = 1, 2, \ldots, d$, in (5.10) gives

$$\partial_i \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) = \partial_0 \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) \partial_i \bar{\beta}(\mathbf{z}), \tag{5.11}$$

which implies

$$\partial_i \bar{\beta}(\mathbf{z}) = \int g_i \, d\mu \Big/ \int f \, d\mu, \tag{5.12}$$

where $\mu = \mu_{\langle \mathbf{z}|\mathbf{g}\rangle - \bar{\beta}(\mathbf{z})f}$ is the Gibbs measure for $\langle \mathbf{z}|\mathbf{g}\rangle - \bar{\beta}(\mathbf{z})f$. Equation (5.12) follows from (5.11) by property (e), Section 4. Property ($\bar{\text{e}}$) follows now from (5.12) and (5.3). □

LEMMA 1. *Under Hypothesis* $\bar{\text{A}}$, *the Hessian matrix* $\nabla^2 \bar{\beta}(\mathbf{z})$ *is strictly positive definite, for every* $\mathbf{z} \in \mathscr{R}^d$. *Moreover, if* $\mathbf{x} = \nabla \bar{\beta}(\mathbf{z})$, $\mathbf{x}^* = (1, \mathbf{x})$, *and* $t_{\mathbf{x}}^{-1} = \int f \, d\mu$, *where* μ *is the Gibbs measure for* $\langle \mathbf{z}|\mathbf{g}\rangle - \bar{\beta}(\mathbf{z})f$, *then*

$$\det \nabla^2 \bar{\beta}(\mathbf{z}) = t_{\mathbf{x}}^d \det \nabla^2 \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) \cdot \langle \mathbf{x}^* | \nabla^2 \gamma(\mathbf{x}^*/t_{\mathbf{x}}) | \mathbf{x}^* \rangle. \tag{5.13}$$

Proof. Taking the partial derivative with respect to z_j in (5.11) gives

$$\begin{aligned}
&\big(\partial_0 \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z})\big)\big(\partial_{ij} \bar{\beta}(\mathbf{z})\big) \\
&\quad = \big(\partial_{00} \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z})\big)\big(\partial_i \bar{\beta}(\mathbf{z})\big)\big(\partial_j \bar{\beta}(\mathbf{z})\big) \\
&\qquad - \big(\partial_{0i} \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z})\big)\big(\partial_j \bar{\beta}(\mathbf{z})\big) \\
&\qquad - \big(\partial_{j0} \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z})\big)\big(\partial_i \bar{\beta}(\mathbf{z})\big) \\
&\qquad + \partial_{ij} \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}), \qquad i, j = 1, 2, \ldots, d.
\end{aligned} \tag{5.14}$$

Set

$$\begin{aligned}
a_{ij} &= \big(\partial_0 \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z})\big)\big(\partial_{ij} \bar{\beta}(\mathbf{z})\big); && i, j = 1, \ldots, d, \\
b_{ij} &= \big(\partial_{ij} \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z})\big); && i, j = 0, 1, \ldots, d, \\
x_i &= \partial_i \bar{\beta}(\mathbf{z}); && i = 1, 2, \ldots, d, \\
c_{ij} &= \begin{cases} -x_j, & i = 0 \text{ and } j = 1, 2, \ldots, d \\ \delta_{ij}, & i, j = 1, 2, \ldots, d, \end{cases} \\
A &= (a_{ij}), && i, j = 1, 2, \ldots, d, \\
B &= (b_{ij}), && i, j = 0, 1, \ldots, d, \\
C &= (c_{ij}), && i = 0, 1, \ldots, d, \\
& && j = 1, 2, \ldots, d.
\end{aligned}$$

Here δ_{ij} is the Kronecker delta. Note that $A = A^T$, $B = B^T$. Now (5.14) may be written in matrix form as

$$A = C^T B C. \tag{5.15}$$

By property (c), Section 4, B is strictly positive definite; consequently by (5.15) A is also strictly positive definite. But

$$\partial_0 \beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) = \int f \, d\mu > 0, \tag{5.16}$$

where μ is the Gibbs measure for $\langle \mathbf{z} | \mathbf{g} \rangle - \bar{\beta}(\mathbf{z})\mathbf{f}$, so it follows that $\nabla^2 \bar{\beta}(\mathbf{z})$ is strictly positive definite.

Let $x_0 = 1$ and $\mathbf{x}^* = (x_0, x_1, \ldots, x_d)$; let $\mathbf{v} = \mathbf{B}^{-1}(\mathbf{x}^*)^T$. It is easily verified that

$$C^T B \mathbf{v} = 0.$$

Consequently,

$$\left[\frac{\mathbf{v}^T}{C^T}\right] B [\mathbf{v} | C] = \left[\begin{array}{c|c} \mathbf{x}^* B^{-1} \mathbf{x} & 0 \\ \hline 0 & A \end{array}\right].$$

The determinant of $[\mathbf{v}|C]$ is easy to evaluate by row–column operations: thus one finds

$$\det[\mathbf{v}|C] = \mathbf{x}^* B^{-1} (\mathbf{x}^*)^T,$$

$$\left(\mathbf{x}^* B^{-1} (\mathbf{x}^*)^T\right) \det B = \det A. \tag{5.17}$$

By property (h), Section 4, $B^{-1} = \nabla^2 \gamma(\mathbf{x}^*/t_{\mathbf{x}})$, therefore (5.13) follows from (5.17) and (5.16). □

The Legendre transform $\bar{\gamma}$ of $\bar{\beta}$ is convex on $\mathscr{R}^d$. For $\mathbf{x} \in \bar{\mathscr{B}} = \nabla \bar{\beta}(\mathscr{R}^d)$, the supremum in (5.9) is attained uniquely at the value of $\mathbf{z}$ for which $\nabla \bar{\beta}(\mathbf{z}) = \mathbf{x}$. The inverse function theorem implies that $\bar{\gamma}$ is smooth on $\bar{\mathscr{B}}$, and

($\bar{\text{f}}$) $\nabla \bar{\gamma} \circ \nabla \bar{\beta} =$ *identity on* $\mathscr{R}^d$;

($\bar{\text{g}}$) $\nabla \bar{\beta} \circ \nabla \bar{\gamma} =$ *identity on* $\bar{\mathscr{B}}$;

($\bar{\text{h}}$) $\nabla \bar{\beta}(\mathbf{z}) = \mathbf{z} \Leftrightarrow \nabla \bar{\gamma}(\mathbf{x}) = \mathbf{z} \Rightarrow \nabla^2 \bar{\gamma}(\mathbf{x}) = (\nabla^2 \bar{\beta}(\mathbf{z}))^{-1}$;

($\bar{\text{i}}$) $\nabla \bar{\beta}(\mathbf{z}) = \mathbf{x} \Rightarrow \bar{\gamma}(\mathbf{x}) + \bar{\beta}(\mathbf{z}) = \langle \mathbf{x} | \mathbf{z} \rangle$;

($\bar{\text{j}}$) $\nabla \bar{\beta}(\mathbf{z}) \neq \mathbf{x} \Rightarrow \bar{\gamma}(\mathbf{x}) + \bar{\beta}(\mathbf{z}) > \langle \mathbf{x} | \mathbf{z} \rangle$;

($\bar{\text{k}}$) $\nabla \bar{\beta}(\mathbf{z}) = \mathbf{x} \Rightarrow \bar{\gamma}(\mathbf{x}) = -H(\bar{\mu}_{\langle \mathbf{z} | \mathbf{G} \rangle}, \sigma^f) < 0.$

The proofs of $(\bar{\text{f}})$–$(\bar{\text{j}})$ are routine. Property $(\bar{\text{k}})$ follows from $(\bar{\text{e}})$, $(\bar{\text{i}})$, and Theorem F.

LEMMA 2. *For* $\mathbf{x} \in \bar{\mathscr{B}}$, $\mathbf{x} = \nabla\bar{\beta}(\mathbf{z})$, *let* $\mathbf{x}^* = (1, \mathbf{x})$ *and* $t_{\mathbf{x}}^{-1} = \int f\,d\mu$ *where* μ *is the Gibbs measure for* $\langle \mathbf{z}|\mathbf{g}\rangle - \bar{\beta}(\mathbf{z})\mathbf{f}$. *Then*

$$\begin{aligned} -\bar{\gamma}(\mathbf{x}) &= -t_{\mathbf{x}}\gamma(\mathbf{x}^*/t_{\mathbf{x}}) \\ &= \sup_{t>0}\left(-t\gamma(\mathbf{x}^*/t)\right); \end{aligned} \tag{5.18}$$

$$\det\nabla^2\bar{\gamma}(\mathbf{x}) = t_{\mathbf{x}}^{-d}\left(\det\nabla^2\gamma(\mathbf{x}^*/t_{\mathbf{x}})\right)\langle \mathbf{x}^*|\nabla^2\gamma(\mathbf{x}^*/t_{\mathbf{x}})|\mathbf{x}^*\rangle. \tag{5.19}$$

Proof. Equation (5.19) follows directly from Lemma 1, property (h), Section 4, and property $(\bar{\text{h}})$. The equation $\bar{\gamma}(\mathbf{x}) = t_{\mathbf{x}}\gamma(\mathbf{x}^*/t_{\mathbf{x}})$ follows from properties (k), $(\bar{\text{k}})$, and (5.4). Recall from Section 4 that $-t\gamma(\mathbf{x}^*/t)$ is a concave function of $t > 0$; since $\beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) = 0$ it follows from (4.4) that $-t\gamma(\mathbf{x}^*/t)$ attains its maximum at $t = t_{\mathbf{x}}$. □

Proof of Theorem F. The entropy map $\bar{\mu} \to H(\bar{\mu}, \sigma^f)$ is upper semicontinuous on $\mathscr{I}_f$ (this follows from (5.4) and the fact that $\mu \to H(\mu)$ is upper semicontinuous on $\mathscr{I}$, cf. [14, Theorem 3.10]). Hence, for each real-valued $G \in \mathscr{F}_\rho(\Sigma_A^f)$ there exists $\bar{\mu}_* \in \mathscr{I}_f$ which maximizes $H(\bar{\mu}, \sigma^f) + \int G\,d\bar{\mu}$ (however, this argument does not imply that $\bar{\mu}_*$ is unique).

Let $c = H(\bar{\mu}_*, \sigma^f) + \int G\,d\bar{\mu}_*$, and let $\varphi = g - cf$, where g and G are related by (5.1). By Theorem D,

$$\begin{aligned} P(g - cf) &= H(\mu_\varphi) + \int \varphi\,d\mu_\varphi \\ &= H(\mu_\varphi) + \int g\,d\mu_\varphi - c\int f\,d\mu_\varphi, \end{aligned} \tag{5.20}$$

$$P(g - cf) > H(\mu) + \int g\,d\mu - c\int f\,d\mu, \qquad \forall \mu \neq \mu_\varphi, \qquad \mu \in \mathscr{I}. \tag{5.21}$$

Consequently,

$$P(g - cf) > 0 \Rightarrow \frac{H(\mu_\varphi)}{\int f\,d\mu_\varphi} + \frac{\int g\,d\mu_\varphi}{\int f\,d\mu_\varphi} > c;$$

but this contradicts the fact that $\bar{\mu}_*$ maximizes $H(\bar{\mu}, \sigma^f) + \int G\,d\bar{\mu}$ (cf. (5.3) and (5.4), substituting μ_φ for μ). On the other hand,

$$\begin{aligned} &P(g - cf) < 0 \\ &\qquad \Rightarrow H(\mu_\varphi) + \int \varphi \, d\mu_\varphi < 0 \\ &\qquad \Rightarrow H(\mu_\varphi) + \int \varphi \, d\mu_\varphi < \left(H(\bar\mu_*, \sigma^f) + \int G \, d\bar\mu_* - c \right) \int f \, d\mu_* \\ &\qquad \Rightarrow H(\mu_\varphi) + \int \varphi \, d\mu_\varphi < H(\mu_*) + \int \varphi \, d\mu_*, \end{aligned}$$

where $\mu_* \in \mathscr{I}$ is the measure corresponding to $\bar\mu_*$ as in (5.3). But this inequality contradicts Theorem D.

This proves that $P(g - cf) = 0$, hence that $c = P_f(G)$. The inequality (5.5) follows immediately. To complete the proof of the Theorem, it suffices to show that $\bar\mu_* = \bar\mu_G$, where $\bar\mu_G$ is defined by (5.6). Since $P(g - cf) = 0$,

$$H(\mu_\varphi) + \int \varphi \, d\mu_\varphi = H(\mu_*) + \int \varphi \, d\varphi_*$$

where $\mu_* \leftrightarrow \bar\mu_*$ as above. But this implies that $\mu_\varphi = \mu_*$, by Theorem D. Thus $\bar\mu_* = \bar\mu_G$. □

6. Periodic Orbits of Symbolic Flows

Let $f \in \mathscr{F}_\rho$ be strictly positive, and let (Σ_A^f, σ^f) be the symbolic flow under f. For $G, G_i \in B(\Sigma_A^f)$, let $g, g_i \in (\Sigma_A)$ be the functions defined by (5.1). Periodic orbits of (Σ_A^f, σ^f) will be denoted by τ; $\tau(G)$ denotes the integral of G (with respect to time) over one period of τ. Thus $\tau(1)$ is the period of τ, and $\tau(G)/\tau(1)$ is the mean value of G over τ.

Define measures $Q^f, Q^{f;G_1,\dots,G_d}$ on $\mathscr{R}^+$ and $\mathscr{R}^+ \times \mathscr{R}^d$, respectively, by

$$\begin{aligned} Q^f(I) &= \#\{\tau\colon \tau(1) \in I\}, \\ Q^{f;G_1,\dots,G_d}\left(\textstyle\prod_{i=0}^d I_i\right) &= \#\{\tau\colon \tau(1) \in I_0, \tau(G_i) \in I_i, i = 1, \dots, d\}. \end{aligned} \tag{6.1}$$

The main results of this paper concern the asymptotic behavior of these measures when the flow is weakly mixing. It is well known (cf. [11, Section 5]) that (Σ_A^f, σ^f) is weakly mixing iff f is not homologous to any function valued in a proper closed subgroup of $\mathscr{R}$.

Hypothesis B. *If $a_0 f + \sum_{i=1}^d a_i g_i$ is homologous to an integer-valued function for constants $a_0, a_1, \dots, a_d \in \mathscr{R}$, then $a_0 = a_1 = \cdots = a_d = 0$.*

Hypothesis B may be reformulated so that the role of the shift is submerged. For $G \in B(\Sigma_A^f)$, let $S_t G = \int_0^t G \circ \sigma_s^f \, ds$, and define the *skew product* flow T_t^G on $\mathscr{K} \times \Sigma_A^f$, where $\mathscr{K} = \{e^{i\theta}: -\pi \leq \theta < \pi\}$, by

$$T_t^G(e^{i\theta}, (\xi, s)) = \left(\exp\{i\theta + 2\pi i S_t G(\xi, s)\}, \sigma_t^f(\xi, s)\right).$$

HYPOTHESIS $\overline{\mathrm{B}}$. *If $a_0, a_1, \dots a_d \in \mathscr{R}$ are constants such that the skew product flow T_t^G, $G = a_0 + \sum_{i=1}^d a_i G_i$, is not topologically ergodic, then $a_0 = a_1 = \cdots = a_d = 0$.*

It is not difficult to show that Hypothesis $\overline{\mathrm{B}}$ implies Hypothesis B, and both imply Hypothesis A.

Let $\bar{\beta} = \beta_{\mathbf{G}}$ and $\bar{\gamma} = \gamma_{\mathbf{G}}$ be the thermodynamic functions associated with $\mathbf{G} = (G_1, G_2, \dots, G_d)$ when $G_i \in \mathscr{F}_p(\Sigma_A^f)$, and let $\overline{\mathscr{B}} = \nabla\bar{\beta}(\mathscr{R}^d)$.

THEOREM 1. *Assume that $G_i \in \mathscr{F}_p(\Sigma_A^f)$, $i = 1, \dots, d$, and that Hypothesis $\overline{\mathrm{B}}$ is satisfied. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \overline{\mathscr{B}}$, $\delta_i > 0$, as $a \to \infty$,*

$$Q\left([a, a + \delta_0] \times \prod_{i=1}^d [ax_i, ax_i + \delta_i]\right)$$
$$\sim e^{-a\bar{\gamma}(\mathbf{x})} a^{-(d+2)/2} (2\pi)^{-d/2} \det\left(\nabla^2 \bar{\gamma}(\mathbf{x})\right)^{1/2} C(\mathbf{x}; \delta), \quad (6.2)$$

where $Q = Q^{f; G_1, \dots, G_d}$ and

$$C(\mathbf{x}; \delta) = \left\{\int_0^{\delta_0} e^{\bar{\beta}(\nabla\bar{\gamma}(\mathbf{x}))t} \, dt\right\}\left\{\int_0^{\delta_d}\int_0^{\delta_{d-1}} \cdots \int_0^{\delta_1} e^{\langle \nabla\bar{\gamma}(\mathbf{x}) | t\rangle} \, dt_1 \dots dt_d\right\} \quad (6.3)$$

moreover, this approximation holds uniformly on any compact subset of $\overline{\mathscr{B}}$.

Recall that $-\bar{\gamma}(\mathbf{x})$ is the maximum entropy achieved by a σ^f-invariant measure $\bar{\mu}$ such that $\int G_i \, d\bar{\mu} = x_i$, $i = 1, 2, \dots, d$. Also, $\nabla^2\bar{\gamma}(\mathbf{x})$ is strictly positive definite, by ($\bar{\mathrm{c}}$) and ($\bar{\mathrm{h}}$), Section 5.

THEOREM 2. *Assume that (Σ_A^f, σ^f) is weakly mixing. Then as $a \to \infty$,*

$$Q^f((0, a]) \sim \frac{e^{aH^*(\sigma^f)}}{aH^*(\sigma^f)}, \quad (6.4)$$

where $H^*(\sigma^f)$ is the topological entropy of the flow (Σ_A^f, σ^f).

Theorem 2 is just the special case $d = 0$ if Theorem 1, because in this case the thermodynamic functions $\bar{\gamma}, \bar{\beta}$ degenerate to $-\bar{\gamma} = H^*(\sigma^f) = \bar{\beta}$. A similar result may be proved for flows which are *not* weakly mixing: the exponential rate is still $H^*(\sigma^f)$ (cf. also [12, Theorem 2]).

Let $G_1, G_2, \ldots, G_d, G \in \mathscr{F}_\rho(\Sigma_A^f)$ satisfy Hypothesis B, and let $\bar{\gamma}_G$, $\gamma_{(G,\mathbf{G})}$, etc., be the associated thermodynamic functions, where $\mathbf{G} = (G_1, G_2, \ldots, G_d)$. For $\mathbf{x} = (x_1, \ldots, x_d) \in \bar{\mathscr{B}} = \nabla\bar{\beta}_{\mathbf{G}}(\mathscr{R}^d)$, let $\bar{\mu}_{\mathbf{x}}$ be the σ^f-invariant measure which maximizes entropy subject to the constraints $\int G_i \, d\mu_{\mathbf{x}} = x_i$, $i = 1, 2, \ldots, d$. Define

$$\sigma_{\mathbf{x}}^2 = \frac{\det \nabla^2 \gamma_{(G,\mathbf{G})}(\mathbf{x}, \int G \, d\bar{\mu}_{\mathbf{x}})}{\det \nabla^2 \bar{\gamma}_{\mathbf{G}}(\mathbf{x})}$$

$$= \left(\frac{d^2}{dy^2} \gamma_{(G,\mathbf{G})}(\mathbf{x}, y) \right)_{y = \int G \, d\bar{\mu}_{\mathbf{x}}}. \tag{6.5}$$

The second equation is an easy consequence of the fact that $-\bar{\gamma}_{(G,\mathbf{G})}(\mathbf{x}, y)$, considered as a function of y has its maximum at $y = \int G \, d\bar{\mu}_{\mathbf{x}}$, for all $\mathbf{x}$.

THEOREM 3. *Assume that $G, G_1, \ldots, G_d \in \mathscr{F}_\rho(\Sigma_A^f)$ satisfy Hypothesis* B. *Then for any* $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathscr{B}$, $\delta_i < 0$, *and nonempty intervals* $J \subset \mathscr{R}$,

$$\frac{\tilde{Q}\left([a, a+\delta_0] \times \Pi_{i=1}^d [ax_i, ax_i + \delta_i] \times \left(a\int G \, d\bar{\mu}_{\mathbf{x}} + a^{1/2}\sigma_{\mathbf{x}}^{-1} J\right)\right)}{Q\left([a, a+\delta_0] \times \Pi_{i=1}^d [ax_i, ax_i + \delta_i]\right)}$$

$$\to \int_J e^{-t^2/2} \, dt / (2\pi)^{1/2} \tag{6.6}$$

as $a \to \infty$, where $Q = Q^{f; G_1, \ldots, G_d}$ and $\tilde{Q} = Q^{f; G_1, \ldots, G_d, G}$.

In other words, if you choose a periodic orbit τ^* at random from among all periodic orbits τ satisfying $a \le \tau(1) \le a + \delta_0$ and $ax_i \le \tau(G_i) \le ax_i + \delta_i$, then $(\tau^*(G) - a\int G \, d\bar{\mu}_{\mathbf{x}})/a^{1/2}\sigma_{\mathbf{x}}$ will be (approximately) distributed according to the standard normal distribution. Theorem 3 is a direct consequence of Theorem 1: the details of the argument are entirely routine and are therefore omitted. It is clear that one can also obtain a multidimensional central limit theorem.

THEOREM 4. *Assume that $G_1, G_2, \ldots, G_d \in \mathscr{F}_\rho(\Sigma_A)$ satisfy Hypothesis* B, *and let $G \in C(\Sigma_A)$. Then for any* $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathscr{B}$, $\delta_i > 0$, $\varepsilon > 0$,

$$\frac{\tilde{Q}\left(([a, a+\delta_0] \times \Pi_{i=1}^d [ax_i, ax_i + \delta_i] \times a[\int G \, d\bar{\mu}_{\mathbf{x}} - \varepsilon, \int G \, d\bar{\mu}_{\mathbf{x}} + \varepsilon]\right)}{Q\left(([a, a+\delta_0] \times \Pi_{i=1}^d [ax_i, ax_i + \delta_i]\right)} \to 1 \tag{6.7}$$

as $a \to \infty$, where $Q = Q^{f; G_1, \ldots, G_d}$ and $\tilde{Q} = Q^{f; G_1, \ldots, G_d, G}$.

The upstart of this is that if you choose τ^* at random from the set of periodic orbits τ satisfying $a \leq \tau(1) \leq a + \delta_0$, $ax_i \leq \tau(G_i) \leq ax_i + \delta_i$, then τ^* is with high probability close to being distributed according to $\bar{\mu}_{\mathbf{x}}$. In the special case $d = 0$, this says that "almost every" periodic orbit is approximately distributed as the maximum entropy measure $\bar{\mu}_{\max}$.

Note that if $G \in \mathscr{F}_\rho(\Sigma_A^f)$ and $G_1, G_2, \ldots, G_d, G$ satisfy Hypothesis B then (6.7) follows immediatly from Theorem 3. Theorem 4 now follows from the observation that $G \in C(\Sigma_A^f)$ may be uniformly approximated by $G^{(n)} \in \mathscr{F}_\rho(\Sigma_A^f)$ for which $G_1, G_2, \ldots, G_d, G^{(n)}$ satisfies Hypothesis B, $n = 1, 2, \ldots$. The proof of this observation is given in Appendix 2.

The main result, Theorem 1, will be proved in Sections 7 and 8.

7. Counting Cycles of the Shift

For real-valued $f_0, f_1, \ldots, f_d \in \mathscr{F}_\rho$ and $n \geq 1$, define a measure $N_n = N_n^{\mathbf{f}}$ on $\mathscr{R}^{d+1}$ by

$$N_n\left(\prod_{i=0}^{d} I_i\right) = \#\{\xi \in \Sigma_A : \sigma^n\xi = \xi;\ S_n f_i(\xi) \in I_i,\ i = 0, 1, \ldots, d\}, \quad (7.1)$$

where I_i is a subinterval of $\mathscr{R}$, $i = 0, 1, \ldots, d$. The measure N_n plays a role for the shift similar to that played by $Q^{f;G_1,\ldots,G_d}$ for the flow (Σ_A^f, σ^f). However, there are several differences: (i) N_n counts all n-cycles ξ (i.e., $\sigma^n\xi = \xi$), not just those whose minimal period is n; (ii) N_n counts $\sigma^i\xi$ separately from ξ for $i = 1, 2, \ldots, n-1$, hence each *orbit* is actually counted p times, where p is its minimal period.

Observe that if $f_i \cong f_i^*$, $i = 0, 1, \ldots, d$, then $N_n^f = N_n^{f^*}$. It follows from Lemma A that it suffices to consider $f_0, f_1, \ldots, f_d \in \mathscr{F}_\rho^+$. In this case

$$N_n\left(\prod_{i=0}^{d} I_i\right) = \#\{\xi \in \Sigma_A^+ : \sigma^n\xi = \xi;\ S_n f_i \in I_i,\ i = 0, 1, \ldots, d\}. \quad (7.2)$$

PROPOSITION 1. *For each* $\mathbf{z} \in \mathscr{R}^{d+1}$ *there is a constant* $K_{\mathbf{z}} < \infty$ *having the following property. If* $\langle \mathbf{z} | \mathbf{y} \rangle \geq \langle \mathbf{z} | \mathbf{x} \rangle$ *for all* $\mathbf{y} \in U \subset \mathscr{R}^{d+1}$, *then*

$$N_n(U) \leq K_{\mathbf{z}} \exp\{n\beta(\mathbf{z}) - \langle \mathbf{z} | \mathbf{x} \rangle\} \quad (7.3)$$

for all $n \geq 1$. *Moreover*, $K_{\mathbf{z}}$ *may be chosen so as to depend continuously on* $\mathbf{z}$.

Here $\beta(\mathbf{z}) = \beta_{\mathbf{f}}(\mathbf{z})$ is the pressure function associated with $\mathbf{f}$.

If the function $\mathbf{f}$ is valued in a closed subgroup Γ of R^{d+1}, then the support of N_n is contained in Γ. I shall distinguish between two cases.

HYPOTHESIS C. *There do not exist constants* $b_0, b_1, \ldots, b_d \in \mathscr{R}$ *such that* $(f_0 - b_0, f_1 - b_1, \ldots, f_d - b_d)$ *is homologous to* $(g_0, g_1, \ldots, g_d)$ *valued in a proper closed subgroup of* $\mathscr{R}^{d+1}$.

HYPOTHESIS D. *The function* $\mathbf{f} = (f_0, f_1, \ldots, f_d)$ *takes values in* $\Gamma_r = \mathscr{Z}^{r+1} \oplus \mathscr{R}^{d-r}$; *there do not exist constants* $b_0, b_1, \ldots, b_d \in \mathscr{R}$ *such that* $(f_0 - b_0, \ldots, f_d - b_d)$ *is homologous to* $(g_0, g_1, \ldots, g_d)$ *valued in a proper closed subgroup of* Γ_r.

PROPOSITION 2. *If* $f_0, f_1, \ldots, f_d \in \mathscr{F}_\rho^+$ *satisfy Hypothesis* C, *if* $\mathbf{x} = \nabla\beta(\mathbf{z})$, *and if* u: $\mathscr{R}^{d+1} \to \mathscr{R}$ *is* C^∞, *nonnegative, and has compact support, then*

$$\int u(\mathbf{y} - n\mathbf{x}) N_n(d\mathbf{y}) \sim e^{-n\gamma(\mathbf{x})}(2\pi n)^{-(d+1)/2}(\det \nabla^2\gamma(\mathbf{x}))^{1/2} \times \int u(\mathbf{y}) e^{-\langle \mathbf{z}|\mathbf{y}\rangle} d\mathbf{y} \tag{7.4}$$

as $n \to \infty$. *Furthermore*, (7.4) *holds uniformly for* $\mathbf{x}$ *in any compact subset of* $\mathscr{B} = \nabla\beta(\mathscr{R}^d)$.

For $x \in \mathscr{R}$, let $\{\{x\}\}$ denote the fractional part of x, i.e., $\{\{x\}\} = x - [[x]]$, where $[[x]]$ is the greatest integer in x.

PROPOSITION 3. *Suppose* $f_0, f_1, \ldots, f_d \in \mathscr{F}_\rho^+$ *satisfy Hypothesis* D. *Let* $u_0, u_1, \ldots, u_d$ *be nonnegative*, C^∞ *functions on* $\mathscr{R}$, *each with support contained in* $(0, 1)$, *and let* $u(\mathbf{y}) = \prod_{i=0}^d u_i(y_i)$. *If* $\mathbf{x} = \nabla\beta(\mathbf{z})$, *then*

$$\int u(\mathbf{y} - n\mathbf{x}) N_n(d\mathbf{y}) \sim e^{-n\gamma(\mathbf{x})}(2\pi n)^{-(d+1)/2}(\det \nabla^2\gamma(\mathbf{x}))^{1/2} C_n(\mathbf{x}; u) \tag{7.5}$$

as $n \to \infty$, *where*

$$C_n(\mathbf{x}; u) = \prod_{i=0}^{r} [u_i(1 - \{\{nx_i\}\})\exp\{-z_i(1 - \{\{nx_i\}\})\}] \times \prod_{i=r+1}^{d} \int_0^1 u_i(t) e^{-z_i t}\, dt. \tag{7.6}$$

Furthermore, (7.5) *holds uniformly for* $\mathbf{x}$ *in any compact subset of* $\mathscr{B} = \nabla\beta(\mathscr{R}^{d+1})$.

The rest of this section is devoted to the proofs.

Let $\Sigma_A^{(k)}$ denote the set of *finite* sequences of length k with transitions allowed by A, i.e.,

$$\Sigma_A^{(k)} = \left\{ \alpha \in \prod_{i=0}^{k} \{1,2,\ldots,l\} : A(\alpha_i, \alpha_{i+1}) = 1, \forall i = 0,1,\ldots,k \right\}$$

$$= \{ \alpha^{k,1}, \alpha^{k,2}, \ldots, \alpha^{k,m(k)} \}.$$

For each sequence $\alpha^{k,i} \in \Sigma_A^{(k)}$ there exists a periodic sequence $\xi^{k,i} \in \Sigma_A^+$ which agrees with $\alpha^{k,i}$ in the first $k+1$ coordinates, i.e., $\xi_n^{k,i} = \alpha_n^{k,i}$, $\forall n = 0,1,\ldots,k$. Define $\psi_{k,i} \in C(\Sigma_A^+)$ and positive measures $M_{n,k,i}(d\mathbf{x})$, $M_{n,k}(d\mathbf{x})$ by

$$\psi_{k,i}(\xi) = \begin{cases} 1 & \text{if } \xi_j = \xi_j^{k,i} = \alpha_j^{k,i}, \forall j = 0,1,\ldots,k \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{n,k,i}(d\mathbf{x}) = \#\{ \xi \in \Sigma_A^+ : \sigma^n \xi = \xi^{k,i};\ \psi_{k,i}(\xi) = 1;\ S_n\mathbf{f}(\xi) \in d\mathbf{x} \},$$

$$M_{n,k}(d\mathbf{x}) = \sum_{i=1}^{m(k)} M_{n,k,i}(d\mathbf{x}).$$

For any finite Borel measure $F(d\mathbf{x})$ on $\mathscr{R}^{d+1}$ let $\hat{F}(\mathbf{z})$ be the Laplace–Fourier transform

$$\hat{F}(\mathbf{z}) = \int_{\mathscr{R}^{d+1}} e^{\langle \mathbf{z} | \mathbf{x} \rangle} F(d\mathbf{x}):$$

this is an entire analytic function of $\mathbf{z} \in C^{d+1}$. For any open rectangle $U = \prod_{i=0}^{d}(x_i, y_i)$, where $-\infty \le x_i < y_i \le \infty$, let $U_\varepsilon = \prod_{i=0}^{d}(x_i - \varepsilon, y_i + \varepsilon)$.

LEMMA 3. *There exists a decreasing sequence of constants $\varepsilon(k)$ such that $\lim_{k \to \infty} \varepsilon(k) = 0$ and such that for all n sufficiently large, $\mathbf{z} \in \mathscr{R}^{d+1}$, and all open rectangles U in $\mathscr{R}^{d+1}$*

$$M_{n,k}(U) \le N_n(U_{\varepsilon(k)}) \le M_{n,k}(U_{2\varepsilon(k)}),$$

$$\hat{M}_{n,k}(\mathbf{z}) e^{-|\mathbf{z}|\varepsilon(k)} \le \hat{N}_n(\mathbf{z}) \le \hat{M}_{n,k}(\mathbf{z}) e^{|\mathbf{z}|\varepsilon(k)}.$$

Proof. Since f_j is Hölder continuous for each $j = 0,1,\ldots,d$, the constants

$$\varepsilon_j(k) = \sum_{n=k}^{\infty} \mathrm{var}_n(f_j) \downarrow 0$$

as $k \to \infty$. Let $\varepsilon(k) = \sum_{j=0}^{d} \varepsilon_j(k)$; then

$$\sup\{|S_n\mathbf{f}(\xi) - S_n\mathbf{f}(\zeta)|: n \geq 0, \xi_i = \zeta_i, \forall 0 \leq i \leq n+k\} \leq \varepsilon(k).$$

Now there is a 1-to-1 correspondence between the sets

$$\bigcup_{i=1}^{m(k)} \{\xi \in \Sigma_A^+ : \sigma^n \xi = \xi^{k,i} \text{ and } \psi_{k,i}(\xi) = 1\}$$

and

$$\{\zeta \in \Sigma_A^+ : \sigma^n \zeta = \zeta\}$$

having the property that any two corresponding sequences ξ, ζ agree in the first $n + k + 1$ coordinates (i.e., $\xi_i = \zeta_i$, $\forall 0 \leq i \leq n + k$). The lemma follows easily. □

Lemma 3 reduces the study of the asymptotic behavior of the measures N_n for large n to the study of $M_{n,k}$ for fixed but large k. The study of $M_{n,k} = \sum_i M_{n,k,i}$ proceeds by way of the Laplace–Fourier transform. The key is that $\hat{M}_{n,k,i}$ can be written in terms of Ruelle's *PF* operators,

$$\begin{aligned}\hat{M}_{n,k,i}(\mathbf{z}) &= \sum_{\zeta:\, \sigma^n\zeta = \xi^{k,i}} \exp\{\langle \mathbf{z} | S_n\mathbf{f}(\zeta)\rangle\}\psi_{k,i}(\zeta) \\ &= \left(\mathscr{L}^n_{\langle \mathbf{z}|\mathbf{f}\rangle}\psi_{k,i}\right)(\xi^{k,i}) \qquad (7.7)\end{aligned}$$

for all $\mathbf{z} \in \mathscr{C}^{d+1}$. By Ruelle's *PF* Theorem (cf. Theorems A, B of Sect. 2), $\mathscr{L}^n_{\langle \mathbf{z}|\mathbf{f}\rangle}\varphi/\lambda^n_{\langle \mathbf{z}|\mathbf{f}\rangle}$ converges in $\|\,\|_\infty$ to $(\int\varphi\, d\nu_{\langle \mathbf{z}|\mathbf{f}\rangle})h_{\langle \mathbf{z}|\mathbf{f}\rangle}$, $\forall \mathbf{z} \in \mathscr{R}^{d+1}$, and by Appendix 1 this convergence also holds for $\mathbf{z}$ in an open neighborhood of $\mathscr{R}^{d+1}$ in $\mathscr{C}^{d+1}$. Hence,

$$\begin{aligned}&\hat{M}_{n,k}(\mathbf{z}) \sim e^{n\beta(\mathbf{z})}C_k(\mathbf{z}), \\ &C_k(\mathbf{z}) \triangleq \sum_{i=1}^{m(k)} \left(\int \psi_{k,i}\, d\nu_{\langle \mathbf{z}|\mathbf{f}\rangle}\right) h_{\langle \mathbf{z}|\mathbf{f}\rangle}(\xi^{k,i})\end{aligned} \qquad (7.8)$$

for all $\mathbf{z}$ in an open neighborhood of $\mathscr{R}^{d+1}$.

LEMMA 4. *For any real-valued* $\varphi \in \mathscr{F}_\rho^+$

$$\lim_{k\to\infty} \sum_{i=1}^{m(k)} \left(\int \psi_{k,i}\, d\nu_\varphi\right) h_\varphi(\xi^{k,i}) = 1.$$

Proof. If $\psi_{k,i}(\xi) = 1$ then ξ and $\xi^{k,i}$ agree in the first $(k+1)$ coordinates. Since h_φ is uniformly continuous

$$\lim_{k\to\infty} \sum_{i=1}^{m(k)} \left| h_\varphi(\xi) - h_\varphi(\xi^{k,i}) \right| \psi_{k,i}(\xi) = 0$$

uniformly for $\xi \in \Sigma_A^+$. Since

$$\int \left(\sum_{i=1}^{m(k)} \psi_{k,i} \right) h_\varphi \, d\nu_\varphi = \int h_\varphi \, d\nu_\varphi = 1,$$

the lemma follows. □

Formula (7.8) and Lemmas 3 and 4 imply that for each $\mathbf{z} \in \mathscr{R}^{d+1}$

$$\hat{N}_n(\mathbf{z}) \sim e^{n\beta(\mathbf{z})} \tag{7.9}$$

as $n \to \infty$, uniformly for $\mathbf{z}$ in any compact set. Unfortunately, it is not so easy to get a handle on the behavior of $\hat{N}_n(\mathbf{z})$ for $\mathbf{z} \in \mathscr{C}^{d+1} \setminus \mathscr{R}^{d+1}$; it is easier to study the measures $M_{n,k}$ directly, since (7.8) holds for $\mathbf{z}$ in a neighborhood of $\mathscr{R}^{d+1}$. However, it is possible to use (7.9) to give a direct proof of Proposition 1.

Proof of Proposition 1. If $\langle \mathbf{z} | \mathbf{y} \rangle \geq \langle \mathbf{z} | \mathbf{x} \rangle$ for all $\mathbf{y} \in U$ then

$$\begin{aligned} N_n^{\mathbf{f}}(U) &\leq \int_{\mathscr{R}^{d+1}} e^{\langle \mathbf{z}|\mathbf{y}\rangle} N_n^{\mathbf{f}}(d\mathbf{y}) e^{-\langle \mathbf{z}|\mathbf{x}\rangle} \\ &= \hat{N}_n^{\mathbf{f}}(\mathbf{z}) e^{-\langle \mathbf{z}|\mathbf{x}\rangle}. \end{aligned}$$

By (7.9) there is a constant $K_{\mathbf{z}} < \infty$ such that $\hat{N}_n^{\mathbf{f}}(\mathbf{z}) \leq \mathrm{K}_{\mathbf{z}} e^{n\beta(\mathbf{z})}$ for all $n \geq 1$. Because of the uniformity in (7.9), $K_{\mathbf{z}}$ can be chosen continuously. □

Let $\mathscr{S} = \mathscr{S}^{d+1}$ denote the set of Schwartz class (rapidly decreasing) functions on $\mathscr{R}^{d+1}$, i.e., $u \in \mathscr{S}$ iff $u \in C^\infty(\mathscr{R}^{d+1})$ and

$$\sup_{\mathbf{x} \in \mathscr{R}^{d+1}} |\mathbf{x}|^m |D^\alpha u(\mathbf{x})| < \infty$$

for all $m = 0, 1, \ldots,$ and for multi-indices $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$. It is well known that $\mathscr{S}$ is preserved by the Fourier transform, i.e., if $u \in \mathscr{S}$ then $\hat{u}(i\Theta), \Theta \in \mathscr{R}^{d+1}$, is also in $\mathscr{S}$. Recall the *Parseval identity*, which states that for $u \in \mathscr{S}$ and $F(d\mathbf{x})$ any finite Borel measure on $\mathscr{R}^{d+1}$,

$$\int_{\mathscr{R}^{d+1}} u(\mathbf{x}) F(d\mathbf{x}) = \int_{\mathscr{R}^{d+1}} \hat{u}(i\Theta)\hat{F}(-i\Theta)\, d\Theta/(2\pi)^{d+1},$$

where $d\Theta$ denotes Lebesgue measure on $\mathscr{R}^{d+1}$.

Heuristic Proof of Proposition 2. In view of Lemmas 3 and 4, it suffices to show that for any $u \in \mathscr{S}$, $u \geq 0$, with compact support, $k = 1, 2, \ldots,$ and $i = 1, 2, \ldots, m(k)$,

$$\int_{\mathscr{R}^{d+1}} u(\mathbf{y} - n\mathbf{x}) M_{n,k,i}(d\mathbf{y}) \sim e^{-n\gamma(\mathbf{x})}(2\pi n)^{-(d+1)/2}(\det \nabla^2\gamma(\mathbf{x}))^{1/2}\hat{u}(-\mathbf{z})C_{k,i}(\mathbf{z}) \tag{7.10}$$

as $n \to \infty$, where $\mathbf{x} = \nabla\beta(\mathbf{z})$ and $C_{k,i}(\mathbf{z}) = (\int\psi_{k,i}\, d\nu_{\langle \mathbf{z}|\mathbf{f}\rangle})h_{\langle \mathbf{z}|\mathbf{f}\rangle}(\xi^{k,i})$, and that (7.10) holds uniformly for $\mathbf{x}$ in any compact subset of $\mathscr{B}$.

Apply Parseval's identity to the function $u(\mathbf{y} - n\mathbf{x})\exp\{-\langle \mathbf{z}|\mathbf{y}\rangle\}$ and the measure $\exp\{\langle \mathbf{z}|\mathbf{y}\rangle\}M_{n,k,i}(dy)$ to obtain

$$\begin{aligned}(2\pi)^{d+1}\int_{\mathscr{R}^{d+1}} u(\mathbf{y} - n\mathbf{x}) M_{n,k,i}(d\mathbf{y}) &= \int_{\mathscr{R}^{d+1}} \hat{u}(i\Theta - \mathbf{z})\exp\{n\langle \mathbf{x}|i\Theta - \mathbf{z}\rangle\}\hat{M}_{n,k,i}(\mathbf{z} - i\Theta)\, d\Theta \\ &= \int_{\mathscr{R}^{d+1}} \hat{u}(i\Theta - \mathbf{z})\exp\{n\langle \mathbf{x}|i\Theta - \mathbf{z}\rangle\}\mathscr{L}^n_{\langle \mathbf{z}-i\Theta|\mathbf{f}\rangle}\psi_{k,i}(\xi^{k,i})\, d\Theta \end{aligned} \tag{7.11}$$

(cf. (7.7)). Since $\mathbf{f}$ satisfies Hypothesis C, it follows from Theorem C and the Spectral Radius Theorem (cf. also Proposition 6, Appendix 1) that for each $\Theta \neq \mathbf{0}$ there is a $\delta > 0$ such that

$$(1+\delta)^n e^{-n\beta(\mathbf{z})}\|\mathscr{L}^n_{\langle \mathbf{z}-i\Theta|\mathbf{f}\rangle}\psi_{k,i}\|_\infty \to 0. \tag{7.12}$$

Moreover, by Proposition 6, this convergence is uniform for Θ in any compact set not containing $\mathbf{0}$. On the other hand, for Θ near $\mathbf{0}$ Proposition 5 (Appendix 1) implies that

$$\begin{aligned}\mathscr{L}^n_{\langle \mathbf{z}-i\Theta|\mathbf{f}\rangle}\psi_{k,i}(\xi^{k,i}) &\sim e^{n\beta(\mathbf{z}-i\Theta)}\left(\int\psi_{k,i}\, d\nu_{\langle \mathbf{z}-i\Theta|\mathbf{f}\rangle}\right)h_{\langle \mathbf{z}-i\Theta|\mathbf{f}\rangle}(\xi^{k,i}) \\ &= e^{n\beta(\mathbf{z}-i\Theta)}C_{k,i}(\mathbf{z} - i\Theta).\end{aligned} \tag{7.13}$$

Now (7.12) and (7.13) suggest that for large n the major contribution to the last integral in (7.11) comes from Θ near zero, i.e.,

$$(2\pi)^{d+1}\int_{\mathscr{R}^{d+1}} u(\mathbf{y}-n\mathbf{x})M_{n,k,i}(d\mathbf{y})$$
$$\sim \int_{|\Theta|\le\varepsilon} \exp\{n\langle\mathbf{x}|i\Theta-\mathbf{z}\rangle + n\beta(\mathbf{z}-i\Theta)\}$$
$$\times C_{k,i}(\mathbf{z}-i\Theta)\hat{u}(i\Theta-\mathbf{z})\,d\Theta. \tag{7.14}$$

Unfortunately, (7.14) is not easy to justify. The difficulty is that the convergence in (7.12) is uniform only for Θ in compact sets of $\mathscr{R}^{d+1}\setminus\{\Theta\}$. Circumventing this problem requires an "unsmoothing" argument.

Given (7.14), one may derive (7.10) by a routine use of Laplace's method of asymptotic expansion (cf. [6, Chap. II]). The Taylor series approximation

$$\langle\mathbf{x}|i\Theta-\mathbf{z}\rangle + \beta(\mathbf{z}-i\Theta)$$
$$= -\langle\mathbf{x}|\mathbf{z}\rangle + \beta(\mathbf{z}) - \langle\Theta|\nabla^2\beta(\mathbf{z})|\Theta\rangle/2 + o(|\Theta|^2)$$
$$= -\gamma(\mathbf{x}) - \langle\Theta|\nabla^2\gamma(\mathbf{x})^{-1}|\Theta\rangle/2 + o(|\Theta|^2) \tag{7.15}$$

holds uniformly for $|\Theta|\le\varepsilon$ (recall properties (e), (h), Sect. 4). Substituting this quadratic expression into the integral in (7.14) and evaluating the resulting asymptotic Gaussian integral yields (7.10). □

A rigorous proof of Proposition 2 will be given later in this section.

Heuristic Proof of Proposition 3. As in Proposition 2, it is enough to show that

$$\int_{\mathscr{R}^{d+1}} u(\mathbf{y}-n\mathbf{x})M_{n,k,i}(d\mathbf{y})$$
$$\sim e^{-n\gamma(\mathbf{x})}(2\pi n)^{-(d+1)/2}\left(\det\nabla^2\gamma(\mathbf{x})\right)^{1/2}C_{k,i}(\mathbf{z})C_n(\mathbf{x};u) \tag{7.16}$$

as $n\to\infty$, where $C_{k,i}(\mathbf{z}) = (\int\psi_{k,i}\,d\nu_{\langle\mathbf{z}|\mathbf{f}\rangle})h_{\langle\mathbf{z}|\mathbf{f}\rangle}(\xi^{k,i})$ and $C_n(\mathbf{x};u)$ is as in (7.6). As usual, $\mathbf{x} = \nabla\beta(\mathbf{z})$.

Using Parseval's identity as before, one obtains

$$(2\pi)^{d+1}\int_{\mathscr{R}^{d+1}} u(\mathbf{y}-n\mathbf{x})M_{n,k,i}(d\mathbf{y})$$
$$= \int_{\mathscr{R}^{d+1}} \hat{u}(i\Theta-\mathbf{z})\exp\{n\langle\mathbf{x}|i\Theta-\mathbf{z}\rangle\}\cdot\mathscr{L}^n_{\langle\mathbf{z}-i\Theta|\mathbf{f}\rangle}\psi_{k,i}(\xi^{k,i})\,d\Theta. \tag{7.17}$$

Since $\mathbf{f}$ satisfies Hypothesis D, the operator $\mathscr{L}_{\langle \mathbf{z}-i\Theta|\mathbf{f}\rangle}$ is 2π-periodic in $\theta_0, \theta_1, \ldots, \theta_r$. Moreover, since each $u_j \in C^\infty$ and has support in $(0,1)$, the function $u_j(y)\exp\{i\theta_j - z_j)y\}$ has a uniformly, absolutely convergent Fourier series for each $j = 0, 1, \ldots, r$. Hence, it follows from (7.17) that

$$\begin{aligned}(2\pi)^{d+1} &\int_{\mathscr{R}^{d+1}} u(\mathbf{y} - n\mathbf{x}) M_{n,k,i}(d\mathbf{y}) \\ &= \prod_{j=0}^{r} \left(u_j(1 - \{\{nx_j\}\})\exp\{-z_j(1 - \{\{nx_j\}\})\}\right) \\ &\times \int_{\Gamma_r^*} \left(\prod_{j=0}^{r} \exp\{i\theta_j(1 - \{\{nx_j\}\})\}\right) \\ &\times \left(\prod_{j=r+1}^{d} \hat{u}_j(i\theta_j - z_j)\right) \exp\{n\langle x|i\Theta - \mathbf{z}\rangle\} \\ &\times \left(\mathscr{L}^n_{\langle \mathbf{z}-i\Theta|f\rangle}\psi_{k,i}(\xi^{k,i})\right) d\Theta, \end{aligned} \tag{7.18}$$

where

$$\Gamma_r^* = \left\{(\theta_0, \theta_1, \ldots, \theta_d) \in \mathscr{R}^{d+1} \colon |\theta_j| \le \tfrac{1}{2},\ j = 0, 1, \ldots, r\right\}$$

is the dual group of Γ_r, and $d\Theta$ is Lebesgue measure.

The rest of the argument is essentially the same as in the heuristic proof of Proposition 2. By Hypothesis D and Proposition 7, Appendix 1, the primary contribution to the integral in (7.18) should come from $|\Theta| \le \varepsilon$, $\varepsilon > 0$ small. The asymptotic evaluation of this integral may be accomplished by Laplace's method, using (7.15) as in the proof of Proposition 2. This leads to (7.16). □

Rigorous proofs of Propositions 2 and 3 call for some kind of "unsmoothing" procedure. We shall use a general unsmoothing theorem due to Stone [17].

Let $\mathscr{S}_0^+$ denote the set of all compactly supported, nonnegative functions $u \in \mathscr{S}$. Let $\{K_m\}_{m \ge 1}$ be a sequence of probability measures on $\mathscr{R}^{d+1}$ which converge weakly to the unit point mass at the origin (i.e., $K_m(\{|\Theta| \le \varepsilon\}) \to 1$ as $m \to \infty$ for all $\varepsilon > 0$), and such that each Fourier transform $\hat{K}_m(i\Theta)$ has compact support. Let $\mathscr{A} \subset \mathscr{R}^{d+1}$ be an index set.

THEOREM F. *Let $\{\mu_n^{\mathbf{x}}, \mathbf{x} \in \mathscr{A}, n \in \mathscr{N}\}$ be a family of nonnegative Borel measures on $\mathscr{R}^{d+1}$. Suppose that for each $m \ge 1$ and each $u \in \mathscr{S}_0^+$,*

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathscr{A}} \left| \int u(\mathbf{y})(K_m * \mu_n^{\mathbf{x}})(d\mathbf{y}) - \int u(\mathbf{y})\, d\mathbf{y} \right| = 0. \tag{7.19}$$

Then for each $u \in \mathscr{S}_0^+$

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathscr{A}} \left| \int u(\mathbf{y}) \mu_n^{\mathbf{x}}(d\mathbf{y}) - \int u(\mathbf{y})\, d\mathbf{y} \right| = 0. \tag{7.20}$$

Here $*$ denotes convolution. Theorem F follows from Theorem 2.1 of [17].

Suppose $\mathscr{A}$ is a compact subset of $\mathscr{R}^{d+1}$, and suppose (7.20) holds for all $u \in \mathscr{S}_0^+$. Let $u_{\mathbf{x}}(\mathbf{y})$, $\mathbf{x} \in \mathscr{A}$, be a family of functions in $\mathscr{S}_0^+$ such that (i) $\bigcup_{\mathbf{x}\in\mathscr{A}}$ support $(u_{\mathbf{x}})$ is compact, and (ii) $\mathbf{x} \to u_{\mathbf{x}}$ is continuous in $\|\cdot\|_\infty$. Then

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathscr{A}} \left| \int u_{\mathbf{x}}(\mathbf{y}) \mu_n^{\mathbf{x}}(d\mathbf{y}) - \int u_{\mathbf{x}}(\mathbf{y})\, d\mathbf{y} \right| = 0. \tag{7.21}$$

This follows from (7.20) by a straightforward approximation argument. Thus, to prove (7.21) it suffices to prove (7.19) for each $m \geq 1$ and each $u \in \mathscr{S}_0^+$.

Proof of Proposition 2. It suffices to establish (7.10) for each $u \in \mathscr{S}_0^+$ uniformly for $\mathbf{x} \in \mathscr{A}$ where $\mathscr{A}$ is any compact subset of $\mathscr{B}$. Let $\nabla\beta(\mathbf{z}_{\mathbf{x}}) = \mathbf{x}$, and

$$u_{\mathbf{x}}(\mathbf{y}) = u(\mathbf{y}) e^{\langle \mathbf{z}_{\mathbf{x}} | \mathbf{y}\rangle}, \tag{7.22}$$

$$\begin{aligned} \mu_n^{\mathbf{x}}(d\mathbf{y}) = {} & (2\pi n)^{(d+1)/2} e^{n\gamma(\mathbf{x})} \left(\det \nabla^2 \gamma(\mathbf{x})\right)^{-1/2} C_{k,i}(\mathbf{z}_{\mathbf{x}})^{-1} \\ & \times e^{\langle \mathbf{z}_{\mathbf{x}} | \mathbf{y}\rangle} M_{n,k,i}(d(\mathbf{y} + n\mathbf{x})), \end{aligned} \tag{7.23}$$

where $C_{k,i}(\mathbf{z}) = (\int \psi_{k,i}\, d\nu_{\langle \mathbf{z}|\mathbf{f}\rangle}) h_{\langle \mathbf{z}|\mathbf{f}\rangle}(\xi^{k,i})$. Then (7.10) is equivalent to (7.21). Thus it suffices to prove (7.19) for each $m \geq 1$ and $u \in \mathscr{S}_0^+$.

Let K_m be as in Theorem F, and assume that $\hat{K}_m(i\Theta)$ has compact support F_m. By Parseval's identity and (7.7)

$$\begin{aligned} & \int u(\mathbf{y})(K_m * \mu_n^{\mathbf{x}})(d\mathbf{y}) \\ & = \int_{F_m} \hat{u}(i\Theta) \hat{K}_m(-i\Theta) \hat{\mu}_n^{\mathbf{x}}(-i\Theta)\, d\Theta/(2\pi)^{d+1} \\ & = (n/2\pi)^{(d+1)/2} e^{n\gamma(\mathbf{x})} \left(\det \nabla^2 \gamma(\mathbf{x})\right)^{-1/2} C_{k,i}(\mathbf{z}_{\mathbf{x}})^{-1} \\ & \quad \times \int_{F_m} \hat{u}(i\Theta) \hat{K}_m(-i\Theta) e^{n\langle \mathbf{x} | i\Theta - \mathbf{z}_{\mathbf{x}}\rangle} \mathscr{L}^n_{\langle \mathbf{z}_{\mathbf{x}} - i\Theta | \mathbf{f}\rangle} \psi_{k,i}(\xi^{k,i})\, d\Theta \\ & \sim (n/2\pi)^{(d+1)/2} \left(\det \nabla^2 \gamma(\mathbf{x})\right)^{-1/2} C_{k,i}(\mathbf{z}_{\mathbf{x}})^{-1} \\ & \quad \times \int_{\{|\Theta| \leq \varepsilon\}} \hat{u}(i\Theta) \hat{K}_m(-i\Theta) C_{k,i}(\mathbf{z}_{\mathbf{x}} - i\Theta) \\ & \quad \times \exp\{n(\langle \mathbf{x} | i\Theta\rangle + \beta(\mathbf{z}_{\mathbf{x}} - i\Theta) - \beta(\mathbf{z}_{\mathbf{x}}))\}\, d\Theta. \end{aligned} \tag{7.24}$$

The last step follows from (7.12) and (7.13). Note that since F_m is compact, (7.12) holds uniformly for $\Theta \in F_m \setminus \{|\Theta| < \varepsilon\}$ and $\mathbf{x} \in \mathscr{A}$, by Proposition 6; also (7.13) holds uniformly for $|\Theta| \le \varepsilon$ and $\mathbf{x} \in \mathscr{A}$, by Proposition 5. Thus (7.24) holds uniformly for $\mathbf{x} \in \mathscr{A}$.

The relation (7.19) follows from (7.24) by Laplace's method: the uniformity in $\mathbf{x} \in \mathscr{A}$ follows from the local uniformity in $\mathbf{z}$ of the Taylor series expansion for $\beta(\mathbf{z} - i\Theta) - \beta(\mathbf{z})$ near $\Theta = \mathbf{0}$. □

For the proof of Proposition 3, an unsmoothing theorem for the group $\Gamma_r = \mathscr{Z}^{r+1} \oplus \mathscr{R}^{d-r}$ is necessary. Let $H(dy)$ denote Haar measure on Γ_r, and let $\{K_m\}_{m \ge 1}$ be a sequence of probability measures on Γ_r which converge weakly to the pointmass at $\mathbf{0}$, and such that each Fourier transform $\hat{K}_m(i\Theta)$ has compact support on Γ_r^*. Theorem 2.1 of [17] implies

THEOREM G. *Let $(\mu_n^{\mathbf{x}}, \mathbf{x} \in \mathscr{A}, N \in \mathscr{N}\}$ be a family of nonnegative Borel measures on Γ_r. Suppose that for each $m \ge 1$ and each $u \in \mathscr{S}_0^+$,*

$$\lim_{n\to\infty} \sup_{\mathbf{x} \in \mathscr{A}} \left| \int u(\mathbf{y})(K_m * \mu_n^{\mathbf{x}})(d\mathbf{y}) - \int u(\mathbf{y}) H(d\mathbf{y}) \right| = 0. \tag{7.25}$$

Then for each $u \in \mathscr{S}_0^+$,

$$\lim_{n\to\infty} \sup_{\mathbf{x} \in \mathscr{A}} \left| \int u(\mathbf{y})\mu_n^{\mathbf{x}}(d\mathbf{y}) - \int u(\mathbf{y}) H(d\mathbf{y}) \right| = 0. \tag{7.26}$$

As before, $\mathscr{S}_0^+$ denotes $\{u \in \mathscr{S}: u \ge 0, \operatorname{supp}(u) \text{ compact}\}$.

Suppose $\mathscr{A}$ is a compact subset of $\mathscr{R}^{d+1}$, and suppose (7.26) holds for all $u \in \mathscr{S}_0^+$. Let $u_{\mathbf{x}}(\mathbf{y})$, $\mathbf{x} \in \mathscr{A}$, be functions in $\mathscr{S}_0^+$ such that (i) $\bigcup_{\mathbf{x} \in \mathscr{A}}$ support $(u_{\mathbf{x}})$ is compact; (ii) $\mathbf{x} \to u_{\mathbf{x}}$ is continuous in $\|\cdot\|_\infty$. Then

$$\lim_{n\to\infty} \sup_{\mathbf{x} \in \mathscr{A}} \left| \int u_{\mathbf{x}}(\mathbf{y})\mu_n^{\mathbf{x}}(d\mathbf{y}) - \int u_{\mathbf{x}}(\mathbf{y}) H(d\mathbf{y}) \right| = 0. \tag{7.27}$$

Proof of Proposition 3. It must be shown that (7.16) holds uniformly for $\mathbf{x} \in \mathscr{A}$, where $\mathscr{A} \subset \mathscr{B}$ is compact. As before, let $\nabla\beta(\mathbf{z}_{\mathbf{x}}) = \mathbf{x}$, and let $u_{\mathbf{x}}$, $\mu_n^{\mathbf{x}}$ be defined by (7.22) and (7.23). Then (7.16) is equivalent to (7.27). Hence it is enough to prove (7.25) for each $m \ge 1$ and $u \in \mathscr{S}_0^+$.

Using the Parseval identity *for the group* Γ_r one obtains

$$\begin{aligned} &\int_{\Gamma_r} u(\mathbf{y})(K_m * \mu_n^{\mathbf{x}})(d\mathbf{y}) \\ &\quad = (n/2\pi)^{(d+1)/2} e^{n\gamma(\mathbf{x})} (\det \nabla^2 \gamma(\mathbf{x}))^{-1/2} C_{k,i}(\mathbf{z}_{\mathbf{x}})^{-1} \\ &\qquad \times \int_{\Gamma_r^*} \hat{u}(i\Theta) \hat{K}_m(-i\Theta) e^{n\langle \mathbf{x} | -\mathbf{z}_{\mathbf{x}} \rangle} \mathscr{L}^n_{\langle \mathbf{z}_{\mathbf{x}} - i\Theta | \mathbf{f} \rangle} \psi_{k,i}(\xi^{k,i}) H(d\Theta). \end{aligned} \tag{7.28}$$

(Note: $\hat{u}(i\Theta)$, $\hat{K}_m(-i\Theta)$ denote the Fourier transforms relative to Γ_r^*, not $\mathscr{R}^{d+1}$. Thus $\hat{u}(\mathbf{0}) = \int_{\Gamma_r} u(\mathbf{y}) H(d\mathbf{y})$. Also, $\hat{K}_m(\mathbf{0}) = 1$, since K_m is a probability measure.)

The relation (7.25) follows from (7.28) by the same argument as in the proof of Proposition 2, since $\hat{K}_m(i\Theta)$ has compact support in Γ_r^*. □

8. Counting Periodic Orbits of a Symbolic Flow

The periodic orbits of a symbolic flow are in 1-to-1 correspondence with the periodic orbits of the underlying shift. In particular, $(\xi, t) \in \Sigma_A^f$ lies on a periodic orbit of σ^f iff $\sigma^n \xi = \xi$ for some $n = 1, 2, \ldots$. Hence, the Q measures of section 6 may be expressed in terms of the N_n measures of section 7.

Let $f \in \mathscr{F}_\rho$ be strictly positive, let $G_1, G_2, \ldots, G_d \in \mathscr{F}_\rho(\Sigma_A^f)$, and let $g_i \in \mathscr{F}_\rho$ be given by (5.1). Let $Q = Q^{f; G_1, \ldots, G_d}$ and $N_n = N_n^{(f, g_1, \ldots, g_d)}$. Recall that N_n counts all ξ such that $\sigma^n \xi = \xi$, including those ξ whose (minimal) period $m | n$, $m < n$. Consequently,

$$Q = \sum_{n=1}^{\infty} n^{-1} \sum_{m | n} \mu\left(\frac{n}{m}\right) N_m \tag{8.1}$$

where μ is the Moebius function of number theory.

Let $\bar{\beta}, \bar{\gamma}$ be the thermodynamic functions for $G_1, G_2, \ldots, G_d$, and let β, γ be the thermodynamic functions for $f, g_1, g_2, \ldots, g_d$. If $\mathbf{x} \in \bar{\mathscr{B}} = \nabla\bar{\beta}(\mathscr{R}^d)$ then there is a unique $\mathbf{z} \in \mathscr{R}^d$ such that $\nabla\bar{\beta}(\mathbf{z}) = \mathbf{x}$. If this is the case, let

$$\mathbf{x}^* = (1, \mathbf{x}), \tag{8.2}$$

$$t_{\mathbf{x}} = 1 \Big/ \int f \, d\mu_{\langle \mathbf{z} | \mathbf{g} \rangle - \bar{\beta}(\mathbf{z}) f}. \tag{8.3}$$

Recall that $\beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) = 0$ and $\nabla\beta(-\bar{\beta}(\mathbf{z}), \mathbf{z}) = \mathbf{x}^*/t_{\mathbf{x}}$; also, $-t\gamma(\mathbf{x}^*/t)$ is a concave function of t which achieves its maximum uniquely at $t = t_{\mathbf{x}}$, and $t_{\mathbf{x}}\gamma(\mathbf{x}^*/t_{\mathbf{x}}) = \bar{\gamma}(\mathbf{x})$ (cf. Lemma 2).

Lemma 5. *For each $\varepsilon > 0$ there exist $\eta = \eta(\mathbf{x}, \varepsilon) > 0$ and $K = K_{\mathbf{x}} < \infty$ such that*

$$\begin{aligned}
&\sum_{n \geq a(t_{\mathbf{x}} + \varepsilon)} N_n\left([a, a + \delta_0] \times \prod_{i=1}^{d} [a x_i, a x_i + \delta_i]\right) \\
&\quad + \sum_{n \leq a(t_{\mathbf{x}} - \varepsilon)} N_n\left([a, a + \delta_0] \times \prod_{i=1}^{d} [a x_i, a x_i + \delta_i]\right) \\
&\quad \leq K\left(\exp\{-a(\bar{\gamma}(\mathbf{x}) + \eta)\}\right)
\end{aligned} \tag{8.4}$$

as $a \to \infty$. Furthermore, $K_{\mathbf{x}}$ and $\eta(\mathbf{x}, \varepsilon)$ may be chosen so as to depend continuously on $\mathbf{x}$.

Proof. The plan is to use Proposition 1. Let $t_0 = t_{\mathbf{x}} - \varepsilon$, $t_1 = t_{\mathbf{x}} + \varepsilon$, $\mathbf{z}^{(i)} = \nabla\gamma(\mathbf{x}^*/t_i)$, $i = 0, 1$. (Note that $\mathbf{z}^{(i)} \in \mathscr{R}^{d+1}$.) Since $-t\gamma(\mathbf{x}^*/t)$ achieves its maximum uniquely at $t = t_{\mathbf{x}}$, it follows that

$$\beta(\mathbf{z}^{(1)}) < 0 < \beta(\mathbf{z}^{(0)}), \tag{8.5}$$

$$t_1\beta(\mathbf{z}^{(i)}) - \langle \mathbf{x}^* | \mathbf{z}^{(i)} \rangle < -\bar{\gamma}(\mathbf{x}), \qquad i = 0, 1. \tag{8.6}$$

(Here I have used property (j), Sect. 4.)

The linear functional $\langle \mathbf{z}^{(1)} | \mathbf{y} \rangle$, $\mathbf{y} \in [1, 1 + \delta_0/a] \times \prod_{i=1}^d [x_i, x_i + \delta_i/a]$, attains its maximum at one of the corners of the rectangle: call this corner $\mathbf{x}'(a)$. Obviously $|\mathbf{x}^* - \mathbf{x}'(a)| = O(a^{-1})$. By Proposition 1,

$$\begin{aligned} N_n\left([a, a + \delta_0] \times \prod_{i=1}^d [ax_i, ax_i + \delta_i]\right) &\\ \leq K \exp\{ n\beta(\mathbf{z}^{(1)}) - a\langle \mathbf{z}^{(1)} | \mathbf{x}'(a) \rangle \} &\\ \leq K' \exp\{ n\beta(\mathbf{z}^{(1)}) - a\langle \mathbf{z}^{(1)} | \mathbf{x}^* \rangle \} & \end{aligned}$$

for suitable constants $K, K' < \infty$. Therefore, by (8.5),

$$\begin{aligned} \sum_{n \geq at_1} N_n\left([a, a + \delta_0] \times \prod_{i=1}^d [ax_i, ax_i + \delta_i]\right) &\\ \leq K'' \exp\{ a(t_1\beta(\mathbf{z}^{(1)}) - \langle \mathbf{z}^{(1)} | \mathbf{x}^* \rangle) \} & \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{n \leq at_0} N_n\left([a, a + \delta_0] \times \prod_{i=1}^d [ax_i, ax_i + \delta_i]\right) &\\ \leq K''' \exp\{ a(t_0\beta(\mathbf{z}^{(0)}) - \langle \mathbf{z}^{(0)} | \mathbf{x}^* \rangle) \}. & \end{aligned}$$

By (8.6), these inequalities prove (8.4). The fact that the constants in (8.4) may be chosen continuously in $\mathbf{x}$ follows from Proposition 1 and the continuity of the thermodynamic functions. □

Assume now that $f, g_1, g_2, \ldots, g_d$ satisfy Hypothesis B (cf. Sect. 6). We shall distinguish between two cases: (i) $f, g_1, g_2, \ldots, g_d$ satisfy Hypothesis C; (ii) $f, g_1, \ldots, g_d$ do not satisfy Hypothesis C. Case (i) is easier to handle: We shall consider if first. Let $C(J)$ denote the set of real-valued continuous functions on J.

LEMMA 6. *Let $F \in C([t_1, t_2])$, where $t_1 < t < t_2$ and let $\sigma^2 > 0$. Then for $\varepsilon \le \min(t - t_1, t_2 - t_1)$,*

$$\lim_{a \to \infty} \sum_{a(t-\varepsilon) < n < a(t+\varepsilon)} a^{-1/2} F(n/a) e^{-(n-at)^2/2a\sigma^2} = F(t)(2\pi)^{1/2}\sigma. \quad (8.7)$$

Moreover, (8.7) *holds uniformly for t, σ in any compact subset of* $(0, \infty)$ *and F in any compact subset of $C([t_1, t_2])$, $t_1 < t < t_2$.*

Proof. It is easily seen that for any $\varepsilon' > 0$,

$$\sum_{n:\ |n-at| < a\varepsilon'} (2\pi a)^{-1/2} \sigma^{-1} e^{-(n-at)^2/2a\sigma^2}$$
$$\sim \int_{-\varepsilon' a^{1/2}\sigma^{-1}}^{\varepsilon' a^{1/2}\sigma^{-1}} e^{-x^2/2}\, dx (2\pi)^{-1/2} \sim 1,$$

with local uniformity in σ, t. The Lemma is an easy consequence of this and the Arzela–Ascoli Theorem. □

Proof of Theorem 1: Case (i). Let

$$U = [a, a + \delta_0] \times \prod_{i=1}^{d} [ax_i, ax_i + \delta_i].$$

Since $N_n(U) \ge \Sigma_{m|n} \mu(n/m) N_m(U)$ (by the Moebius Inversion Formula) it follows from Lemma 5 that

$$\limsup_{a \to \infty} a^{-1} \log \sum_{n:\ |n - at_{\mathbf{x}}| > a\varepsilon} n^{-1} \sum_{m|n} \mu\left(\frac{n}{m}\right) N_m(U) < -\bar{\gamma}(\mathbf{x}), \quad (8.8)$$

$$\limsup_{a \to \infty} a^{-1} \log \sum_{n:\ |n - at_{\mathbf{x}}| \le a\varepsilon} n^{-1} \sum_{\substack{m|n \\ m \ne n}} N_m(U) < -\bar{\gamma}(\mathbf{x}). \quad (8.9)$$

Moreover, Lemma 5 implies that the lim sups in (8.8) and (8.9) stay uniformly bounded away from $-\bar{\gamma}(\mathbf{x})$ for $\mathbf{x}$ in any compact subset of $\bar{\mathscr{B}}$. Consequently, to prove (6.2) it suffices to show that

$$\sum_{n:\ |n - at_{\mathbf{x}}| \le a\varepsilon} n^{-1} N_n(U)$$
$$\sim e^{-a\bar{\gamma}(\mathbf{x})} a^{-(d+2)/2} (2\pi)^{-d/2} \left(\det \nabla^2 \bar{\gamma}(\mathbf{x})\right)^{1/2} C(\mathbf{x}, \delta) \quad (8.10)$$

for small $\varepsilon > 0$, uniformly for $\mathbf{x}$ in any compact subset of $\bar{\mathscr{B}}$.

Now the indicator function of the rectangle $\prod_{i=0}^{d} [0, \delta_i]$ may be approximated from above and from below by nonnegative C^∞ functions with compact support. To prove (8.10) it therefore suffices to show that for every C^∞ u: $\mathscr{R}^{d+1} \to \mathscr{R}$ with compact support

$$\sum_{|n-at_{\mathbf{x}}|\le a\varepsilon} n^{-1}\int u(\mathbf{y}-a\mathbf{x}^*)N_n(d\mathbf{y})$$
$$\sim e^{-a\bar{\gamma}(\mathbf{x})}a^{-(d+2)/2}(2\pi)^{-d/2}\big(\det\nabla^2\bar{\gamma}(\mathbf{x})\big)^{1/2}C'(\mathbf{x},u), \quad (8.11)$$
$$C'(\mathbf{x},u)=\int u(\mathbf{y})\exp\{-\langle\nabla(\mathbf{x}^*/t_{\mathbf{x}})|\mathbf{y}\rangle\}\,d\mathbf{y}, \quad (8.12)$$

uniformly for $\mathbf{x}$ in any compact subset of $\bar{\mathscr{B}}$. But it follows from Proposition 2 that

$$n^{-1}\int u(\mathbf{y}-a\mathbf{x}^*)N_n(d\mathbf{y}) \sim \exp\{-n\gamma(a\mathbf{x}^*/n)\}n^{-(d+3)/2}$$
$$\times(2\pi)^{-(d+1)/2}\big(\det\nabla^2\gamma(a\mathbf{x}^*/n)\big)^{1/2}$$
$$\times\int u(\mathbf{y})\exp\{-\langle\nabla\gamma(a\mathbf{x}^*/n)|\mathbf{y}\rangle\}\,d\mathbf{y} \quad (8.13)$$

uniformly for n in the range $|n-at_{\mathbf{x}}|\le a\varepsilon$, provided $\varepsilon>0$ is sufficiently small.

Recall that $-s\gamma(\mathbf{x}^*/s)$ is a concave function of $s>0$ which achieves its maximum uniquely at $s=t_{\mathbf{x}}$; for s near $t_{\mathbf{x}}$, $-s\gamma(\mathbf{x}^*/s)$ may be approximated by its Taylor series, giving (cf. (4.4)),

$$-n\gamma(a\mathbf{x}^*/n) = -a\bar{\gamma}(\mathbf{x})-\tfrac{1}{2}a^{-1}(n-at_{\mathbf{x}})^2t_{\mathbf{x}}^{-3}\langle\mathbf{x}^*|\nabla^2\gamma(\mathbf{x}^*/t_{\mathbf{x}})|\mathbf{x}^*\rangle$$
$$+a^{-1}o\big((n-at_{\mathbf{x}})^2\big). \quad (8.14)$$

Recall (Lemma 2) that

$$\det\nabla^2\bar{\gamma}(\mathbf{x}) = t_{\mathbf{x}}^{-d}\big(\det\nabla^2\gamma(\mathbf{x}^*)\big)\langle\mathbf{x}^*|\nabla^2\gamma(\mathbf{x}^*/t_{\mathbf{x}})|\mathbf{x}^*\rangle. \quad (8.15)$$

Also,

$$\nabla\gamma(\mathbf{x}^*/t_{\mathbf{x}}) = (-\bar{\beta}(\mathbf{z}),\mathbf{z}). \quad (8.16)$$

Combining (8.13)–(8.16) and applying Lemma 6 gives (8.11). The local uniformity in $\mathbf{x}$ follows from Lemma 6 and the local uniformity in $\mathbf{x}$ of (8.13) and (8.14). □

Consider now Case (ii), in which $f, g_1, g_2, \dots, g_d$ satisfy Hypothesis B but not Hypothesis C. The primary difference between this case and Case (i) is that the terms in $\Sigma_n n^{-1}N_n(U)$ oscillate rapidly rather than slowly.

Fix $t_1 < t_2$, and let $\mathscr{G}$ denote the set of real-valued continuous $F(x_1, x_2, \ldots, x_k, t)$ defined on $[0,1]^k \times [t_1, t_2]$ which are periodic with period 1 in each of the variables $x_1, x_2, \ldots, x_k$, i.e., such that

$$F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k, t) = F(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_k, t)$$

for all $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \in [0,1]$ and $t \in [t_1, t_2]$.

LEMMA 7. *Let $F \in \mathscr{G}$ and $t_1 < t < t_2$. Suppose $b_1, b_2, \ldots, b_k \in \mathscr{R}$ are such that $1, b_1, \ldots, b_k$ are linearly independent over the rational numbers. Then for all $x_1, x_2, \ldots, x_k \in \mathscr{R}$ and $\varepsilon \le \min(t - t_1, t_2 - t)$*

$$\lim_{a \to \infty} \sum_{n:\ |n-at| \le a\varepsilon} a^{-1/2} \exp\{-(n-at)^2/2a\sigma^2\} \times F(\{\{ax_1 - nb_1\}\}, \ldots, \{\{ax_k - nb_k\}\}, n/a) = (2\pi)^{1/2}\sigma \int_0^1 \int_0^1 \cdots \int_0^1 F(y_1, y_2, \ldots, y_k, t)\, dy_1 \ldots dy_k. \quad (8.17)$$

Moreover, (8.17) *holds uniformly for $t, x_1, \ldots, x_k$, and $\sigma > 0$ locally, and F in any compact subset of $\mathscr{G}$.*

Note. There is no uniformity in $b_1, b_2, \ldots, b_k$. Recall that $\{\{x\}\}$ denotes the fractional part of x.

Proof. As n varies over any interval of length $O(\log a)$ contained in $\{n: |n - at| \le Ka^{1/2}$, where $K < \infty$ is a large but fixed constant, (n/a) and $\exp\{-(n-at)^2/2a\sigma^2\}$ remain nearly constant, provided a is large. By an equidistribution theorem of Weyl, together with the continuity of F,

$$\sum_{n \in J} F(\{\{ax_1 - nb_1\}\}, \ldots, \{\{ax_k - nb_k\}\}, n/a) \sim |J| \int_0^1 \int_0^1 \cdots \int_0^1 F(y_1, y_2, \ldots, y_k, t)\, dy_1 \ldots dy_k$$

uniformly for intervals J contained in $\{y: |y - at| \le Ka^{1/2}\}$ of length $|J| \ge C \log a$. There is also uniformity in $x_1, x_2, \ldots, x_k$ and for F in any compact subset of $\mathscr{G}$. It follows that the factor $F(\ldots)$ may be replaced by $\int\int \ldots \int F$ in the sum in (8.17), at least for n in the range $|n - at| \le Ka^{1/2}$.

The argument may now be completed by following the line of reasoning used in Lemma 6. □

Proof of Theorem 1: Case (ii). As in Case (i), it suffices to show that (6.2) holds for small $\varepsilon > 0$, uniformly for $\mathbf{x}$ in any compact subset of $\overline{\mathscr{B}}$.

For this it suffices to show that if $u_0, u_1, \dots, u_d$ are nonnegative C^∞ functions on $\mathscr{R}$, each with compact support, and if $u(\mathbf{y}) = \prod_{i=0}^d u_i(y_i)$, then

$$\sum_{n:\ |n - at_{\mathbf{x}}| \le a\varepsilon} n^{-1} \int u(\mathbf{y} - a\mathbf{x}^*) N_n(d\mathbf{y})$$
$$\sim e^{-a\bar{\gamma}(\mathbf{x})} a^{-(d+2)/2} (2\pi)^{-d/2} \left(\det \nabla^2 \bar{\gamma}(\mathbf{x})\right)^{1/2} C^*(\mathbf{x}, u), \quad (8.18)$$

where $\mathbf{x}^* = (1, \mathbf{x})$ and

$$C^*(\mathbf{x}, u) = \int e^{\bar{\beta}(\mathbf{z})t} u_0(t)\, dt \prod_{i=1}^{d} \int e^{z_i t} u_i(t)\, dt,$$

uniformly for $\mathbf{x}$ in any compact subset of $\bar{\mathscr{B}}$. This is because the indicator function of $[0, \delta_i]$ may be approximated from above and below by nonnegative C^∞ functions with compact support.

In Case (ii) $f, g_1, g_2, \dots, g_d$ satisfy Hypothesis B but not Hypothesis C. Therefore, these exist $b_0, b_1, \dots, b_d \in \mathscr{R}$ such that $f - b_0 \cong f_0$, $g_1 - b_1 \cong f_1, \dots, g_d - b_d \cong f_d$ where $\mathbf{f} = (f_0, f_1, \dots, f_d)$ takes its values in a proper closed subgroup Γ of $\mathscr{R}^{d+1}$ not contained in any d-dimensional linear subspace of $\mathscr{R}^{d+1}$. (If it were the case that $\mathbf{f}$ took its values in a d-dimensional subspace then there would be constants $a_0, a_1, \dots, a_d \in \mathscr{R}$, not all zero, such that $\sum_{i=0}^d a_i f_i \equiv 0$. But then $a_0 f + \sum_{i=1}^d a_i g_i$ would be homologous to a constant, contradicting Hypothesis B.)

We shall assume that $f_0, f_1, \dots, f_d$ *satisfy Hypothesis* D. There is no loss of generality in this, because one may always make a nonsingular linear transformation of $\mathscr{R}^{d+1}$ which maps Γ onto Γ_r for some r. Making a linear transformation does not change the orbit counting problem in any essential way (if T: $\mathscr{R}^{d+1} \to \mathscr{R}^{d+1}$ is a nonsingular linear transformation then $N_n^{T\mathbf{f}}(TU) = N_n^{\mathbf{f}}(U)$ and $\gamma_{T\mathbf{f}}(T\mathbf{x}') = \gamma_{\mathbf{f}}(\mathbf{x}')$ for all $n, U, \mathbf{x}'$).

If $f_0, f_1, \dots, f_d$ satisfy Hypothesis D then $b_{r+1} = b_{r+2} = \cdots = b_d = 0$ and $1, b_0, b_1, \dots, b_r$ are linearly independent over the rationals. For if not then there would exist integers $k_0, k_1, \dots, k_r$, not all zero, such that $\sum_{i=0}^r k_i b_i \in \mathscr{Z}$; but then $k_0 f + \sum_{i=1}^r k_i g_i$ would be homologous to an integer-valued function, contradicting Hypothesis B.

Let $N_n = N_n^{f, g_1, \dots, g_d}$ as before, and let $N_n^* = N_n^{f_0, f_1, \dots, f_d}$. Then for any C^∞ function u on $\mathscr{R}^{d+1}$ with compact support

$$\int \varphi(\mathbf{y}) N_n(d\mathbf{y}) = \int \varphi(\mathbf{y} + n\mathbf{b}) N_n^*(d\mathbf{y}), \quad (8.19)$$

where $\mathbf{b} = (b_0, b_1, \dots, b_d)$. Moreover, if β, γ are the thermodynamic functions for $f, g_1, \dots, g_d$ and β^*, γ^* are the thermodynamic functions for

$f_0, f_1, \dots, f_d$, then for all $\mathbf{z}', \mathbf{x}' \in \mathscr{R}^{d+1}$,

$$\begin{aligned}\beta(\mathbf{z}') &= \beta^*(\mathbf{z}') + \langle \mathbf{z}' | \mathbf{b} \rangle,\\ \gamma(\mathbf{x}') &= \gamma^*(\mathbf{x}' - \mathbf{b}),\\ \nabla\gamma(\mathbf{x}') &= \nabla\gamma^*(\mathbf{x}' - \mathbf{b}),\\ \nabla^2\gamma(\mathbf{x}') &= \nabla^2\gamma^*(\mathbf{x}' - \mathbf{b}).\end{aligned} \tag{8.20}$$

By (8.19), (8.20), and Proposition 3,

$$\begin{aligned} n^{-1}\int u(\mathbf{y} - a\mathbf{x}^*)N_n(d\mathbf{y}) &\sim \exp\{-n\gamma(a\mathbf{x}^*/n)\}n^{-(d+3)/2}\\ &\quad\times (2\pi)^{-(d+1)/2}(\det\nabla^2\gamma(a\mathbf{x}^*/n))^{1/2} \times C_n(a\mathbf{x}^*, u),\end{aligned} \tag{8.21}$$

where

$$\begin{aligned} C_n(a\mathbf{x}^*, u) = &\prod_{i=0}^{r} u_i(1 - \{\{ax_i^* - nb_i\}\})\\ &\times\exp\left\{-\sum_{i=0}^{r}(\partial_i\gamma(a\mathbf{x}^*/n))(1 - \{\{ax_i^* - nb_i\}\})\right\}\\ &\times\prod_{i=r+1}^{d}\int_0^1 u_i(t)\exp\{-t\partial_i\gamma(a\mathbf{x}^*/n)\}\,dt,\end{aligned}$$

uniformly for n in the range $|n - at_{\mathbf{x}}| \le a\varepsilon$ and uniformly for $\mathbf{x}$ in any compact subset of $\overline{\mathscr{B}}$.

Now $-n\gamma(a\mathbf{x}^*/n)$ may be approximated from above and below by a quadratic expression (cf. (8.14)). Thus, summing (8.21) over n in the range $|n - at_{\mathbf{x}}| \le a\varepsilon$ gives a sum of the same type considered in Lemma 7. From this, one easily obtains (8.18). □

9. Axiom *A* Flows

The results of Section 6 may be carried over with small changes to Axiom *A* flows, in view of Bowen's results [4]. Bowen showed that the periodic orbits of a weakly mixing Axiom *A* flow restricted to a basic set are nearly in 1-to-1 correspondence with those of a certain symbolic flow (cf. (9.2)).

Let (Ω, ϕ_t) be a weakly mixing Axiom *A* flow restricted to a basic set. For C^∞ functions $G_1, G_2, \dots, G_d$: $\Omega \to \mathscr{R}$ and $x_1, x_2, \dots, x_d \in \mathscr{R}$ let $\bar{\mu}_{\mathbf{x}}$ be the maximum entropy measures for ϕ_t subject to $\int G_i\, d\bar{\mu}_{\mathbf{x}} = x_i$, provided such a measure exists, and let $-\bar{\gamma}(\mathbf{x})$ be its entropy. Let $\bar{\mu}_{\max}$ be the maximum entropy measure for ϕ_t, and let $H^*(\phi_t)$ be its entropy.

THEOREM 5. *Assume that $G_1, \dots, G_d \in C^\infty$ satisfy Hypothesis $\bar{\mathrm{B}}$. Then for all $(x_1, x_2, \dots, x_d)$ in a neighborhood of $(\int G_1\, d\bar{\mu}_{\max}, \dots, \int G_d\, d\bar{\mu}_{\max})$,*

$$\#\{\tau: 0 \le \tau(1) - a \le \delta;\ 0 \le \tau(G_i) - ax_i \le \delta\}$$
$$\sim e^{-a\bar{\gamma}(\mathbf{x})} a^{-(d+2)/2}(2\pi)^{-d/2}\left(\det \nabla^2\bar{\gamma}(\mathbf{x})\right)^{1/2} C(\mathbf{x}, \delta),$$
$$C(\mathbf{x}, \delta) = \left\{\int_0^\delta e^{\bar{\beta}(\nabla\bar{\gamma}(\mathbf{x}))t}\, dt\right\}\left\{\int_0^\delta \dots \int_0^\delta e^{\langle\nabla\bar{\gamma}(\mathbf{x})|\mathbf{t}\rangle}\, dt_1 \dots dt_d\right\}, \tag{9.1}$$

where $\bar{\beta}$ is the Legendre transform of $\bar{\gamma}$. Moreover, this approximation is uniform in $\mathbf{x}$ *locally.*

THEOREM 6. *Assume that $G_1, \dots, G_d \in C^\infty$ satisfy Hypothesis $\bar{\mathrm{B}}$, and that $(x_1, x_2, \dots, x_d)$ lies in a neighborhood of $(\int G_1\, d\bar{\mu}_{\max}, \dots, \int G_d \bar{\mu}_{\max})$ in which* (9.1) *holds. Choose a periodic orbit τ at random from among $\{\tau: 0 \le \tau(1) - a \le \delta;\ 0 \le \tau(G_i) - ax_i \le \delta\}$. Then for any continuous G, any $\varepsilon > 0$,*

$$\operatorname{Prob}\left\{\left|\frac{\tau(G)}{\tau(1)} - \int G\, d\bar{\mu}_{\mathbf{x}}\right| > \varepsilon\right\} \to 0$$

as $a \to \infty$. Moreover, if G is C^∞ and $G_1, G_2, \dots, G_d, G$ satisfy Hypothesis $\bar{\mathrm{B}}$ then there is a constant $\sigma_{\mathbf{x}} > 0$ such that

$$\operatorname{Prob}\left\{a^{1/2}\sigma_{\mathbf{x}}^{-1}\left\{\frac{\tau(G)}{\tau(1)} - \int G\, d\bar{\mu}_{\mathbf{x}}\right\} > y\right\} \to \int_y^\infty e^{-t^2/2}\, dt/(2\pi)^{1/2}$$

as $a \to \infty$, for all $y \in \mathscr{R}$.

The salient features of the construction in [4] are as follows: There exist finitely many symbolic flows $(\Sigma_{A_i}^{f_i}, \sigma_t^{(i)})$ and Lipschitz maps π_i: $\Sigma_{A_i}^{f_i} \to \Omega$ such that $\pi_i \circ \sigma_t^{(i)} = \phi_t \circ \pi_i$ for $t \ge 0$ and $i = 0, 1, \dots, N$. The map π_0 is surjective; the maps π_i, $i = 1, \dots, N$, are *not* surjective, and are at most k-to-one. For any invariant measure $\bar{\mu}$ of $(\Sigma_{A_0}^{f_0}, \sigma_t^{(0)})$ whose entropy is sufficiently near the topological entropy $H^*(\sigma_t^{(0)})$, π_0 is a.e. one-to-one with respect to $\bar{\mu}$. If $G_1, G_2, \dots, G_d$: $\Omega \to \mathscr{R}$ are C^∞ and $G_j^{(i)} = G_j \circ \pi_i$, and if $Q_i = Q^{f_i; G_1^{(i)}, \dots, G_d^{(i)}}$ (cf. (6.1)), then for certain integers $l_1, l_2, \dots, l_N$,

$$\#\{\tau: \tau(1) \in J_0;\ \tau(G_j) \in J_j,\ j = 1, 2, \dots, d\}$$
$$= Q_0\left(\prod_{j=0}^d J_j\right) + \sum_{i=1}^N (-1)^{l_i} Q_i\left(\prod_{j=1}^d J_j\right), \tag{9.2}$$

for all intervals $J_0, J_1, \dots, J_d \subset \mathscr{R}$. Here τ refers to a periodic orbit of (Ω, ϕ_t).

The main point is this. Since π_i is *not* surjective but at most k-to-one, $i = 1, \ldots, N$, its topological entropy $H^*(\sigma_t^{(i)})$ is strictly less than $H^*(\sigma_t^{(0)})$. Consequently, for the asymptotics considered in (9.1) the terms $Q_i(\cdot)$, $i = 1, 2, \ldots, N$, are of smaller exponential order of growth than the term $Q_0(\cdot)$ (cf. Theorem 2). Thus Theorems 5 and 6 follow from the corresponding results for $(\Sigma_{A_0}^{f_0}, \sigma_t^{(0)})$, cf. Theorems 1, 3, 4.

APPENDIX 1: PERTURBATION THEORY FOR RPF OPERATORS

Let $f_1, f_2, \ldots, f_k \in \mathscr{F}_\rho^+$ be real-valued. For $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in \mathscr{C}^k$, let

$$\mathscr{L}_\mathbf{z} = \mathscr{L}_{\langle \mathbf{z} | \mathbf{f} \rangle},$$
$$h_\mathbf{z} = h_{\langle \mathbf{z} | \mathbf{f} \rangle},$$
$$\nu_\mathbf{z} = \nu_{\langle \mathbf{z} | \mathbf{f} \rangle},$$
$$\mu_\mathbf{z} = \mu_{\langle \mathbf{z} | \mathbf{f} \rangle},$$
$$\lambda_\mathbf{z} = \lambda_{\langle \mathbf{z} | \mathbf{f} \rangle},$$

for all $\mathbf{z}$ at which these quantities exist. Observe that $\mathscr{L}_\mathbf{z}$ is defined for all $\mathbf{z} \in \mathscr{C}^k$: in fact, $\mathbf{z} \to \mathscr{L}_\mathbf{z}$ is an entire holomorphic function of $\mathbf{z} \in \mathscr{C}^k$, and

$$(\partial/\partial z_j)\mathscr{L}_\mathbf{z} g = \mathscr{L}_\mathbf{z}(f_j g).$$

By Theorems A and B, $\lambda_\mathbf{z}$, $h_\mathbf{z}$, $\nu_\mathbf{z}$, and $\mu_\mathbf{z}$ are well defined for all $\mathbf{z} \in \mathscr{R}^k$.

PROPOSITION 4. *The functions $\mathbf{z} \to \lambda_\mathbf{z}, \mathbf{z} \to h_\mathbf{z}$ have analytic extensions to a neighborhood $\Omega = \Omega(f_1, \ldots, f_k)$ of $\mathscr{R}^k$ in $\mathscr{C}^k$, such that*

$$\mathscr{L}_\mathbf{z} h_\mathbf{z} = \lambda_\mathbf{z} h_\mathbf{z}, \qquad \mathbf{z} \in \Omega; \tag{A1.1}$$

$$\int h_\mathbf{z}\, d\nu_0 = 1, \qquad \mathbf{z} \in \Omega. \tag{A1.2}$$

The function $\mathbf{z} \to \nu_\mathbf{z}$ extends to a weak- analytic measure-valued function on Ω such that*

$$\mathscr{L}_\mathbf{z}^* \nu_\mathbf{z} = \lambda_\mathbf{z} \nu_\mathbf{z}, \qquad \mathbf{z} \in \Omega; \tag{A1.3}$$

$$\int h_\mathbf{z}\, d\nu_\mathbf{z} = 1, \qquad \mathbf{z} \in \Omega. \tag{A1.4}$$

For each $\mathbf{z}^* \in \mathscr{R}^k$ and each $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, \mathbf{z}^*) > 0$ such that if $\mathbf{z} \in \Omega$ and $|\mathbf{z} - \mathbf{z}^*| \le \varepsilon$, then

$$\text{spectrum } \mathscr{L}_\mathbf{z} \setminus \{\lambda_\mathbf{z}\} \subset \{x \in \mathscr{C} : |x| \le \lambda_{\mathbf{z}_*} - \delta\}. \tag{A1.5}$$

Note. weak-$*$ analytic means that for each $g \in \mathscr{F}_\rho$ $\mathbf{z} \to \int g\,d\nu_\mathbf{z}$ is analytic. It is generally impossible for $\mathbf{z} \to \nu_\mathbf{z}$ to be analytic in the total variation norm topology, since usually $\nu_\mathbf{z} \perp \nu'_\mathbf{z}$ for $\mathbf{z} \neq \mathbf{z}'$.

Proposition 4 follows from Theorem A and B and standard results in regular perturbation theory (cf. [8, Chap. 7, No. 1, Chap. 4, No. 3]).

PROPOSITION 5. *There exists a neighborhood Ω' of $\mathscr{R}^k$ in $\mathscr{C}^k$ with the following property: for every compact $K \subset \Omega'$ there exists $\varepsilon = \varepsilon(K) > 0$ such that*

$$(1+\varepsilon)^n \left\| \lambda_\mathbf{z}^{-n} \mathscr{L}_\mathbf{z}^n g - \left(\int g\,d\nu_\mathbf{z} \right) h_\mathbf{z} \right\|_\rho \to 0 \tag{A1.6}$$

uniformly for $\mathbf{z} \in K$ and g such that $\|g\|_\rho \leq 1$.

Proof. It suffices to show that for each $\mathbf{z}^* \in \mathscr{R}^k$ there is a neighborhood $\{\mathbf{z} \in \mathscr{C}^k \colon |\mathbf{z} - \mathbf{z}^*| \leq \alpha\} = D_\alpha$ on which (A1.5) holds. Choose $\alpha > 0$ so small that (A1.5) holds for $\mathbf{z} \in D_\alpha$, and such that $|\lambda_\mathbf{z} - \lambda_\mathbf{z}^*| < \delta/2$ for $\mathbf{z} \in D_\alpha$. By (A1.1) and (A1.3) the subspaces $\{x h_\mathbf{z}, x \in \mathscr{C}\}$ and $\{g \in \mathscr{F}_\rho^+ \colon \int g\,d\nu_\mathbf{z} = 0\}$ of $\mathscr{F}_\rho^+$ are invariant under $\mathscr{L}_\mathbf{z}$. Consequently (cf. [8, III. 5.6]),

$$\mathscr{L}_\mathbf{z} = \mathscr{L}_\mathbf{z}' + \mathscr{L}_\mathbf{z}'', \qquad \mathbf{z} \in D_\alpha,$$

$$\mathscr{L}_\mathbf{z}' g = \left(\int g\,d\nu_\mathbf{z} \right) \lambda_\mathbf{z} h_\mathbf{z},$$

$$\text{spectrum } \mathscr{L}_\mathbf{z}'' = \text{spectrum } \mathscr{L}_\mathbf{z} \setminus \{\lambda_\mathbf{z}\},$$

and the map $\mathbf{z} \to \mathscr{L}_\mathbf{z}''$ is analytic in the operator norm topology (cf. [8, VIII. 1.3, Theorem 1.7]).

It now follows that if $\mathbf{z} \in D_\alpha$, then

$$\text{spectrum } \mathscr{L}_\mathbf{z}'' \subset \{x \in \mathscr{C} \colon |x| \leq |\lambda_\mathbf{z}| - \delta/2\}.$$

By the spectral radius formula, if $\mathbf{z} \in D_\alpha$ then

$$\lim_{n \to \infty} \left\| (\mathscr{L}_\mathbf{z}'')^n \right\|^{1/n} \leq |\lambda_\mathbf{z}| - \delta/2, \qquad \mathbf{z} \in D_\alpha.$$

The result (A1.6) follows easily from this. □

PROPOSITION 6. *Assume that if $\sum_{i=1}^k a_i f_i - c$ is homologous to an integer-valued function then $a_1 = a_2 = \cdots = a_k = 0$. Then for every compact subset K of $\mathscr{C}^k$ disjoint from $\mathscr{R}^k$ there exists $\varepsilon = \varepsilon(K) > 0$ such that*

$$\lim_{n \to \infty} (1+\varepsilon)^n \|\lambda_{\mathrm{Re}\,\mathbf{z}}^{-n} \mathscr{L}_\mathbf{z}^n g\|_\rho = 0 \tag{A1.7}$$

uniformly for $\mathbf{z} \in K$ and $g \in \mathscr{F}_\rho^+$ such that $\|g\|_\rho \leq 1$.

Proof. By Theorem C the spectral radius of $\mathscr{L}_{\mathbf{z}}$ is strictly less than $\lambda_{\operatorname{Re} \mathbf{z}}$ whenever $\operatorname{Im} \mathbf{z} \neq \mathbf{0}$. Since the spectral radius of $\mathscr{L}_{\mathbf{z}}$ is an upper semicontinuous function of $\mathbf{z}$ (cf. [8, IV. 3.1, Theorem 3.1]), (A1.7) follows from the spectral radius formula. □

The set of vectors $(a_1, a_2, \ldots, a_k) \in \mathscr{R}^k$ such that $\sum_{i=1}^k a_i f_i \cong c + \psi$ for some $c \in \mathscr{R}$ and integer-valued ψ forms a subgroup of $\mathscr{R}^k$ which I shall denote by $\Gamma(\mathbf{f})$. It can be shown that $\Gamma(\mathbf{f})$ is closed in $\mathscr{R}^k$.

PROPOSITION 7. *For every compact subset K of $\mathscr{C}^k$ disjoint from $\Gamma(\mathbf{f})$ there exists $\varepsilon = \varepsilon(K) > 0$ such that*

$$\lim_{n \to \infty} (1 + \varepsilon)^n \|\lambda_{\operatorname{Re} \mathbf{z}}^{-n} \mathscr{L}_{\mathbf{z}}^n g\|_\rho = 0 \tag{A1.8}$$

uniformly for $\mathbf{z} \in K$ and $\|g\|_\rho \leq 1$.

The proof is essentially the same as that of Proposition 6.

APPENDIX 2: AN APPROXIMATION THEOREM

The following result completes the proof of Theorem 4.

PROPOSITION 8. *Suppose $G_1, G_2, \ldots, G_d \in \mathscr{F}_\rho(\Sigma_A^f)$ are real-valued functions satisfying Hypothesis* B, *and $G \in C(\Sigma_A^f)$ is real-valued. Then there exist real-valued $G^{(n)} \in \mathscr{F}_\rho(\Sigma_A^f)$ such that $\|G^{(n)} - G\|_\infty \to 0$ and such that for each n, $G_1, G_2, \ldots, G_d, G^{(n)}$ satisfy Hypothesis* B.

Without loss of generality it may be assumed that $G \in \mathscr{F}_\rho(\Sigma_A^f)$, because any continuous function may be uniformly approximated by functions in $\mathscr{F}_\rho(\Sigma_A^f)$. Let $g, g_1, g_2, \ldots, g_d \in \mathscr{F}_\rho$ be defined by (5.1); observe that the map $G \to g$ is linear and continuous with respect to the $\|\cdot\|_\infty$ topologies. Recall that $G_1, \ldots, G_d$ satisfy Hypothesis B iff $g_1, \ldots, g_d$ satisfy Hypothesis B. If $G, G_1, G_2, \ldots, G_d$ do not satisfy Hypothesis B then there exist constants $a_0, a_1, \ldots, a_d \in \mathscr{R}$, not all zero, such that $a_0 g + \sum_{i=1}^d a_i g_i$ is homologous to an integer-valued function.

LEMMA 8. *If $g_1, g_2, \ldots, g_d \in \mathscr{F}_\rho$ are real-valued functions satisfying Hypothesis* B *then there exists real-valued $g_0 \in \mathscr{F}_\rho$ such that $g_0, g_1, \ldots, g_d$ satisfy Hypothesis* B.

Observe that if $g_0, g_1, \ldots, g_d$ satisfy Hypothesis B but $a_0 g + \sum_{i=1}^d a_i g_i$ is homologous to an integer-valued function, then for every $n = 1, 2, \ldots$, $(g + n^{-1} g_0), g_1, g_2, \ldots, g_d$ satisfy Hypothesis B. (If $a_0' g + a_0' n^{-1} g_0 + \sum_{i=1}^d a_i' g_i$ is homologous to an integer-valued function, then $a_0' n^{-1} g_0 + \sum_{i=1}^d (a_i' - a_0' a_i / a_0) g_i$ is homologous to an integer-valued function, contradicting Hypothesis B.) Thus Proposition 8 will follow from Lemma 8 provided it is shown that there exists $G_0 \in \mathscr{F}_\rho(\Sigma_A^f)$ such that

$$g_0(\xi) = \int_0^{f(\xi)} G_0(\xi, t)\, dt.$$

This is easy to arrange: let $0 < \varepsilon < \min f(\xi)$ and define

$$G_0(\xi, t) = \begin{cases} t(\varepsilon - t) g_0(\xi)/C(\varepsilon), & 0 \le t \le \varepsilon \\ 0, & t > \varepsilon, \end{cases}$$

$$C(\varepsilon) = \int_0^\varepsilon t(\varepsilon - t)\, dt.$$

It remains to prove Lemma 8. Let V be the vector space over $\mathscr{R}$ consisting of all real-valued functions $\varphi \in \mathscr{F}_\rho$, and let U be the linear subspace of V consisting of all functions $\varphi \in V$ such that a nonzero multiple of φ is homologous to an integer-valued function in V. Then $g_0, g_1, \dots, g_d \in V$ satisfy Hypothesis B iff $g_0, g_1, \dots, g_d$ project to linearly independent vectors in the quotient space V/U. Therefore, Lemma 8 is equivalent to the assertion that V/U is infinite-dimensional.

To prove that V/U is infinite-dimensional it suffices to exhibit an infinite set of linearly independent vectors. Now $\varphi_1, \varphi_2, \dots, \varphi_n \in V$ project to linearly *dependent* vectors in V/U only if there exist $a_1, a_2, \dots, a_n \in \mathscr{R}$ not all zero such that

$$\sum_{i=1}^{n} a_i S_m \varphi_i(\xi) \in \mathscr{Z} \tag{A2.1}$$

for all $\xi \in \Sigma_A$ such that $\sigma^m \xi = \xi$, some $m = 1, 2 \dots$. Therefore, to prove that V/U is infinite-dimensional it suffices to show that for all $n = 1, 2, \dots$, there exist $\varphi_1, \varphi_2, \dots \in V$ such that (A2.1) cannot be achieved simultaneously for all periodic $\xi \in \Sigma_A$.

Let $\alpha \in \mathscr{R}$ be irrational, and let $\xi^{k,i} \in \Sigma_A$, $k = 1, 2, \dots$, $i = 0, 1$ be sequences such that

(i) $\xi^{k,i}$ is periodic with period $m(k)$;

(ii) the pattern $(\xi_1^{k,i}, \xi_2^{k,i}, \dots, \xi_{m(k)}^{k,i})$ does not appear in the sequence ξ^{k^*, i^*} unless $(k, i) = (k^*, i^*)$.

It is easily shown that such sequences always exist. Define $\varphi_1, \varphi_2, \dots$ by

$$\varphi_k(\xi) = \begin{cases} 1 & \text{if } \xi_n = \xi_n^{k,0},\ n = 1, 2, \dots, m(k) \\ \alpha & \text{if } \xi_n = \xi_n^{k,1},\ n = 1, 2, \dots, m(k) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\sum_{k=1}^{n} a_k \varphi_k \in U$; since $\xi^{n,i}$ is periodic with period $m(n)$,

$$\sum_{k=1}^{n} a_k S_{m(n)} \varphi_k(\xi^{n,i}) \in \mathscr{Z}, \qquad i = 0, 1.$$

By construction

$$S_{m(n)}\varphi_k(\xi^{n,i}) = \begin{cases} 1 & \text{if } k = n,\ i = 0 \\ \alpha & \text{if } k = n,\ i = 1 \\ 0 & \text{if } k \neq n; \end{cases}$$

hence it follows that a_n, $a_n\alpha \in \mathscr{Z}$. Since α is irrational, $a_n = 0$. Similarly, $a_1 = a_2 = \cdots = a_n = 0$. Therefore $\varphi_1, \varphi_2, \ldots,$ project to linearly independent vectors in V/U.

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