Return Probabilities for Random Walk on a Half-Line

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Abstract

A *random walk with reflecting zone* on the nonnegative integers is a Markov chain whose transition probabilities $q(x,y)$ are those of a random walk (i.e., $q(x,y) = p(y-x)$) outside a finite set $\{0,1,2,\ldots,K\}$, and such that the distribution $q(x,\cdot)$ stochastically dominates $p(\cdot - x)$ for every $x \in \{0,1,2,\ldots,K\}$. Under mild hypotheses, it is proved that when $\sum xp_x > 0$, the transition probabilities satisfy $q^n(x,y) \sim C_{xy}R^{-n}n^{-\frac{3}{2}}$ as $n \to \infty$, and when $\sum xp_x = 0$, $q^n(x,y) \sim C_{xy}n^{-\frac{1}{2}}$.

1 Introduction

Let X_n be an aperiodic, irreducible Markov chain on the state space $\mathcal{Z}_+ = \{0,1,2,\ldots\}$ with transition probabilities $q(x,y)$. Call X_n a *random walk with a reflecting zone* if there exists a probability distribution $p_x = p(x)$ on the integers and a finite integer $K \geq 0$ such that:

$$q(x,y) = p(y-x) \text{ if } x > K; \tag{1}$$

$$\textstyle\sum_{y \geq w} q(x,y) \geq \sum_{y \geq w} p(y-x) \text{ for all } x,w \in \mathcal{Z}_+. \tag{2}$$

This condition necessitates that $p(x) = 0$ for all $x < -K-1$. The subject of this note is the detailed asymptotics of the return probabilities $P^0\{X_n = 0\}$ as $n \to \infty$. We shall only consider the cases $\sum xp_x = 0$ and $\sum xp_x > 0$, because when $\sum xp_x < 0$ the Markov chain X_n is positive recurrent and so Kolmogorov's theorem implies that $\lim_{n\to\infty} P^0\{X_n = 0\} = \pi_0$ where π_x is the unique stationary probability distribution for the chain. To avoid trivialities and minor complications arising from periodicities,

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we assume that the support of p_x is not contained in a coset of any proper subgroup of the integers, and that there is at least one *negative* integer x such that $p_x > 0$.

The simplest example of a random walk with a reflecting zone on $\mathcal{Z}_+$ is the simple random walk on the integers with reflection at the origin. In this case the asymptotics of the return probabilities are easily deduced from Stirling's formula, since exact formulas involving the binomial coefficients are available (using the "reflection principle"). In general, however, exact combinatorial formulas are not easily obtained. When the reflecting zone consists of more than one point the analysis of return probabilities seems to require much more sophisticated methods.

Random walks with reflecting zones occur in a number of contexts. Two very basic and simple examples are as follows.

Example 1: *Queueing Systems.* Consider a system in which jobs enter a network of servers and remain in a queue until they are begun. Assume that the number of jobs that can be completed in a unit time is bounded above. If the numbers of jobs entering the system at times $1, 2, \ldots$ are independent, identically distributed $\mathcal{Z}_+$-valued random variables and if the operations of the servers are "memoryless" and independent of the queue length then the number X_n of jobs in the system at time n is a random walk with a reflecting zone. More generally, if for some k the distribution of the number of entering jobs and the behavior of the servers is different when $X_n \leq k$ than it is when $X_n > k$ the process X_n is still a random walk with a reflecting zone.

Example 2: *Isotropic Random Walk in a Homogeneous Tree.* Let $\mathcal{T}$ be the set of vertices of a homogeneous tree of degree $d \geq 3$. (A homogeneous tree of degree d is a graph with no cycles in which every vertex has exactly d neighbors). An isotropic random walk in $\mathcal{T}$ is a Markov chain Y_n with state space $\mathcal{T}$ that behaves as follows. At each time $n = 0, 1, 2, \ldots$ the random traveller chooses a nonnegative integer N_n from a fixed probability distribution $\{r_x\}$ on $\mathcal{Z}_+$; given that $Y_n = y$ for some vertex y, the traveller randomly chooses one of the vertices of the tree at distance N_n from y and moves there at time $n+1$. Suppose that the distribution r_x has finite support. If Y_n is an isotropic random walk on $\mathcal{T}$, then the distance X_n from the initial vertex Y_0 is a random walk with a reflecting zone on $\mathcal{Z}_+$. It is easily verified that $\sum xp_x > 0$, since the degree of the tree is greater than 2 (there are more ways to move away from the root node than ways to move closer to it). Thus, the return probability $P^0\{X_n = 0\}$ is a "large deviation" probability, and should be expected to decay at an exponential rate.

The second example suggests a natural generalization of the definition. Instead of requiring that $q(x, y) = p(y - x)$ whenever $x > K$, require only that the total variation distance $\|q(x - \cdot) - p(\cdot - x)\|$ be small for large x; e.g., less than $C\alpha^x$ for some constants $C > 0$ and $\alpha < 1$. Call such a process a *random walk with nonlocalized reflection.* If Y_n is an isotropic random walk on a homogeneous tree such that the distribution $\{r_x\}$ has exponential decay then the distance X_n from the initial vertex is a random walk with nonlocalized reflection on $\mathcal{Z}_+$. Unfortunately, our techniques lead to interesting results only for random walks with a (finite) reflecting zone.

The definition of a random walk with nonlocalized reflection has an obvious analogue on the positive half-line $\mathbf{R}_+$. A *random walk with nonlocalized reflection on* $\mathbf{R}_+$ is a Markov chain on $\mathbf{R}_+$ whose transition densities $q(x,y)$ are such that $\|q(x,\cdot)-p(\cdot-x)\| \leq C\alpha^x$ for some probability density $p(x)$ on $\mathbf{R}$ and some constants $C>0$ and $\alpha<1$, and such that the integral analogue of condition (2) holds. Such processes arise naturally in the study of continuous-time queueing networks. Another interesting class of examples is as follows.

Example 3: *Isotropic Random Walk on the Hyperbolic Plane.* Let $\mathcal{H}$ be the hyperbolic plane, i.e., the upper half-plane $\{(x,y) \in \mathbf{R}^2 : y > 0\}$ equipped with the Poincaré metric ds/y where ds is the usual Euclidean arc length element (see [1]). An isotropic random walk on $\mathcal{H}$ is a Markov chain Y_n with state space $\mathcal{H}$ whose transition probabilities depend only on hyperbolic distance: given that $Y_n = y$, the traveller chooses a positive real D_n from the probability density $r(x)$, then chooses a point Y_{n+1} at random from the uniform distribution on the (hyperbolic) circle of radius D_n centered at y. If the density $r(x)$ decays exponentially at infinity, then the distance X_n from Y_n to the initial point Y_0 is a random walk with nonlocalized reflection on $\mathbf{R}_+$.

The proof is elementary, using only basic properties of hyperbolic geometry. Two points are worth noting. First, even if the density $r(x)$ has compact support, the radial process X_n will not be a random walk with a reflecting zone, but only a random walk with nonlocalized reflection. Second, the radial process always has positive drift: i.e., $\int_{\mathbf{R}} xp(x)\,dx > 0$. This has to do with the "exponential growth" of hyperbolic space.

The hyperbolic plane is a homogeneous space of the group $SL_2(\mathbf{R})$ of two by two real matrices with determinant 1. Specifically, $\mathcal{H} \approx SL_2(\mathbf{R})/K$ where K is the group of rotation matrices

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus, example 3 may be reinterpreted as an isotropic random walk on the matrix group $SL_2(\mathbf{R})$. Other examples may be obtained by looking at isotropic random walks on other semisimple Lie groups of real rank 1.

Henceforth we shall only consider random walks with a (finite) reflecting zone on $\mathcal{Z}_+$. It is almost certain that our main result is true more generally for RW with nonlocalized reflection on either $\mathcal{Z}_+$ or $\mathbf{R}_+$, but our method of proof works only in the simpler case. We suspect that new and undoubtedly very interesting methods will be required for the analysis of random walks with nonlocalized reflection, and one of our reasons for publishing this note is to draw attention to this problem.

Recall that for RW with a reflecting zone $\{0,1,2,\ldots,K\}$ on $\mathcal{Z}_+$ the probability distribution p_x controlling the jumps when $X_n > K$ has support $[-K,\infty) \cap \mathcal{Z}$.

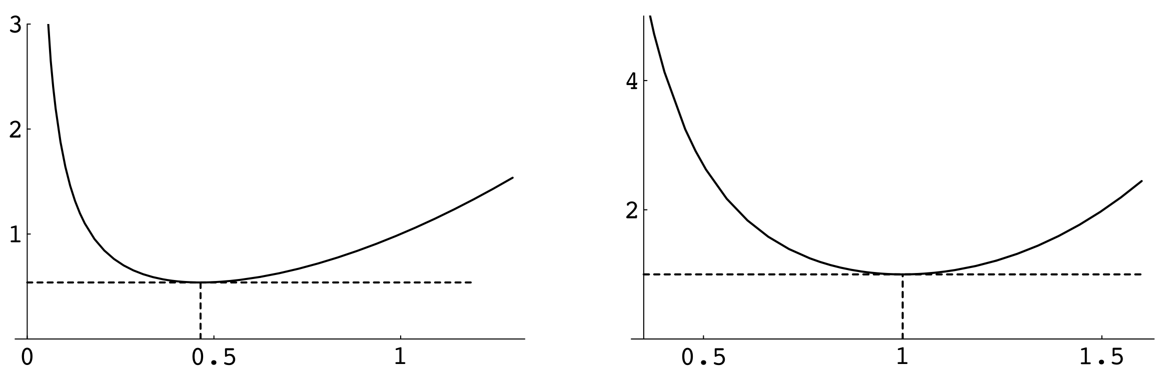
Therefore, the generating function

$$\varphi(t) = \sum p_x t^x \tag{3}$$

is meromorphic in the open unit disk $\mathcal{D} = \{t : |t| < 1\}$ and continuous in the closed disk $\{t : |t| \leq 1\}$, with a pole at the point $t = 0$. We assume that the mean of p_x is nonnegative (but not necessarily finite). Because of our standing assumption that the support of p_x is not contained in a coset of any proper subgroup of the integers, $\lim_{t\to 0+} \varphi(t) = +\infty$. Moreover, since the mean of p_x is nonnegative, the second derivative of φ is positive at every $0 < t < 1$, and hence φ is strictly convex on the unit interval. Define

$$R^{-1} = \min_{0<t\leq 1} \varphi(t). \tag{4}$$

The figure below shows typical shapes for the graph of φ in the cases $\sum x p_x > 0$ and $\sum x p_x = 0$, respectively. The points $(0, R^{-1})$ and $(t_*, 0)$ are the intersections of the dashed horizontal line with the $y-$axis and the dashed vertical line with the $x-$ axis, respectively.



The minimum is attained at only one $t \in (0, 1)$, by the strict convexity of φ; denote this point by t_*. If the mean of p_x is positive or infinite, then $0 < t_* < 1$ and $R^{-1} < 1$, because $\varphi'(1) = \sum x p_x > 0$, but if $\sum x p_x = 0$ then $t_* = R = 1$. Our main result is

Theorem 1.1 *If $\sum x p_x > 0$ then there exist constants $0 < C_{xy} < \infty$ such that as $n \to \infty$,*

$$P^x\{X_n = y\} \sim C_{xy} R^{-n} n^{-\frac{3}{2}}. \tag{5}$$

If $\sum x p_x = 0$ and if for some $t > 1$, $\sum t^y q(x, y) < \infty$ $\forall x \in \mathcal{Z}_+$ then there exist constants $0 < C_{xy} < \infty$ such that as $n \to \infty$

$$P^x\{X_n = y\} \sim C_{xy} n^{-\frac{1}{2}}. \tag{6}$$

Note: The hypothesis $\sum t^y q(x,y) < \infty$ for some $t > 1$ implies that $\sum t^x p_x < \infty$, in particular, that the distribution p_x has exponentially decaying right tail. It is likely that in the case $\sum x p_x = 0$ the conclusion (6) holds under somewhat weaker moment hypotheses.

For isotropic random walk in a homogeneous tree the local limit theorem (5) was proved by Sawyer [7] by entirely different techniques (Cartier's theory of spherical functions for homogeneous trees). For isotropic random walk on a rank one semisimple Lie group an analogous local limit theorem was proved by Bougerol [3]; Bougerol's methods even extend to the nonisotropic case, but use *very* heavy machinery, to wit, the Plancherel formula for such groups. Of course, in general a random walk with a reflecting zone on $\mathcal{Z}_+$ cannot be realized as the radial part of an isotropic random walk on a tree, so Sawyer's result does not imply (5), nor can his techniques be adapted to prove it.

Our method of analysis is based on generating functions. We will use the Markov property to obtain functional equations for the Green's function(s) of the random walk, and Wiener-Hopf factorization to show that R is a branch point of order 2. This use of Wiener-Hopf factorization seems limited to problems with a finite number of scatterers.

We have discussed the role of the hypothesis (1) and commented on the desirability of weakening it, but as yet we have not mentioned hypothesis (2). This assures that whenever X_n visits one of the "reflective" sites $0, 1, \ldots, K$ it gets a (nonnegative) "kick" at least as great as the increment that would result from simply choosing from the distribution p_x. Does the theorem really require it? Yes! Suppose , for instance, that $K = 0$, so that 0 is the only reflective site and $p_x = 0$ for all $x < -1$. If $q(0,0)$ were greater than R^{-1} then the probability of being at 0 after n steps would decay like $Cq(0,0)^n$ rather than $CR^{-n}n^{-\frac{3}{2}}$. One may easily check that $q(0,0) > R^{-1}$ is impossible if hypothesis (2) is satisfied.

2 Green's Functions

Information about the transition probabilities will be recovered from their generating functions, the so-called *Green's functions.* These are defined as follows: for $x, y \in \mathcal{Z}_+$ and $0 \leq |s| < 1$,

$$G_{xy}(s) = \sum_{n=0}^{\infty} P^x\{X_n = y\} s^n = \sum_{n=0}^{\infty} q^n(x,y) s^n. \tag{7}$$

Each series is absolutely convergent in the open unit disk, since its coefficients are probabilities, and defines an analytic function. When $\sum x p_x > 0$ the radius R of convergence of G_{xy} is actually larger than 1, as will be shown; the analytic continuation has the same power series in $\{0 \leq |s| < R\}$.

Recovering information about the asymptotic behavior of probabilities from their generating functions is often accomplished by resorting to a Tauberian theorem, e.g., that of Karamata. (See, for example, the chapters on random walk problems in [4], and the final chapter of [8], where numerous uses of the Karamata theorem are made.) Unfortunately, we have no way of obtaining the necessary *a priori* information about the probabilities to apply such a Tauberian theorem. Instead, we shall employ the following theorem of Darboux (see [2]).

Theorem 2.1 *(Darboux) Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients a_n and radius of convergence R. Suppose that G has no singularities in the closed disk $|z| \leq R$* except $z = R$ *(i.e.,G has an analytic continuation to an open domain Ω such that $\{|z| \leq R\} \subseteq \Omega \cup \{R\}$), and that in a neighborhood of $z = R$,*

$$G(z) = A(z)(R - z)^{\alpha} + B(z) \tag{8}$$

where $A(z)$ and $B(z)$ are analytic functions. Then as $n \to \infty$,

$$a_n \sim \frac{-A(R)R^{1-n}}{\Gamma(-\alpha)n^{1+\alpha}} \tag{9}$$

Note: In equation (8), the positive branch of the function z^{α} is meant. The implied branch cut is along the negative axis. Thus, the branch cut for the function $G(z)$ in (8) is along the halfline $[R, \infty)$.

The use of the Darboux theorem requires more regularity of the generating function in a neighborhood of the singular point $z = R$ than does Karamata's theorem, and the bulk of the paper will be devoted to verifying its hypotheses. Showing that a function G has the behavior indicated in (8) in the vicinity of a point R with $\alpha = \frac{1}{2}$ is tantamount to showing that it has a branch point of order 2. This, in turn, is equivalent to showing that it is (locally) a branch of the inverse of a function that is locally two-to-one in a neighborhood of $G(R)$.

The main steps in the argument will be to show that all the Green's functions G_{xy} have the same radius R_c of convergence, that none has singularities on the circle of convergence other than that at $s = R_c$, and that near $s = R_c$ each satisfies (8) with $\alpha = \frac{1}{2}$ when $R_c > 1$ and $\alpha = -\frac{1}{2}$ when $R_c = 1$. Since each of the power series G_{xy} has nonnegative coefficients, the radius R_c of convergence is itself *necessarily* a singular point, by a well known result of complex variable theory.

That all of the power series G_{xy} have the same radius of convergence is elementary, requiring only a "Harnack" type inequality. Keep in mind that for a power series $\sum a_x s^x$ with nonnegative coefficients the radius of convergence is $\inf\{s > 0 : \sum a_x s^x = \infty\}$.

Lemma 2.2 *For any $x, y, x', y' \in \mathcal{Z}_+$ there exist a positive constant ε and integers k, n_* such that for all $n \geq n_*$,*

$$q^{n+k}(x, y) \geq \varepsilon q^n(x', y'). \tag{10}$$

Consequently, all of the power series G_{xy} have the same radius of convergence.

Proof: By assumption, the Markov chain X_n is irreducible. Thus, there exist positive probability paths from x to x' and from y' to y. By the Chapman-Kolmogorov equations, $q^{n+i+j}(x,y) \geq q^i(x,x')q^n(x',y')q^j(y',y)$. Since i,j can be chosen so as to make $q^i(x,x')q^j(y',y) = \varepsilon > 0$, the inequality (10) follows. ///

The remaining steps in the argument will exploit certain simple interrelationships among the Green's functions that derive from the Markov property. The first of these is a consequence of the one-step Markov property. Recall that the Green's functions record the expected number of discounted visits to the various sites. In order that $X_{n+1} = y$ it is necessary that $X_n = w$ for some state w and that the chain makes the jump from state w to state y at time $n+1$. Thus,

$$\boxed{G_{xy}(s) = \delta_{xy} + \textstyle\sum_{w=0}^{\infty} sG_{xw}(s)q(w,y).} \tag{11}$$

The second set of interrelationships among the Green's functions also follow from the Markov property, but in a slightly more sophisticated way. Fix an integer $L \geq K \vee x$. In order that $X_n = y$ for some integer $y \leq L$ it must be the case either that $n = 0$ or that for some $m < n$, $X_m = w$ for some $w \leq L$ *and* $X_k > L$ for all $m < k < n$. Consequently, for each $y \leq L$,

$$\boxed{G_{xy}(s) = \delta_{xy} + \textstyle\sum_{0 \leq w \leq L} G_{xw}(s)H_{wy}(s)} \tag{12}$$

where

$$H_{wy}(s) = H^L_{wy}(s) = E^w s^T 1_{\{X_T = y\}}, \tag{13}$$

$$T = T_L = \inf\{n \geq 1 \,:\, X_n \leq L\} \tag{14}$$

In fact, each of the systems of equations (11) and (12) may be written as a single matrix equation involving matrix-valued generating functions. It is especially advantageous to consider the system (12) this way, because the matrices involved are finite. Thus, for any x and any $L \geq K \vee x$,

$$\mathcal{G}_x(s) = u_x + \mathcal{H}(s)\mathcal{G}_x(s)$$

where u_x is the $(L+1) \times 1$ unit vector with entry 1 in the x^{th} position and zeroes elsewhere; $\mathcal{G}_x(s)$ is the $(L+1) \times 1$ vector-valued function with $G_{xy}(s)$ as the y^{th} entry; and $\mathcal{H}(s)$ is the $(L+1) \times (L+1)$ matrix-valued function with entries $H_{wy}(s)$. This shows that the Green's functions $G_{xy}(s)$ are in fact completely determined by the matrix-valued function $\mathcal{H}(s)$:

$$\boxed{\mathcal{G}_x(s) = (\mathcal{I} - \mathcal{H}(s))^{-1}u_x} \tag{15}$$

at least for sufficiently small s, because the power series for $H_{wy}(s)$ contains no constant term, and hence the matrix norm of $\mathcal{H}(s)$ is small for s near 0, which implies that $\mathcal{I} - \mathcal{H}(s)$ is invertible.

In the subsequent sections we will investigate the analyticity properties of the entries of $\mathcal{H}(s)$ using the machinery of Wiener-Hopf factorization. We will see that $\mathcal{H}(s)$ is analytic in the disk $|s| < R = 1/\varphi(t_*)$, and has a branch point singularity at $s = 1/\varphi(t_*)$. This does not, of course, preclude the possibility that $\mathcal{I} - \mathcal{H}(s)$ is noninvertible for some s in the disk. However, the relations (11) and the hypothesis (2) assure that $\mathcal{I} - \mathcal{H}(s)$ must be invertible in the disk $|s| < \varphi(t_*)$, as we now show.

Proposition 2.3 *Let R_c be the (common) radius of convergence of the power series G_{xy}. If $R_c < R = \varphi(t_*)^{-1}$ or if $R_c = R$ and $t_* < 1$ then for each pair $x, y \in \mathcal{Z}_+$,*

$$G_{xy}(R_c) = \lim_{s \to R_c-} G_{xy}(s) < \infty, . \tag{16}$$

and consequently the singularity $s = R_c$ is not a pole.

Note: Later we will see that when $t_* = 1$ (equivalently, when $\sum x p_x = 0$), the radius of convergence of the Green's functions is $R = 1$ and $G_{xy}(R) = \infty$.

Proof: Define new generating functions

$$\begin{aligned} \mathcal{A}_x(s,t) &= \sum_{y=0}^{\infty} t^y G_{xy}(s) & (17) \\ Q_y(t) &= \sum_{w \in \mathcal{Z}_+} q(y,w) t^w & (18) \end{aligned}$$

(recall that $q(y, w)$ are the transition probabilities of the Markov chain X_n). Observe that these power series converge absolutely for s, t in the open unit disk. Multiplying equations (11) by t^y and summing over y, one obtains a functional equation for $\mathcal{A}_x$:

$$\mathcal{A}_x(s,t)(1 - s\varphi(t)) = t^x + \sum_{0 \le y \le K} s G_{xy}(s)(Q_y(t) - t^y \varphi(t)). \tag{19}$$

The sum on the right side extends only over $y = 0, 1, 2, \ldots, K$ because of (1). By the law of permanence for functional equations, (19) remains valid for any analytic continuations of the generating functions $\mathcal{A}_x$ and G_{xy}.

In Lemma 2.4 below we will show that (2) implies that for all $0 < t < 1$,

$$Q_y(t) \le t^y \varphi(t)$$

with strict inequality unless $q(y, y') = p(y' - y)$ for all $y' \in \mathcal{Z}$. Strict inequality must hold for some of the sites y, for instance $y = 0$, because we have assumed that $p_x > 0$ for at least one negative integer x, and $q(y, y') = 0$ for $y' < 0$. Now consider what happens in (19) as s increases. Assume first that $t_* < 1$; then for at least one y between 0 and K, $Q_y(t_*) - t_*^y \varphi(t_*) < 0$. As long as s remains less than both R_c and R, both factors on the left side of (19) are positive; consequently, the right side is also positive:

$$t_*^x + \sum_{0 \le y \le K} sG_{xy}(s)(Q_y(t_*) - t_*^y \varphi(t_*)) > 0.$$

But each factor $(Q_y(t_*) - t_*^y \varphi(t_*)) \le 0$, and at least one is strictly negative, by the previous paragraph. Hence, there is at least one y for which $G_{xy}(s)$ remains bounded for $0 \le s < 1/\varphi(t_*)$. But (10) then implies that $G_{xy}(s)$ remains bounded for *all* $x, y \in \mathcal{Z}_+$.

A similar argument works when $t_* = 1$ and $R < 1$. In this case, choose $t < 1$ sufficiently close to 1 that $R\varphi(t) < 1$; such a t exists because $\varphi(1) = 1$ and φ is continuous. As above, $Q_y(t) \le t^y \varphi(t)$ for all $y = 0, 1, \ldots, K$, and there is strict inequality for at least one y. Let s increase to R; note that both factors on the right side of (19) are positive. This implies that

$$t^x + \sum_{0 \le y \le K} sG_{xy}(s)(Q_y(t) - t^y \varphi(t)) > 0$$

for all $s < R$. Hence, $G_{xy}(R) < \infty$ for at least one, and therefore every y. ///

Lemma 2.4 *Let $Q_y(t)$ and $\varphi(t)$ be defined by (18) and (3), respectively. Under hypothesis (2), for every $0 < t < 1$,*

$$Q_y(t) \le t^y \varphi(t) \tag{20}$$

with strict inequality unless $q(y, y') = p(y' - y)$ for every $y' \in \mathcal{Z}$. Moreover, for every $0 < t < 1$,

$$Q'_y(t) > 0. \tag{21}$$

Proof: Suppose that $\{a_n\}_{n \in \mathcal{Z}}$ and $\{b_n\}_{n \in \mathcal{Z}}$ are two probability distributions on the integers such that $\{a_n\}$ stochastically dominates $\{b_n\}$, i.e., $\sum_{n \ge m} a_n \ge \sum_{n \ge m} b_n$ for every integer m. Then a standard and elementary result in stochastic ordering implies that on some probability space there exists a random vector (U, V) such that

$$\begin{aligned} P\{U = n\} &= a_n, \\ P\{V = n\} &= b_n, \quad \text{and} \\ P\{U \ge V\} &= 1. \end{aligned}$$

Clearly, $P\{U > V\} > 0$ unless $a_n = b_n$ for all n. It is then obvious that for every $t \in (0, 1)$, $Et^U \le Et^V$, with strict inequality unless $a_n = b_n$ for all n. This may be rewritten as $\sum a_n t^n \le \sum b_n t^n$. Equation (20) follows.

That $Q'_y(t) \geq 0$ for all $t \in (0,1)$ is a consequence of the hypothesis that $q(y,y') = 0$ for $y' < 0$. That strict inequality holds follows because if $q(y,0) = 1$ for some y then hypothesis (2) would be impossible, since the distribution $p(x)$ puts positive mass on both the positive and negative integers. ///

Although we can go no further with the line of reasoning used in the proof of Proposition 2.3 in the general case, it is worth mentioning that in the special case where $K = 0$ (i.e., where there is only one reflector) relation (19) can be used to give a relatively simple proof of (5)-(6) and to give a completely explicit formula for the generating function G_{x0}.

Proof of (5)-(6) for $K = 0$**:** When $K = 0$, (19) simplifies to

$$\mathcal{A}_x(s,t)(1 - s\varphi(t)) = t^x + sG_{x0}(s)(Q_0(t) - \varphi(t)). \tag{22}$$

For each $0 < s < 1$ there is a unique solution $t = \zeta_1(s)$ of the equation $\varphi(t) = 1/s$ satisfying $0 < t < t_*$ (recall that $\varphi(t)$ is continuous and strictly decreasing for $0 < t < t_*$, and $\lim_{t \to 0+} \varphi(t) = \infty$). Clearly, $\mathcal{A}_x(s, \zeta_1(s)) < \infty$, since $\mathcal{A}_x$ is a power series in s, t with coefficients between 0 and 1, and since $0 < s < 1$ and $0 < t < t_* \leq 1$. Consequently, the right side of (22) must be zero at $(s, \zeta_1(s))$; solving for $G_{x0}(s)$ yields

$$sG_{x0}(s) = \frac{\zeta_1(s)^x}{\varphi(\zeta_1(s)) - Q_0(\zeta_1(s))}. \tag{23}$$

Observe that explicit formulas for all of the other Green's functions may now be obtained from (22) by equating coefficients of powers of t.

By the law of permanence, this functional equation persists up to the smallest positive singularity of the right side. (NOTE: Keep in mind that this is only valid for the case $K = 0$.) But both φ and Q_0 are analytic in the unit disk and $\varphi(t) - Q_0(t) > 0$ for all $0 < t < 1$, so the smallest positive singularity of G_{x0} occurs at the smallest positive singularity of $\zeta_1(s)$, if $\zeta_1(s) < 1$, otherwise at $s = 1$. Now $\zeta_1(s)$ is defined implicitly by the equation $s\varphi(\zeta_1(s)) = 1$; by the implicit function theorem (complex version) a singularity of $\zeta_1(s)$ can occur only when $\varphi'(\zeta_1(s)) = 0$. The *only* positive s for which $\varphi'(\zeta_1(s)) = 0$ is $s = R$, since the only positive value of t such that $\varphi'(t) = 0$ is $t = t_*$. Thus, the radius of convergence of G_{x0} is $R = 1/\varphi(t_*)$.

There are now two cases. First, suppose that $t_* < 1$. Then since $\varphi''(t_*) > 0$, the point $s = R$ is a branch point of $\zeta_1(s)$ of order 2 (see the proof of Proposition 3.8 below). Because $\varphi(t) - Q_0(t) > 0$ and $\varphi'(t) - Q'_0(t) < 0$ in a neighborhood of $t = t_*$, it follows that $s = R$ is a branch point of G_{x0} of order 2, in particular, (8) holds with $\alpha = \frac{1}{2}$. Moreover, (22) now implies that each of the Green's functions G_{xy} has a branch point at $s = R$ of order 2. In light of Darboux's theorem, this proves (5) in the special case where there is only one reflective site.

Next, suppose that $t_* = 1$. Again, since $\varphi'(t_*) = 0$ and $\varphi''(t_*) > 0$, the point $s = R$ is a branch point of $\zeta_1(s)$ of order 2. However, in this case the denominator in (23) is zero at the singular point. But note that $\varphi'(1) - Q'_0(1) \neq 0$, because $\varphi'(1) = 0$

and $Q_0'(t) > 0 \; \forall t > 0$.Thus, it follows from (23) that near the singular point $s = 1$ the equation (8) holds for $G = G_{x0}$ with $\alpha = -\frac{1}{2}$. By (22), the same is true for all the Green's functions G_{xy}. By Darboux's theorem, this proves (6) in the case where there is only one reflective site. ///

3 Wiener-Hopf Factorization

Further exploitation of the equation(s) (15) requires more information about the analyticity properties of the entries of the matrix-valued function $\mathcal{H}(s)$. As we will see, these entries may be expressed in terms of the generating functions of the *ladder variables* of the random walk on $\mathcal{Z}$ with step distribution p_x. In this section we will investigate the generating functions of various first passage times for this random walk, beginning with the ladder indices. The key to this analysis is the use of *Wiener-Hopf factorization* ([8], ch. 4).

Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. random variables each with distribution p_x, and set $S_n = \sum_{j=1}^{n} Y_j$, with $S_0 = 0$. Define the *ladder times* τ_+, τ_- and *ladder heights* Z_+, Z_- as follows:

$$\tau_+ = \inf\{n > 0 \; : \; S_n \geq 0\}; \tag{24}$$
$$\tau_- = \inf\{n > 0 \; : \; S_n < 0\}; \tag{25}$$
$$Z_+ = S_{\tau_+} \text{ on } \{\tau_+ < \infty\}; \tag{26}$$
$$Z_- = S_{\tau_-} \text{ on } \{\tau_- < \infty\}. \tag{27}$$

The Wiener-Hopf factorization of the potential kernel is the following relation:

$$1 - s\varphi(z) = (1 - Es^{\tau_+}z^{Z_+})(1 - Es^{\tau_-}z^{Z_-}), \tag{28}$$

valid for all $|s| < 1$ and $|z| = 1$. (See [8], ch. 4, sec. 17, P4-5. Spitzer's definition of the ladder variables is slightly different from that used here.) The expectations on the right are integrals over the events where the relevant ladder variables are defined, e.g., $Es^{\tau_+}z^{Z_+} = Es^{\tau_+}z^{Z_+}1\{\tau_+ < \infty\}$. Observe that for each fixed s the factors on the right side of (28) extend analytically to bounded functions of z in $|z| \leq 1$ and $|z| \geq 1$, respectively. Moreover, for $0 < s < 1$, $1 - Es^{\tau_+}z^{Z_+}$ has no zeroes in $|z| < 1$ and $1 - Es^{\tau_-}z^{Z_-}$ has no zeroes in $|z| > 1$. The factorization (28) into bounded functions analytic in the interior and exterior of the unit circle is essentially unique in the following sense (see [8], ch. 4, sec. 17, P3): If, for some s,

$$1 - s\varphi(z) = g_+(z)g_-(z) = h_+(z)h_-(z)$$

are factorizations into bounded interior and exterior functions such that $h_+(z) = 0$ implies $g_+(z) = 0$ for $|z| < 1$ and such that $g_-(z) = 0$ implies $h_-(z) = 0$ for $|z| > 1$, then for some constant c, $g_+ = ch_+$ and $h_- = cg_-$. This is an elementary consequence of Liouville's theorem.

Recall that the distribution p_x is concentrated on the set of integers greater than $-K-2$. Define M to be the greatest integer such that $p_{-M} > 0$; then M is well-defined and $M \leq K+1$. Since we have assumed that $p_x > 0$ for at least one negative integer x, $M > 0$. By definition of M, the random walk S_n makes no negative jumps of size smaller than $-M$. It follows that $Z_- \geq -M$ with probability one, and that $P\{Z_- = -M\} > 0$. This special property of Z_- allows us to give an explicit description of the factor $1 - Es^{\tau_-} z^{Z_-}$ in the Wiener-Hopf factorization.

Proposition 3.1 *For each s satisfying $|s| < 1$ the function $z^M - sz^M\varphi(z)$ has precisely M zeroes $\zeta_1(s), \zeta_2(s), \ldots, \zeta_M(s)$ in the unit disk, counted according to multiplicity, and*

$$1 - Es^{\tau_-} z^{Z_-} = \prod_{i=1}^{M} \left(1 - \frac{\zeta_i(s)}{z}\right). \tag{29}$$

Proof: That $z^M - sz^M\varphi(z)$ has precisely M zeroes in the unit disk is a straightforward consequence of the argument principle (or Rouche's theorem). Since $|s| < 1$ and $\varphi(e^{i\theta}) \leq 1 \, \forall \, \theta \in [0, 2\pi]$, the curve $\theta \to e^{iM\theta} - se^{iM\theta}\varphi(e^{i\theta})$ winds around the origin M times. Therefore, by the argument principle, the function $z^M - sz^M\varphi(z)$ has precisely M zeroes in the unit disk.

Now consider the following factorization of the potential kernel:

$$1 - s\varphi(z) = \left(\prod_{i=1}^{M}(1 - \frac{\zeta_i(s)}{z})\right)\left(\frac{z^M - sz^M\varphi(z)}{\prod_{i=1}^{M}(z^M - \zeta_i(s))}\right),$$

valid for all z on the unit circle. The first factor extends analytically to a bounded function for z in the exterior of the unit circle, while the second factor is analytic for z in the interior of the unit circle. Moreover, the first factor has no zeroes in $|z| > 1$, and the second factor has no zeroes in $|z| < 1$. Consequently, by the essential uniqueness of the Wiener-Hopf factorization, the factors are constant multiples of the factors in (28). That the constant is 1 may be verifed by checking the behavior as $|z| \to \infty$. ///

For each fixed s in the unit disk, formula (29) exhibits $z^M(1 - Es^{\tau_-} z^{Z_-})$ as a polynomial of degree M with zeroes $\zeta_1(s), \zeta_2(s), \ldots, \zeta_M(s)$. Consequently, the coefficients in this polynomial are the elementary symmetric polynomials in the zeroes $\zeta_i(s)$ (see [5], Ch. 5), i.e.,

$$h_j(s) = Es^{\tau_-} 1_{\{Z_- = -j\}} = (-1)^{j-1}\sigma_j(\zeta_1(s), \zeta_2(s), \ldots, \zeta_M(s)) \tag{30}$$

where

$$\begin{aligned}
\sigma_1(t_1, t_2, \ldots, t_M) &= \sum_{1 \leq i \leq M} t_i \\
\sigma_2(t_1, t_2, \ldots, t_M) &= \sum_{1 \leq i < j \leq M} t_i t_j \\
&\cdots \\
\sigma_M(t_1, t_2, \ldots, t_M) &= t_1 t_2 \ldots t_M.
\end{aligned}$$

Clearly, equation (30) persists for s beyond the unit circle along any curve free of singularities of $\zeta_i(s)$. Thus, the location and nature of the singularities of the functions $h_j(s)$ is determined by the singularities of the functions $\zeta_i(s)$. Before studying these zeroes in more detail, we record the following property of the coefficients in the power series

$$h_j(s) = \sum_{n=1}^{\infty} P\{\tau_- = n;\, Z_- = -j\} s^n.$$

Lemma 3.2 *There exist an integer $k \geq 1$ and a real number $\delta > 0$ such that for each pair $i, j \in \{1, 2, \ldots, M\}$, and every $n \geq 1$,*

$$P\{\tau_- = n + k;\, Z_- = -i\} \geq \delta P\{\tau_- = n;\, Z_- = -j\}. \tag{31}$$

Moreover, there exists an integer n_* such that for every $n \geq n_*$ and every j,

$$P\{\tau_- = n;\, Z_- = -j\} > 0 \tag{32}$$

Proof: This is, in essence, a consequence of our hypotheses that the distribution p_x, $x \in \mathcal{Z}$, has nonnegative mean, does not concentrate all of its mass on $[0, \infty)$, and is not concentrated on any *lattice* (a coset of a proper subgroup of $\mathcal{Z}$). These hypotheses imply (a) that for any $x \in \mathcal{Z}$ there exists a positive probability path from 0 to x, and (b) that there exists an integer k_* such that for all $n \geq k_*$ there exists a positive probability path from 0 to 0 of length n. It follows that for *every* sufficiently large integer k_1, there exists, for each pair $i', j' \in \{0, 1, \ldots, M\}$, a positive probability path of length k_1 from i' to j'. Fix k_1, and choose one such path for each pair. Since there are only finitely many paths, they have a finite minimum $-\Gamma$. Consequently, for every interval J of length $M + 1$ contained in $[\Gamma, \infty)$ there exists, for every pair $i, j \in J$, a positive probability path $\gamma''_{ij}$of length k_1 from i to j that does not enter $(-\infty, -1]$.

Because the distribution p_x, $x \in \mathcal{Z}$, has nonnegative mean, there is a positive probability path from 0 to $[\Gamma, \infty)$ that does not enter the negative integers. We may assume this path γ ends at an integer multiple $nM \geq \Gamma$ of M, by suffixing if necessary a path from an i to a j as constructed in the preceding paragraph. Translation of this path by j units to the right, for any $j \in [0, M - 1]$ gives a positive probability path from j to $nM + j$ that does not enter the negative integers. Clearly, there is a positive probability path γ' from nM to 0 that does not enter the negative integers, since $p_{-M} > 0$; translation of this path by j units to the right, for any $j \in [0, M - 1]$, gives a positive probability from $nM + j$ to j that does not enter the negative integers. By concatenating (i) a translate of γ; (ii) an appropriate γ''_{ij}, as constructed in the preceding paragraph; and (iii) a translate of γ', we may build, for any pair $i', j' \in \{0, 1, \ldots, M - 1\}$, a positive probability path from i' to j' that does not enter the negative integers. *Moreover,* all of these paths have the same length k', by construction. Let $\delta' > 0$ be the minimum probability of these paths.

In order that $\tau_- = n$ and $Z_- = -j$, it must be the case that $S_{n-1} \in \{0, 1, \ldots, M - 1\}$. Since there is a path of length k' from any $i' \in \{0, 1, \ldots, M-1\}$ to $M - j$ with probability $\geq \delta'$ that does not enter the negative integers, and since the probability of going from $M - j$ to $-j$ in one step is $p_{-M} > 0$, it follows that (31) holds with $\delta = p_{-M}\delta'$.

Finally, recall that k_1 could be chosen arbitrarily large in the construction of the paths γ''_{ij} (first paragraph of the proof). Therefore, for some k_* there exist positive probability paths of arbitrary length $\geq k_*$ that do not enter the negative integers from any $i' \in \{0, 1, \ldots, M-1\}$ to any $M - j \in \{0, 1, \ldots, M-1\}$. Consequently, an adjustment of the argument of the preceding paragraph proves (32). ///

The next three lemmas provide necessary information about the zeroes of the function $z^M(1 - s\varphi(z))$ for $1 \leq |s| \leq R$.

Lemma 3.3 *For each $0 < s \leq 1/\varphi(t_*)$ there is a unique solution $z = \zeta_1(s)$ of $1 = s\varphi(z)$ satisfying $0 < z \leq t_*$, and there are precisely $M - 1$ zeroes of $1 = s\varphi(z)$ satisfying $0 < |z| < \zeta_1(s)$. There are no zeroes other than $z = \zeta_1(s)$ on the circle $|z| = \zeta_1(s)$.*

Proof: Recall that $\varphi(t)$ is strictly convex for $0 < t < 1$ with a unique minimum at $t = t_* \leq 1$, and that $\varphi(t) \to \infty$ as $t \to 0+$. The existence and uniqueness of $\zeta_1(s)$ for $0 < s \leq 1/\varphi(t_*)$ follow from the intermediate value theorem and the fact that convexity forces $\varphi(t)$ to decrease for $0 < t \leq t_*$. (See the figure in section 1.) Notice that the zero $\zeta_1(s)$ is a simple zero for $0 < s < \varphi(t_*)$, since $\varphi' < 0$ on $(0, t_*)$, but that at $s = \varphi(t_*)$ it becomes a double zero, since $\varphi'(t_*) = 0$ and $\varphi''(t_*) > 0$; this reflects the fact that as $s \to \varphi(t_*)-$ the zero $\zeta_1(s)$ approaches t_*, where it collides with another zero that has been approaching t_* from above.

Next, recall that the probability distribution p_x is not supported by a coset of any proper subgroup of the integers. Consequently, for each $0 < t \leq 1$ and each $0 < \theta < 2\pi$,

$$|\varphi(te^{i\theta})| < \varphi(t),$$

and hence there are no zeroes of $1 = s\varphi(z)$ on the circle $|z| = \zeta_1(s)$ other than $\zeta_1(s)$.

Suppose that $0 < s < 1/\varphi(t_*)$; then $0 < \zeta_1(s) < t_*$ and for all small $\varepsilon > 0$, $\varphi(\zeta_1(s) + \varepsilon) < \varphi(\zeta_1(s))$. It follows that as z traverses the circle $|z| = \zeta_1(s) + \varepsilon$ counterclockwise, the curve $z^M - sz^M\varphi(z)$ winds around the origin M times. The argument principle now implies that there are precisely M zeroes inside this circle. Letting $\varepsilon \to 0+$ and using the results of the preceding paragraphs, one sees that there are $M - 1$ zeroes *strictly* inside the circle of radius $\zeta_1(s)$.

Finally, consider what happens as $s \to 1/\varphi(t_*)$ from below. The zeroes $\zeta_i(s)$ vary continuously with s, by the argument principle, and $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ all lie strictly inside the circle of radius $\zeta_1(s)$ centered at 0. We have shown that $\zeta_i(s)$ cannot approach a point of the circle $|z| = t_*$ other than t_* as $s \to 1/\varphi(t_*)+$; the only other possibility to be ruled out is that some $\zeta_i(s)$ other than $i = 1$ approaches t_* as $s \to 1/\varphi(t_*)+$. But $z = t_*$ is a *double* root of $1 = \varphi(z)/\varphi(t_*)$, since $\varphi'(t_*) = 0$ and $\varphi''(t_*) > 0$. As $s \to 1/\varphi(t_*)-$, the root $\zeta_1(s)$ approaches t_* along the real axis from the left. Thus, it is impossible for any other $\zeta_i(s)$ to approach t_* from *inside* the circle: any other root would have to approach t_* at an asymptote tangent to the real axis from the *outside* of the circle. ///

Lemma 3.4 *For each* $0 < s \leq 1/\varphi(t_*)$ *and each* $\theta \in (0, 2\pi)$ *there are precisely* M *zeroes of* $1 = se^{i\theta}\varphi(z)$ *lying* strictly *inside the circle of radius* $\zeta_1(s)$ *centered at 0. There are no zeroes* on *the circle* $|z| = \zeta_1(s)$.

Proof: Since $|\varphi(\zeta_1(s)e^{i\theta})| < \varphi(\zeta_1(s))$ and $\varphi(\zeta_1(s)) = 1/s$, there are no zeroes of $1 = se^{i\theta}\varphi(z)$ *on* the circle $|z| = \zeta_1(s)$. Fix any complex s' satisfying $0 < |s'| < s$. As z traverses the circle $|z| = \zeta_1(s)$ counterclockwise, $z^M - z^M s'\varphi(z)$ winds around the origin M times, because $|s'\varphi(z)| < 1$. As $s' \to se^{i\theta}$ the curve $z^M - z^M s'\varphi(z)$ deforms continuously to the curve $z^M - z^M se^{i\theta}\varphi(z)$ without passing through 0. Consequently, $z^M - z^M se^{i\theta}\varphi(z)$ winds around the origin M times. ///

Lemma 3.5 *For each complex* s *satisfying* $0 < |s| \leq 1/\varphi(t_*)$ *and* $s \neq 1/\varphi(t_*)$ *there are* only M *zeroes of* $1 = s\varphi(z)$ *in the closed disk* $|z| \leq t_*$.

Proof: The last two lemmas show that there are precisely M zeroes in the closed disk $|z| \leq \zeta_1(|s|)$, so it suffices to show that there are no zeroes in the closed annulus $\zeta_1(|s|) \leq |z| \leq t_*$. But $\varphi(t)$ is strictly decreasing for $0 < t < t_*$, and because of our standing hypothesis on the support of the distribution p_x, $|\varphi(te^{i\theta})| < \varphi(t)$ for all $\theta \in (0, 2\pi)$. Consequently, $|\varphi(z)| < |s^{-1}|$ for all $z \neq \zeta_1(|s|)$ in the annulus $\zeta_1(|s|) \leq |z| \leq t_*$. ///

It is not necessarily the case that the zeroes $\zeta_i(s)$ are analytic functions of s for $|s| \leq R$, and in fact $\zeta_1(s)$ has a singularity at $s = R$, as we will show. But recall that the functions $h_j(s)$ are *symmetric* polynomials of the zeroes $\zeta_i(s)$. The next lemma shows that symmetric polynomials of the roots *in any prescribed region* are analytic.

Lemma 3.6 *Suppose that* $F(s, z)$ *is jointly analytic in* s, z *for* $s \in \Omega$ *and* $|z| < r + \varepsilon$, *for some* $r > 0, \varepsilon > 0$ *and some simply connected domain* Ω. *Suppose further that for each* $s \in \Omega$ *there are exactly* m *zeroes* $z = \zeta_i(s)$ *of* $F(s, z) = 0$ *(not necessarily distinct) satisfying* $|z| < r$, *and that there are* no *zeroes satisfying* $|z| = r$. *Then the elementary symmetric polynomials* $\sigma_1, \sigma_2, \ldots, \sigma_m$ *of the roots* $\zeta_i(s)$ *are analytic functions of* s *for* $s \in \Omega$.

Note: It is not necessarily true that the functions $\zeta_i(s)$ are separately analytic. For example, if $F(s,z) = s - z^2$ then $\zeta_i(s) = \pm\sqrt{s}$.

Proof: Let $\Gamma = \{z : |z| = r\}$ with counterclockwise orientation. Then by the residue theorem, for any entire function $g(z)$,

$$\sum_{j=1}^{m} g(\zeta_j(s)) = \frac{1}{2\pi i} \oint_\Gamma g(z) \frac{F'(s,z)}{F(s,z)}\, dz;$$

in particular, this formula is valid for the functions $g(z) = 1, z, z^2, z^3, \ldots, z^m$. The contour integral on the right side is certainly analytic in s, by a routine application of the Morrera, Fubini, and Couchy theorems, since $F'(s,z)/F(s,z)$ is analytic in s for each $z \in \Gamma$ (here we use the hypothesis that $F(s,z)$ has no zeroes on the contour Γ). Since the elementary symmetric polynomials may be represented as polynomials in the sums $\sum_{j=1}^{m} g(\zeta_j(s))$ for $g(z) = z^j$, $j = 1, 2, 3, \ldots, m$, it follows that they are analytic in s. ///

Corollary 3.7 *In some neighborhood of R, $\zeta_1(s)$ has the form*

$$\zeta_1(s) = a(s)\sqrt{R - s} + b(s), \tag{33}$$

where $a(s), b(s)$ are analytic, $a(R) < 0, b(R) > 0$, and $\sqrt{}$ is the positive branch of the square root. Moreover, the symmetric polynomials of $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ are analytic in a neighborhood of R.

Proof: By Lemmas (3.3-3.5), for each (complex) s satisfying $0 < |s| \le R, s \ne R$, there are exactly M zeroes of $z^M(1 - s\varphi(z))$ in the closed disk $|z| \le t_*$, and none on the circle $|z| = t_*$. These M zeroes vary continuously with s, by the argument principle. As noted in the proof of Lemma 3.3, as $s \to R$ from inside the circle $|s| < R$, only $\zeta_1(s)$ approaches R; hence, there is some $\varepsilon > 0$ such that $|\zeta_j(s)| < t_* - \varepsilon$ for all $|s| \le R, s \ne R$ and $j = 2, 3, \ldots, M$. Since $\zeta_1(s) \to t_*$ as $s \to R$, it follows that for all s in some neighborhood of R in $|s| \le R$, $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ are the *only* zeroes in $|z| \le R - \varepsilon$, and none of them lies on the circle $|z| = R - \varepsilon$. This situation persists for all s satisfying $|s - R| < \delta$, for $\delta > 0$ sufficiently small, since zeroes of $z^M(1 - s\varphi(z))$ vary continuously with s. Therefore, by Lemma 3.6, the symmetric polynomials in $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ are analytic functions of s for $|s - R| < \delta$.

Now recall that $\varphi(z)$ is defined by a power series with nonnegative coefficients, and that at $z = t_*$ it has the form

$$\varphi(z) = \varphi(t_*) + \frac{1}{2}\varphi''(t_*)(z - t_*)^2 + \cdots. \tag{34}$$

Consequently, the inverse function has a branch point of order 2 at $\varphi(t_*) = 1/R$. But $\zeta_1(s) = \varphi^{-1}(1/s)$ for one of these branches. Since $s \to 1/s$ is a univalent analytic function in a neighborhood of $s = R$, it follows that $\zeta_1(s)$ has a branch point of order 2 at $s = R$. Thus, $\zeta_1(s)$ has a Puiseux series expansion at $s = R$ in powers of $\sqrt{R-s}$, with the first coefficient $b(s) = t_* > 0$ and the second coefficient $a(s) < 0$ because $\varphi''(t_*) < 0$. ///

Proposition 3.8 *For each $j = 1, 2, \ldots, M$ the function $h_j(s) = Es^{\tau_-}1_{\{Z_-=-j\}}$ extends to an analytic function in an open domain containing $\{s : |s| \leq 1/\varphi(t_*)\} - \{R\}$. The only singularity of h_j on the circle $|s| = 1/\varphi(t_*)$ is at $R = 1/\varphi(t_*)$, and this is a branch point of order 2. Thus, in a neighborhood of R each $h_j(s)$ has the form*

$$h_j(s) = A_j(s)\sqrt{R-s} + B_j(s) \tag{35}$$

where A_j and B_j are analytic in a neighborhood of $s = R$ and $A_j(R) < 0, B_j(R) > 0$. For all s such that $|s| \leq R$ and $s \neq |s|$,

$$|h_j(s)| < h_j(|s|). \tag{36}$$

Proof: Recall that for $|s| < 1$, the functions $h_j(s)$ are the elementary symmetric polynomials in the zeroes $\zeta_1(s), \zeta_2(s), \ldots, \zeta_M(s)$. By Lemma 3.5, there are *only* M zeroes in the closed disk $|z| \leq t_*$, and by Lemmas 3.3-3.4, for $|s| < R$ none of these zeroes lies on the circle $|z| = t_*$. Consequently, by Lemma 3.6, the functions $h_j(s)$ extend analytically to the disk $|s| < R$. Moreover, for any $|s| = R$, $s \neq R$, there are no zeroes of $z^M(1 - s\varphi(z))$ on the circle $|z| = t_*$, by Lemmas 3.4-3.6, so, by the continuity of $z^M(1 - s\varphi(z))$, this must also be true for all s in a neighborhood of any s' on the circle $|s'| = R$ other than $s' = R$. Thus, by Lemma 3.6, each $h_j(s)$ extends analytically to a neighborhood of $\{s : |s| = R\} - \{R\}$.

Next, consider the situation in a neighborhood of $s = R$. The function $h_j(s)$ is the jth symmetric polynomial of $\zeta_1(s), \zeta_2(s), \ldots, \zeta_M(s)$, which is $\zeta_1(s)$ times the $(j-1)$th symmetric polynomial of $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ plus the jth symmetric polynomial of $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ (with the convention that the 0th symmetric polynomial is 1). But by Corollary 3.7, the symmetric polynomials in $\zeta_2(s), \zeta_3(s), \ldots, \zeta_M(s)$ extend analytically to a neighborhood of $s = R$, and $\zeta_1(s)$ has a branch point of order 2 at $s = R$. Consequently, each of the functions $h_j(s)$ has a Puiseux series in powers of $\sqrt{R-s}$ in a neighborhood of $s = R$, as advertised in (35). It remains to show that $A_j(R) < 0$ and $B_j(R) > 0$.

Recall that $h_j(s) = Es^{\tau_-}1_{\{Z_-=-j\}}$, so $Es^{\tau_-} = \sum_{j=1}^M h_j(s)$. But by Proposition 3.1, $Es^{\tau_-} = 1 - \prod_{i=1}^M(1 - \zeta_i(s))$. Consequently,

$$\sum_{j=1}^M h_j(s) = 1 - \prod_{i=1}^M (1 - \zeta_i(s)) = A(s)\sqrt{R-s} + B(s)$$

for functions A, B analytic in a neighborhood of R, and satisfying $B(R) > 0$ (since $ER^{\tau_-} > 0$) and $A(R) < 0$ (since $a(R) < 0$ in 33). Now the coefficients in the power series $h_j(s) = \sum_{n=1}^{\infty} P\{\tau_- = n;\, Z_- = -j\}s^n$ are nonnegative, and by Lemma 3.2 some are positive, so for each j, $B_j(R) > 0$. Moreover, Lemma 3.2 implies that, if any one of the functions $h_j(s)$ has a square root singularity at $s = R$, then they *all* have square root singularities at $s = R$. We have just seen that their sum has such a singularity at $s = R$; consequently, each h_j has a singularity at $s = R$, and in particular, $A_j(R) \neq 0$. Since $h_j(s)$ is nondecreasing for $s \in (0, R]$, $A_j(R) < 0$.

Finally, (32) implies (36), by a standard argument.

///

Ultimately, we will relate the generating functions $H_{wy}(s)$ appearing in the relations (15) to the functions $h_j(s)$. This is most easily done by introducing the following functions:

$$f_i^{(j)}(s) = Es^{T(i)} 1_{\{S_{T(i)} = -i-j\}} \tag{37}$$

where $i \in \mathcal{Z}_+$, $j = 0, 1, 2, \ldots, M-1$ and

$$T(i) = \inf\{n > 0 :\ S_n \leq -i\}.$$

These are easily expressed in terms of the functions h_j. In order that the random walk make a first visit to the set of integers $\leq -i$, it must achieve a number of successive drops below its record lows, eventually winding up at or below $-i$. At each new record low, the random walk "starts afresh", by the strong Markov property, so the time to the next record low is an independent copy of τ_-. For any i, there are only finitely many possibilities for the sequence of successive record lows recorded before the eventual drop below $-i$. These may be indexed by the set of paths with negative integer increments starting at 0 and ending at $-i-j$:

$$\mathcal{P}_{ij} = \{(j_1, j_2, \ldots, j_r) :\ j_\nu > 0,\ \sum_{\nu=1}^{r} j_\nu = i + j \text{ and } \sum_{\nu=1}^{r-1} j_\nu < i\}.$$

By the strong Markov property,

$$\boxed{f_i^{(j)}(s) = \sum_{\mathcal{P}_{ij}} \prod_{\nu=1}^{r} h_{j_\nu}(s).} \tag{38}$$

Corollary 3.9 *For each pair i, j the function $f_i^{(j)}(s)$ extends to an analytic function in the disk $|s| \leq 1/\varphi(t_*)$ whose only singularity on the circle $|s| = 1/\varphi(t_*)$ is at $R = 1/\varphi(t_*)$. This is a branch point of order 2. Thus, in a neighborhood of R,*

$$f_i^{(j)}(s) = A_{ij}(s)\sqrt{R-s} + B_{ij}(s) \tag{39}$$

where A_{ij} and B_{ij} are analytic and $A_{ij}(R) < 0, B_{ij}(R) > 0$. For each s satisfying $|s| \leq R$ and $s \neq |s|$,

$$|f_i^{(j)}(s)| < f_i^{(j)}(|s|). \tag{40}$$

Proof: This is an immediate consequence of the preceding proposition, since the sum in (38) has only finitely many terms. ///

Note that the Darboux theorem may now be applied to the functions $f_i^{(j)}$, giving the following asymptotic estimates:

$$P\{T(i) = n;\ S_{T(i)} = -i - j\} \sim C_{ij} R^{-n} n^{-\frac{3}{2}}. \tag{41}$$

This should be compared with the results of [8], sec. 32, especially Proposition P3, which gives asymptotic estimates for the *tail* probabilities $P\{T(i) \geq n\}$. We do not know the *minimal* moment conditions necessary for the estimates (41).

Although there are infinitely many generating functions in the collection $\{f_i^{(j)} : i \geq 0,\ 0 \leq j < M\}$, the finite subcollection $\{f_i^{(j)}\}_{0 \leq i,j \leq M-1}$ determines all the rest. For in order that $T(nM+i) < \infty$ for some $0 \leq i \leq M-1$, the random walk must first pass through each of the intervals $[-kM - i, -(k+1)M - i)$, $k = 0, 1, 2, \ldots, n-1$, since the random walk never makes jumps $\leq -M$. At the times of first entry into each of these intervals the random walk starts afresh, and by spatial homogeneity the passages from interval to interval are conditionally independent, depending on where in the intervals the first entries occur. Conditioning on the time $T((n-1)M+i)$ and the corresponding hitting place $S_{T((n-1)M+i)}$ (which must be between $-((n-1)M+i)$ and $-(nM+i)$)) yields

$$f_{nM+i}^{(j)}(s) = \sum_{k=0}^{M-1} f_{(n-1)M+i}^{(k)}(s) f_{M-k}^{(j)}(s). \tag{42}$$

This may be rewritten as a matrix equation. Define the $M \times M$ matrix-valued generating function $\mathcal{F}(s)$ with entries

$$\mathcal{F}_{kj}(s) = f_{M-k}^{(j)}(s),\ \ j, k = 0, 1, \ldots, M-1.$$

Then (42) is equivalent to the matrix equation

$$v_n^i(s) = v_{n-1}^i(s) \mathcal{F}(s) \tag{43}$$

where $v_m^i(s)$ is the vector-valued function

$$v_m^i(s) = \left(f_{mM+i}^{(0)}(s), f_{mM+i}^{(1)}(s), \ldots, f_{mM+i}^{(M-1)}(s) \right). \tag{44}$$

Equation (43) may be iterated (in n), giving

$$\boxed{v_n^i(s) = v_0^i(s) \mathcal{F}(s)^n.} \tag{45}$$

This exhibits every $f_{nM+i}^{(j)}(s)$ as an algebraic function of the M^2 functions $f_i^{(j)}(s)$, where $0 \leq i, j < M$.

Proposition 3.10 *The spectral radius $\|\mathcal{F}(s)\|$ of $\mathcal{F}(s)$ satisfies*

$$\|\mathcal{F}(s)\| < 1 \quad \forall\, |s| \le R,\ s \ne R; \tag{46}$$
$$\|\mathcal{F}(R)\| < 1 \quad \text{if } \sum x p_x > 0; \tag{47}$$
$$\|\mathcal{F}(R)\| = 1 \quad \text{if } \sum x p_x = 0. \tag{48}$$

Note: The *spectral radius* of an $m \times m$ matrix A is the maximum of the moduli of the eigenvalues of A. For any $r > \|A\|$, the series $\sum_{n \ge 0} z^n A^n$ converges uniformly for $|z| < 1/r$ and is analytic in z.

Proof: The entries $f_i^{(j)}(s)$ of the matrix $\mathcal{F}(s)$ are positive and strictly increasing in s for $0 \le s \le R$ (because the probability distribution p_x is not supported by a coset of a proper subgroup, there are positive probability paths in all the sets $\mathcal{P}_{ij}$). Consequently, the spectral radius $\|\mathcal{F}(s)\|$, which coincides with the lead (Perron-Frobenius) eigenvalue is a positive, strictly increasing function of s, and so the maximum must occur at $s = R$. Note also that for any *complex* $s \ne |s|$, $|f_i^{(j)}(s)| < f_i^{(j)}(|s|)$ and hence $\|\mathcal{F}(s)\| < \|\mathcal{F}(|s|)\|$.

Consider first the case where $EY_i = \sum x p_x = 0$. In this case $R = 1$. Moreover, the stopping time $T(i)$ is finite with probability one, since the random walk S_n is recurrent. Consequently, the matrix $\mathcal{F}(1)$ is stochastic. This implies that the lead eigenvalue is 1, and therefore also $\|\mathcal{F}(1)\| = 1$.

Next, suppose that $\sum x p_x > 0$. In this case, $t_* < 1$ and $R = 1/\varphi(t_*) > 1$. Consider the exponentially tilted probability measure $\overline{P}$, with associated expectation operator $\overline{E}$, defined by

$$\overline{P}(A) = E t_*^{S_n} R^n 1_A \qquad \forall A \in \sigma(Y_1, Y_2, \dots, Y_n).$$

By a standard argument ,

$$\begin{aligned} f_i^{(j)}(R) &= E R^{T(i)} 1_{\{S_{T(i)} = -i-j\}} \\ &= t_*^{i+j} \overline{P}\{S_{T(i)} = -i-j\}. \end{aligned}$$

Consequently, $\sum_j f_i^{(j)}(R) = \overline{E} t_*^{-S_{T(i)}} < 1$ for all i. It follows that the lead eigenvalue of the matrix $\mathcal{F}(R)$ is strictly less than 1. ///

4 The Green's Function at $s = R$

Using the results of the preceding section and equation (15), we will determine the nature of the singularity at $s = R$ of the Green's function(s) $G_{xy}(s)$. This necessitates relating the nature of the singularities of the functions $f_i^{(j)}$ at R to those of the functions H_{wy}. These two sets of functions will be shown to be interrelated by a set of matrix equations. Thus, we begin by discussing the singularities of matrix-valued analytic functions.

An $(m \times n)$ matrix-valued function $\Psi(s)$ of a complex variable s will be said to have a branch point of order 2 at $s = R$ if there exist matrix-valued functions $A(s), B(s)$ analytic near $s = R$ such that $A(R) \neq 0$ and in a neighborhood of R,

$$\Psi(s) = A(s)\sqrt{R-s} + B(s)$$

for one of the branches of $\sqrt{R-s}$. (A matrix-valued function is analytic in a domain if each entry is analytic.) Equivalently, $\Psi(s)$ has a branch point of order 2 at $s = R$ if there exists a matrix-valued analytic function $\Phi(t)$ with a nonzero linear term such that in a neighborhood of $s = R$,

$$\Psi(s) = \Phi(\sqrt{R-s}). \tag{49}$$

Clearly, if each of a finite collection of $d \times d$ matrix-valued functions has a branch point of order 2 at R then so does any polynomial expression in these functions (although this may be a "degenerate" branch point, i.e., the polynomial expression may actually be regular at R).

Lemma 4.1 *Suppose that $\Psi(s)$ is a $d \times d$ matrix-valued function with a branch point of order 2 at $s = R$. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $\rho > \|\Psi(R)\|$. Then*

$$\Phi(s) = \sum_{n=0}^{\infty} a_n \Psi(s)^n$$

has a (possibly degenerate) branch point of order 2 at $s = R$.

Note: Recall that $\|A\|$ denotes the spectral radius of A.

Proof: The function $F(M) = \sum_{n=0}^{\infty} a_n M^n$ is an analytic function of the $d \times d$ matrix argument M for $\|M\| < \rho$ (i.e.,the coordinate functions are analytic functions of d^2 variables, one for each entry). Suppose that $\Psi(s) = \Gamma(\sqrt{R-s})$ in a neighborhood of R, where Γ is analytic. Since the composition of analytic mappings is analytic (see any textbook on functions of several complex variables), $F(\Gamma(t))$ is analytic in a neighborhood of $\Gamma(0)$. But $\Phi(s) = F(\Gamma(\sqrt{R-s}))$ near R, so R is a possibly degenerate branch point of order 2. ///

For the case $\sum x p_x = 0$ of Theorem 1, the following result will be needed.

Lemma 4.2 *Suppose that $\Psi(s)$ is a $d \times d$ matrix-valued function with a branch point of order 2 at $s = R$. Suppose that $\Psi(R)$ has an eigenvalue $\lambda(R)$ of multiplicity 1 with left and right eigenvectors $\mathbf{w}(R), \mathbf{v}(R)$ satisfying $\mathbf{w}(R)^t \mathbf{v}(R) = \mathbf{w}(R)^t \mathbf{1} = 1$. Then there are functions $\lambda(s), \mathbf{w}(s), \mathbf{v}(s)$ such that $\lambda(s)$ is an eigenvalue of $\Psi(s)$ of multiplicity 1 with left and right eigenvectors $\mathbf{w}(s), \mathbf{v}(s)$ satisfying $\mathbf{w}(s)^t \mathbf{v}(s) = \mathbf{w}(s)^t \mathbf{1} = 1$, and such that $s = R$ is a (possibly degenerate) branch point of order 2 for each of $\lambda(s), \mathbf{w}(s)$, and $\mathbf{v}(s)$.*

Proof: The fact that the eigenvalue is an isolated eigenvalue of multiplicity 1 allows the use of standard results from regular perturbation theory (see, e.g., [6], ch. XII, secs. 1-2). In particular, there exist holomorphic functions $\lambda(M), \mathbf{w}(M), \mathbf{v}(M)$ of the *matrix* argument M defined in a neighborhood of $M = \Psi(R)$ (in $\mathbf{C}^{d^2}$) such that

$$\begin{aligned} M\mathbf{v} &= \lambda\mathbf{v}; \\ \mathbf{w}^t M &= \lambda\mathbf{w}; \\ \mathbf{w}^t\mathbf{v} &= 1; \\ \mathbf{w}^t\mathbf{1} &= 1. \end{aligned}$$

But $\Psi(s) = \Phi(\sqrt{R-s})$ for an analytic matrix-valued function Φ, so the functions $\lambda(s) = \lambda(\Phi(\sqrt{R-s}))$, etc., have branch points of order 2 at $s = R$ (possibly degenerate). ///

We are now in a position to analyze the singularities of the functions $H_{wy}(s)$ defined by (13).

Proposition 4.3 *If $L - y > M$ then $H_{wy}(s) = q(w, y)s$ and hence is analytic in the entire complex plane. If $L - y \leq M$ then $H_{wy}(s)$ is analytic for $|s| < R = 1/\varphi(t_*)$ and regular at every point of the circle $|s| = R$ except $s = R$, and $s = R$ is a branch point of order 2. Hence, in a neighborhood of R the function H_{wy} has the form*

$$H_{wy}(s) = A_{wy}(s)\sqrt{R-s} + B_{wy}(s) \tag{50}$$

where A_{wy} and B_{wy} are analytic near $s = R$, $B_{wy}(R) > 0$, and $A_{wy}(R) < 0$. If $L - y \leq M$ then for all $0 < |s| \leq R$,

$$|H_{wy}(s)| < H_{wy}(|s|). \tag{51}$$

Proof: The function $H_{wy}(s)$ is the probability generating function of the first time $T = T_L \geq 1$ that the process X_n returns to the interval $[0, L] = \{0, 1, 2, \ldots, L\}$ on the event that this visit occurs at location y. This may happen in one of two ways: either the first jump is from the initial site w to the site y, or the first jump is to the exterior of the interval $[0, L]$ and at some later time the process revisits $[0, L]$. But if the first jump is to the exterior of $[0, L]$ then X_n behaves in accordance with the transition laws of the random walk S_n until the next visit to $[0, L]$, since the scattering sites are all contained in $[0, K] \subseteq [0, L]$. Consequently, if the first jump is to the exterior of the interval $[0, L]$ then the position at time T_L is somewhere in $[L - M + 1, L]$ if $T_L < \infty$, and hence if $L - y > M$ then

$$H_{wy}(s) = q(w, y)s. \tag{52}$$

Moreover, since X_n behaves as S_n up until the time of first return to $[0, L]$ (if the first jump is to the exterior of $[0, L]$), conditional on the location of the state X_1, the time T_L is identical in law to $1 + T(i)$ for the appropriate integer i, where $T(i)$ is the stopping time introduced in (37). Thus, if $L - y \leq M$ then

$$H_{wy}(s) = q(w, y)s + \sum_{u=L+1}^{\infty} q(w, u)sf_{u-L}^{(L-y)}(s). \tag{53}$$

Observe that (51) follows from (53), because by Corollary 3.9 the functions $f_i^{(j)}$ have the same property.

Unfortunately, the sum in (53) has infinitely many terms, so we may not immediately deduce that the functions H_{wy} have the same singularities as do $f_i^{(j)}$. But recall that the functions $f_i^{(j)}$ are determined by the matrix $\mathcal{F}(s)$ in (45). This suggests grouping the terms in (53) in blocks of M to obtain a matrix power series relation for $H_{wy}(s)$. Equation (45) implies

$$\begin{aligned} H_{wy}(s) &= q(w, y)s + \sum_{n=1}^{\infty} \sum_{i=1}^{M} q(w, L + nM + i)sf_{nM+i}^{(L-y)}(s) \\ &= q(w, y)s + \sum_{n=1}^{\infty} \sum_{i=1}^{M} q(w, L + nM + i)s \left(v_0^i(s)\mathcal{F}(s)^n\right)_{L-y} \end{aligned}$$

where the vector-valued generating function $v_0^i(s)$ is as in (44) and $(.)_{L-y}$ indicates the $(L - y)$th entry of the vector. This may be rewritten as

$$H_{wy}(s) = q(w, y)s + \sum_{i=1}^{M} \left(v_0^i(s) \sum_{n=1}^{\infty} q(w, L + nM + i)s\mathcal{F}(s)^n \right)_{L-y}. \tag{54}$$

This exhibits $H_{wy}(s)$ as an entry in a (sum of) matrix-valued power series. Now recall that if $\sum xp_x > 0$ then, by Prop. 3.10, $\|\mathcal{F}(R)\| < 1$, so for any $|s| \leq R$ the matrix $\mathcal{F}(s)$ is (strictly) inside the "circle" of convergence of each of the matrix power series $\sum_{n=1}^{\infty} q(w, L + nM + i)\mathcal{F}^n$ (because this series converges absolutely for any matrix satisfying $\|\mathcal{F}\| \leq 1$). Similarly, if $\sum xp_x = 0$ then by Prop. 3.10, $\|\mathcal{F}(R)\| = 1$, and again for any $|s| \leq R$ the matrix $\mathcal{F}(s)$ is (strictly) inside the "circle" of convergence of each of the matrix power series $\sum_{n=1}^{\infty} q(w, L + nM - i + 1)\mathcal{F}^n$ (because by the hypotheses of Theorem 1 there is a $t > 1$ such that each of the matrix power series $\sum_{n=1}^{\infty} q(w, L+nM-i+1)\mathcal{F}^n$ converges absolutely for any matrix satisfying $\|\mathcal{F}\| \leq t$). In either case it follows that the only singularities of H_{wy} in $|s| \leq R$ are singularities of $\mathcal{F}$, and therefore, by Corollary 3.9, $s = R$ is the only possible singularity. In either case it follows from Lemma 4.1 that $H_{wy}(s)$ either is regular (analytic) or has a branch point of order 2 at $s = R$, since by Corollary 3.9 $\mathcal{F}(s)$ has a branch point of order 2 at $s = R$.

To see that $s = R$ is in fact a branch point of order 2, consider again the relation (53). By (45) and (46) the series converges uniformly and absolutely for s in any compact subset of $|s| < R$, and so the derivative may be computed by summing the derivatives of the terms in (53). Each of the coefficients $q(w, u)$ in the series is nonnegative, and at least one is strictly positive. Each of the functions $f_i^{(j)}(s)$ has a power series with nonnegative coefficients, and each has a branch point of order 2 at $s = R$, so $(f_i^{(j)})'(s) \to \infty$ as $s \to R$. It follows that the same is true for H_{wy}: the derivative $\to \infty$ as $s \to R$, so $s = R$ cannot be a regular point of H_{wy}. ///

Next, consider the matrix-valued function $\mathcal{H}(s)$ with entries $H_{wy}(s)$ indexed by $w, y \in \{0, 1, \dots, L\}$.

Lemma 4.4 *For each $0 < s \leq R$, $\mathcal{H}(s)$ is a Perron-Frobenius matrix. The lead (Perron-Frobenius) eigenvalue $\lambda(s) = \|\mathcal{H}(s)\|$ is a continuous, strictly increasing function of $s \in [0, R]$. For complex s such that $|s| \leq R$ and $s \neq R$,*

$$|\lambda(s)| < \lambda(|s|). \tag{55}$$

Proof: To show that $\mathcal{H}(s)$ is a Perron-Frobenius matrix for $0 < s \leq R$, we must show that it is a nonnegative matrix for which some positive power has strictly positive entries. Note that the entries of $\mathcal{H}(s)$ are analytic for $|s| < R$ and continuous in $|s| \leq R$, by the previous proposition. For $n \geq 1$ and $w, y \in [0, L]$, the wyth entry of $cH(s)^n$ is the generating function of the time of the nth return to $[0, L]$ on the event that this occurs at location y, when the starting point is w. Now the sequence of successive visits to $[0, L]$ by X_n is itself a Markov chain. It is irreducible, because X_n is irreducible. Moreover, it is aperiodic, because $H_{LL}(s) > 0$ (by an argument similar to the proof of Lemma 3.2). It follows that for some $n \geq 1$, $\mathcal{H}(s)^n$ has strictly positive entries. Since the entries are nonconstant polynomials in the functions $H_{wy}(s)$, they are strictly increasing in s for $0 \leq s \leq R$, hence so is the lead eigenvalue. Relation(55) follows from the last statement in Proposition 4.3.

///

Proposition 4.5 *Suppose that $\sum x p_x > 0$. Then the radius of convergence of the Green's function $G_{xy}(s)$ is $R = 1/\varphi(t_*)$. There are no singularities of $G_{xy}(s)$ on the circle of convergence except $s = R$, and this is a branch point of order 2. In a neghborhood of $s = R$,*

$$G_{xy}(s) = A_{xy}(s)\sqrt{R - s} + B_{xy}(s) \tag{56}$$

where $A_{xy}(s), B_{xy}(s)$ are analytic and satisfy $B_{xy}(R) > 0$ and $A_{xy}(R) < 0$.

Proof: Let R_c be the radius of convergence of G_{xy}. Since G_{xy} is defined by a power series with nonnegative coefficients, there can be no singularities in the disk $0 < |s| < R_c$. Moreover, $s = R_c$ is itself a singularity.

For each $s \in (0, R)$ the matrix $\mathcal{H}(s)$ is a Perron-Frobenius matrix, and the lead eigenvalue is increasing and continuous in s, by the preceding lemma. Suppose that for some $s \in (0, R)$, $\lambda(s) = 1$; if $s \leq R_c$ then by (15) $\lim_{s' \uparrow s} G_{xy}(s') = \infty$, since $\mathcal{H}(s')$ is aperiodic and irreducible. But this would contradict Proposition 2.3. Thus, we may conclude that $\lambda(s) < 1$ for every $s \in (0, R)$ satisfying $s \leq R_c$.

By Proposition 4.3, the matrix-valued function $\mathcal{H}(s)$ has no singularities in the disk $|s| < R$ or on the circle $|s| = R$ other than $s = R$. Since $\|\mathcal{H}(s)\| < \|\mathcal{H}(|s|)\|$ for all s satisfying $|s| \leq R$ and $s \neq |s|$, by (55), it now follows from the last paragraph that $(\mathcal{I} - \mathcal{H}(s))^{-1}$ has no singularities in the disk $|s| < R$ nor on the circle $|s| = R$ other than perhaps $s = R$. By (15), the Green's function $G_{xy}(s)$ can have no singularities in $|s| \leq R$ other than $s = R$. This proves that $R_c \geq R$.

To prove that $R_c = R$ it suffices to show that R is a singularity of G_{xy}. Equation (15) implies that

$$\mathcal{G}_x(s) = \sum_{n=0}^{\infty} \mathcal{H}(s)^n u_x,$$

where the vector u_x has nonnegative entries, at least some of which are positive. Since $\|\mathcal{H}(R)\| < 1$ and since $\mathcal{H}(s)$ has nonnegative entries with at least some satisfying $H'_{wy}(s) \to \infty$ as $s \uparrow R$, it follows by an argument similar to that used in the proof of Proposition 4.3 that $\mathcal{G}'_x(s) \to \infty$ as $s \uparrow R$. Hence, R must be a singularity.

Finally, $s = R$ is a branch point of order 2 by Lemma 4.1, since by the above arguments $\|\mathcal{H}(R)\| < 1$ and by Proposition 4.3 $s = R$ is a branch point of order 2 of $\mathcal{H}(s)$. That $A_{xy}(R) < 0$ and $B_{xy}(R) > 0$ follow because $G_{xy}(s)$ is an increasing, positive function of $s \in (0, R]$. ///

Proposition 4.6 *Suppose that $\sum x p_x = 0$ and that there exists a $t > 1$ such that for every $x \in \mathcal{Z}_+$, $\sum_{y \in \mathcal{Z}_+} q(x, y) t^y < \infty$. Then the radius of convergence of the Green's function(s) $G_{xy}(s)$ is $R = 1$. There are no singularities on the circle $|s| = 1$ except $s = 1$, and in a neighborhood of $s = 1$,*

$$G_{xy}(s) = \frac{A_{xy}(s)}{\sqrt{1-s}} + B_{xy}(s) \tag{57}$$

where $A_{xy}(s), B_{xy}(s)$ are analytic and satisfy $A_{xy}(1) > 0$.

Proof: The same argument as in the proof of the previous proposition shows that the radius of convergence is at least 1 and that there is no singularity on the circle $|s| = 1$ except possibly $s = 1$. We must show that $s = 1$ is actually a singularity, and that G_{xy} has the form (57) in a neighborhood of 1.

Recall that for each $s \in (0, 1)$ the matrix $\mathcal{H}(s)$ is a Perron-Frobenius matrix. By the Perron-Frobenius theorem, the lead eigenvalue has one-dimensional left and right eigenspaces, and the basis vectors may be chosen to have *strictly* positive entries. Let $\lambda(s)$ be the lead eigenvalue, and let $\mathbf{v}(s), \mathbf{w}(s)$ be the positive right and left eigenvectors satisfying $\mathbf{w}(s)^t\mathbf{v}(s) = 1$ and $\sum_j w_j(s) = 1$. The functions $\lambda, \mathbf{v}$, and $\mathbf{w}$ are all continuous, in fact analytic, functions of s, with order 2 branch points (possibly degenerate) at $s = 1$, by Lemma 4.2 and Proposition 4.3. The nonzero entries of $\mathcal{H}(s)$ are increasing functions of $s \in (0, 1]$, so the lead eigenvalue $\lambda(s)$ is an increasing function of s.

The assumption $\sum xp_x = 0$ implies that X_n is a recurrent Markov chain, because whenever $X_n \geq K$ the transitions are those of a mean zero random walk on the integers. Consequently, for any integer $w \in [0, L]$, the time of first return to the interval $[0, L]$ is finite with probability 1. Therefore, for any such w,

$$\sum_{y\in[0,L]} H_{wy}(1) = 1$$

(recall the definition (13) of H_{wy}). Thus, the matrix $\mathcal{H}(1)$ is a stochastic matrix, and the lead eigenvalue $\lambda(1) = 1$. Since u_x is a probability vector, it follows that $\lim_{n\to\infty} \mathcal{H}(1)^n u_x = (\mathbf{w}(1)^t u_x)\mathbf{v}(1)$ has positive entries. Hence, by the continuity of $\lambda(s)$,

$$(\mathcal{I} - \mathcal{H}(s))^{-1}u_x = \sum_{n=0}^{\infty} \mathcal{H}(s)^n u_x \to \infty$$

as $s \uparrow 1$. This proves that $s = 1$ is a singularity of the Green's functions $G_{xy}(s)$, and that $G_{xy}(1) = \infty$.

Define a new matrix-valued function $\mathcal{R}(s) = \mathcal{H}(s) - \lambda(s)\mathbf{v}(s)\mathbf{w}^t(s)$; this is well-defined for s in a neighborhood of the segment $[0, 1]$, since the eigenvalue and eigenvectors extend analytically (by regular perturbation theory). Since the lead eigenvalue $\lambda(s)$ is isolated and of multiplicity one, the spectral radii of the matrices $\mathcal{R}(s)$ are bounded above by $1 - \varepsilon$ for some $\varepsilon > 0$ independent of s in a neighborhood of $[0, 1]$. Moreover, by Lemma 4.2 the functions $\lambda, \mathbf{w}$, and $\mathbf{v}$ have order 2 branch points at $s = 1$, so the same is true of $\mathcal{R}(s)$. Thus, the result of Lemma 4.1 applies to $(\mathcal{I} - \mathcal{R}(s))^{-1}$: in particular, $(\mathcal{I} - \mathcal{R}(s))^{-1}$ is finite and has a branch point of order 2 at $s = 1$. Consequently, since $\mathcal{H}(s)^n = \lambda(s)^n\mathbf{v}(s)\mathbf{w}(s)^t + \mathcal{R}(s)^n$ for all s,

$$\begin{aligned}\mathcal{G}_x(s) &= (\mathcal{I} - \mathcal{H}(s))^{-1}u_x \\ &= (1 - \lambda(s))^{-1}(\mathbf{w}(s)^t u_x)\mathbf{v}(s) + (\mathcal{I} - \mathcal{R}(s))^{-1}u_x.\end{aligned}$$

This shows that the Green's function has the form (57). ///

Proof of Theorem 1: Theorem 1 is an immediate consequence of Propositions 4.5 and 4.6, by Darboux's theorem.

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