

# GROWTH PROFILE AND INVARIANT MEASURES FOR THE WEAKLY SUPERCRITICAL CONTACT PROCESS ON A HOMOGENEOUS TREE

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ABSTRACT. It is known that the contact process on a homogeneous tree of degree  $d+1 \geq 3$  has a *weak survival* phase, in which the infection survives with positive probability but nevertheless eventually vacates every finite subset of the tree. It is shown in this paper that in the weak survival phase there exists a spherically symmetric invariant measure whose density decays exponentially at infinity, thus confirming a conjecture of Liggett [10]. The proof is based on a study of the relationships between various thermodynamic parameters and functions associated with the contact process initiated by a single infected site. These include (a) the *growth profile*, which determines the exponential rate of growth in space-time on the event of survival; (b) the exponential rate  $\beta$  of decay of the hitting probability function at infinity (also studied in [7]); and (c) the exponential rate  $\eta$  of decay in time  $t$  of the probability that the initial infected site is infected at time  $t$ . It is shown that  $\beta$  is a strictly increasing function of the infection rate  $\lambda$  in the weak survival phase. It is also shown that  $\eta < 1$  except at the critical value  $\lambda_2$  demarcating the boundary between the weak and strong survival phases, where  $\eta = 1$ .

## 1. INTRODUCTION

This paper is a sequel to [7], in which the *weakly supercritical* phase of an isotropic contact process on an infinite homogeneous tree was studied. Briefly, an *isotropic contact process* on the homogeneous tree  $\mathcal{T} = \mathcal{T}_d$  of degree  $d+1$  is a continuous time Markov process  $A_t$  on the set of finite subsets of  $\mathcal{T}$  (henceforth,  $\mathcal{T}$  will be identified with the vertex set of the tree) that evolves as follows. Infected sites (members of  $A_t$ ) recover at rate 1 and upon recovery are removed from  $A_t$ ; healthy sites (members of  $A_t^c$ ) become infected at rate  $\lambda$  times the number of infected neighbors, and upon infection are added to  $A_t$ . Under the default probability measure  $P$ , the initial state  $A_0$  is the singleton set  $\{e\}$  (where  $e$  is a distinguished element of  $\mathcal{T}$  called the “root”). See [8] and [5] for general information on the contact process, and [7], especially sections 1 and 2, for background information concerning the contact process on a homogeneous tree. Some of the arguments of this paper (see the proofs of Proposition 8 and Theorem 4 below) are borrowed from [7].

It was discovered by Pemantle [13] for trees of degree greater than 3 and by Liggett [11] (see also Stacey [14] for a relatively simple argument) for the tree of degree 3 that the contact process described above has (at least) 3 essentially different “phases”: (i) *extinction*, (ii) *weak survival*, and (iii) *strong survival*. In phase (i),  $A_t = \emptyset$  eventually, with probability 1; in phase (ii),  $|A_t| \rightarrow \infty$  with positive probability but for every *finite* subset  $B$  of  $\mathcal{T}$ ,  $A_t \cap B = \emptyset$  eventually, with probability 1; and in phase (iii), with positive probability  $e \in A_t$  for arbitrarily large values of  $t$ . There exist critical constants  $0 < \lambda_1 < \lambda_2 < \infty$  such

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that  $\lambda \leq \lambda_1$  implies ultimate extinction;  $\lambda_1 < \lambda \leq \lambda_2$  implies weak survival; and  $\lambda > \lambda_2$  implies strong survival. The phase of interest here is *weak survival* (we also refer to the contact process in this phase as *weakly supercritical*). This phase is of interest, among other reasons, because it does not occur for the contact process on Euclidean lattices (e.g., on  $\mathbb{Z}^d$ ). Henceforth we shall discuss only this case. Thus, we make the standing assumption that

$$\lambda \in (\lambda_1, \lambda_2].$$

**1.1. Growth Profile and Other Characteristics.** The main results of this paper concern a function  $V(s)$  which we shall call the *growth profile* of the contact process. Two other parameters also enter into the statements of these results:  $\beta = \beta(\lambda)$ , which determines the size of the “limit set” of the contact process on the event of its survival (see [7]), and  $\eta = \eta(\lambda)$ , the exponential rate of decay of  $P\{e \in A_t\}$ . The definitions are as follows:

$$(1) \quad \beta = \lim_{n=d(x,e) \rightarrow \infty} (P\{x \in A_t \text{ for some } t\})^{1/n};$$

$$(2) \quad \eta = \lim_{t \rightarrow \infty} (P\{e \in A_t\})^{1/t};$$

$$(3) \quad V(s) = \log \lim_{n=d(x,e) \rightarrow \infty} (P\{x \in A_{ns}\})^{1/n}.$$

The existence of these limits follows from simple subadditivity arguments (see below for  $\eta$  and  $V(s)$ , and see [7] for  $\beta$ ). The main result of [7] is that for all values of  $\lambda \in (\lambda_1, \lambda_2]$ ,

$$(4) \quad \beta \leq \frac{1}{\sqrt{d}}.$$

**Theorem 1.** *The growth profile  $V(s)$  is a concave, continuous function of  $s > 0$  that is bounded above by  $\log \beta$  and satisfies*

$$(5) \quad \lim_{s \rightarrow 0^+} V(s) = -\infty,$$

$$(6) \quad \lim_{s \rightarrow \infty} V(s)/s = \log \eta.$$

Theorem 1 will be proved in section 2 below – see Propositions 1-4. We have been unable to prove that  $V(s)$  is *strictly* concave, but we conjecture that it is.

**Theorem 2.** *If  $\lambda < \lambda_2$  and  $\beta < 1/\sqrt{d}$  then  $\eta < 1$ . If  $\beta = 1/\sqrt{d}$  then  $\eta = 0$ .*

In view of equation (6), Theorem 2 implies that if  $\lambda < \lambda_2$  and  $\beta < 1/\sqrt{d}$  then  $V(s) \rightarrow -\infty$  as  $s \rightarrow \infty$ , but that if  $\beta = 1/\sqrt{d}$  then  $V(s)$  is *nondecreasing* in  $s$  (since it is concave). This will be of crucial importance in the proof of Theorem 5 below. Theorem 2 will be proved in section 4.

**Theorem 3.** *In the weak survival phase  $\lambda \in (\lambda_1, \lambda_2]$  the parameter  $\beta$  is a strictly increasing function of  $\lambda$ .*

Theorem 3 will be proved in section 6 below. It follows immediately from theorems 2-3 that  $\eta < 1$  for all  $\lambda \in (\lambda_1, \lambda_2)$ . We have not been able to determine whether  $\beta(\lambda_2) = 1/\sqrt{d}$ , so we do not yet know if it is possible for  $\eta = 1$  in the weakly supercritical phase.

The function  $V(s)$  proscribes the shape and size of the set  $A_t$  of infected sites at large times  $t$ . Define  $r_t$  and  $R_t$  to be the *smallest* and *largest* distances  $d(x, e)$  among the infected sites  $x \in A_t$ . Also, for any integer  $n$  and any  $s > 0$  define  $N_n(ns)$  to be the number of vertices  $x \in A_{ns}$  at distance  $n$  from  $e$  that are infected at time  $ns$ .

**Theorem 4.** *Suppose that  $\beta < 1/\sqrt{d}$ . Then there exist smallest and largest solutions  $0 < s_1 \leq s_2 < \infty$  of  $V(s) = -\log d$ . Almost surely on the event of survival,*

$$(7) \quad \lim_{t \rightarrow \infty} r_t/t = 1/s_2, \quad \text{and}$$

$$(8) \quad \lim_{t \rightarrow \infty} R_t/t = 1/s_1.$$

Moreover, for any  $s > 0$  such that  $V(s) + \log d > 0$ ,

$$(9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log N_n(ns) = V(s) + \log d$$

See section 5 for the proof. This theorem explains the use of the term “growth profile”. The function  $V(s)$  effectively determines how the contact process “spreads out” in space-time, and determines roughly how many vertices at each distance from  $e$  are infected at any large time  $t$ . Together with Theorem 2, it also implies that  $A_t$  recedes linearly from the root vertex  $e$  if and only if  $\lambda \in (\lambda_1, \lambda_2)$ .

**1.2. Invariant Measures for the Contact Process.** The set of invariant measures in the phase of strong survival has a relatively simple structure: every such measure is a mixture of the point mass at  $\emptyset$  and the “upper invariant measure” (see Zhang [15]). For weakly supercritical contact processes the set of invariant measures seems in general to be much more complex. Durrett and Schinazi [3] proved that it has infinitely many extreme points. Liggett [10] showed that, at least for  $\lambda > \lambda_1$  sufficiently close to  $\lambda_1$ , there are invariant measures  $\nu$  that are *spherically symmetric* (about the root vertex  $e$ ) and have exponentially receding densities, i.e., are such that for all  $x \in \mathcal{T}$ ,

$$(10) \quad C_1 z^{d(x,e)} \leq \nu\{A : x \in A\} \leq C_2 z^{d(x,e)},$$

where  $0 < C_1 < C_2 < \infty$  are constants independent of  $x$  and  $d(x, e)$  denotes the distance from  $x$  to  $e$  in  $\mathcal{T}$ . Liggett also conjectured that such invariant measures exist for *all*  $\lambda \in (\lambda_1, \lambda_2]$ , and gave a sufficient condition for their existence. This condition involves the function

$$(11) \quad \varphi(z) = \lim_{t \rightarrow \infty} \left( E \sum_{x \in A_t} z^{L(x)} \right)^{\frac{1}{t}},$$

where  $L(x)$  is the *depth* of the vertex  $x$  in the tree (see [10] or section 3 below for the definition). Liggett proved that if there exists a solution  $z \in (1/\sqrt{d}, 1)$  of the equation

$$(12) \quad \varphi(z) = 1,$$

then there exists a spherically symmetric invariant measure for the contact process that satisfies the exponential decay law (10). The next result shows that the growth profile determines Liggett’s function  $\varphi$ .

**Theorem 5.** *If  $\beta < 1/\sqrt{d}$  then for every  $z > 1/\sqrt{d}$ ,*

$$(13) \quad \varphi(z) = \max_{0 < s < \infty} \exp \left( \frac{V(s) + \log(dz)}{s} \right).$$

The proof will be given in section 3, along with the proof of the following corollary.

**Corollary 1.** *If  $\beta < 1/\sqrt{d}$ , then  $\varphi(1/d\beta) = 1$ .*

In view of (4) and Theorem 3, this implies that Liggett’s conjecture is true:

**Corollary 2.** *For every  $\lambda \in (\lambda_1, \lambda_2)$  there is a spherically symmetric invariant measure for the contact process whose density decays exponentially at infinity.*

## 2. GROWTH PROFILE: BASIC PROPERTIES

By the isotropy of the contact process, the probability that the set  $A_t$  of sites infected at time  $t$  contains a particular vertex  $x \in \mathcal{T}$  depends only on  $t$  and  $|x|$ . Thus, we may define

$$u_n(t) = P\{x \in A_t\},$$

where  $x$  is any vertex at distance  $n$  from the root  $e$ . The Markov and monotonicity properties of the contact process imply that for all nonnegative integers  $m, n$  and all times  $s, t \geq 0$ ,

$$(14) \quad u_{m+n}(s+t) \geq u_m(s)u_n(t).$$

Consequently, by the basic subadditivity lemma, for every  $s > 0$  the limit

$$(15) \quad \lim_{n \rightarrow \infty} u_n(ns)^{\frac{1}{n}} \triangleq U(s) \triangleq \exp\{V(s)\}$$

exists, and for every  $s > 0$  and every integer  $n \geq 0$ ,

$$(16) \quad u_n(ns) \leq U(s)^n.$$

**Proposition 1.**  $V(s) \leq \log \beta$

It is clear that  $u_n(ns) \leq u_n$  for any  $n \in \mathbb{N}$  and any  $s > 0$ . Taking  $n$ th roots on both sides of this inequality and letting  $n \rightarrow \infty$ , one obtains the desired result.

**Proposition 2.**  $V(s)$  is a concave and, therefore, continuous function of  $s > 0$ .

According to the fundamental inequality (14), for all  $0 < s < t < \infty$  and all nonnegative integers  $m, n$ ,

$$u_m(ms)u_n(nt) \leq u_{m+n}(ms+nt).$$

Taking the  $(m+n)$ th root of both sides and letting  $m, n \rightarrow \infty$  in such a way that the ratio  $m/(m+n)$  converges to  $\alpha \in (0, 1)$ , one gets

$$U(s)^\alpha U(t)^{1-\alpha} \leq U(s+t),$$

which implies that  $V$  is concave.

**Proposition 3.**  $\lim_{s \rightarrow 0^+} V(s) = -\infty$ .

This is equivalent to showing that  $\lim_{s \rightarrow 0^+} U(s) = 0$ . Recall that

$$U(s) = \lim_{n \rightarrow \infty} u_n(ns)^{1/n},$$

and that  $u_n(ns)$  is the probability that a particular vertex  $x$  at distance  $n$  from  $e$  will be infected at time  $ns$ . Let  $x_0, x_1, x_2, \dots, x_n$  be the successive vertices on the geodesic segment from  $e = x_0$  to  $x = x_n$ , and for each  $1 \leq m \leq n$  define  $\tau_m$  to be the elapsed time between the first infection of  $x_{m-1}$  and the first infection of  $x_m$ . In order that  $x_n$  be infected at time  $ns$ , it must be the case that at least half of the times  $\tau_m$  are no greater than  $2s$ . But the distribution of  $\tau_m$ , conditional on the history of the contact process up to the time of first infection of  $x_{m-1}$ , is stochastically larger than an exponential random variable with mean  $1/\lambda$ , so the conditional probability that  $\tau_m \leq 2s$  is no larger than  $1 - e^{-2\lambda s}$ . Thus, the probability that at least half of the random variables  $\tau_m$ ,  $1 \leq m \leq n$ , are  $\leq 2s$  is no larger than

$$\sum_{k=[n/2]}^n \binom{n}{k} (1 - e^{-2\lambda s})^k (e^{-2\lambda s})^{n-k} \leq (n+1)2^n (1 - e^{-2\lambda s})^{[n/2]}$$

Taking  $n$ th roots, one finds that

$$U(s) = \limsup_{n \rightarrow \infty} (u_n(ns))^{1/n} \leq 2\sqrt{1 - e^{-2\lambda s}},$$

which converges to 0 as  $s \rightarrow \infty$ .

By definition,  $u_0(t) = P\{e \in A_t\}$  is the probability that the root vertex  $e$  is infected at time  $t$ . If  $e$  is infected at time  $t$  and if there is an infection trail from  $(e, t)$  to  $(e, t + s)$ , then clearly  $e$  is infected at time  $t + s$ . Hence, by the Markov and monotonicity properties of the contact process,  $u_0(t + s) \geq u_0(t)u_0(s)$  for all  $s, t > 0$ . It follows that

$$\eta = \lim_{t \rightarrow \infty} u_0(t)^{\frac{1}{t}} \leq 1$$

exists, and that  $u_0(t) \leq \eta^t$  for all  $t > 0$ .

**Proposition 4.**  $\lim_{s \rightarrow \infty} V(s)/s = \log \eta$ .

The concavity of  $V$  implies that the limit exists and is nonpositive, so it suffices to consider only integer values of  $s$ . Fix  $\varepsilon = 1/m > 0$  smaller than 1, and let  $n > 0$  be an integer such that  $n\varepsilon \in \mathbb{N}$ . Let  $x$  be a vertex at distance  $n\varepsilon$  from the root  $e$ . If there are infection trails that extend from  $(e, 0)$  to  $(e, n - n\varepsilon)$  and from  $(e, n - n\varepsilon)$  to  $(x, n)$ , then their concatenation is an infection trail from  $(e, 0)$  to  $(x, n)$ . Hence, by the Markov and monotonicity properties,

$$u_{n\varepsilon}(n) \geq u_0(n - n\varepsilon)u_{n\varepsilon}(n\varepsilon)$$

Taking  $n$ th roots of both sides and letting  $n \rightarrow \infty$  through integer multiples of  $1/\varepsilon$  gives

$$U(1/\varepsilon)^\varepsilon \geq \eta^{1-\varepsilon}U(1)^\varepsilon.$$

Taking logarithms and letting  $\varepsilon \rightarrow 0$  yields

$$\lim_{s \rightarrow \infty} \frac{V(s)}{s} \geq \log \eta.$$

The reverse inequality is obtained in much the same way. Let  $x$  be a vertex at distance  $n\varepsilon$  from the root  $e$ . If there are infection trails that extend from  $(e, 0)$  to  $(x, n)$  and from  $(x, n)$  to  $(e, n + n\varepsilon)$  then their concatenation is an infection trail from  $(e, 0)$  to  $(e, n + n\varepsilon)$ ; consequently,

$$u_{n\varepsilon}(n)u_{n\varepsilon}(n\varepsilon) \leq u_0(n + n\varepsilon).$$

Taking  $n$ th roots and letting  $n \rightarrow \infty$  gives

$$U(1/\varepsilon)^\varepsilon U(1)^\varepsilon \leq \eta^{1+\varepsilon}$$

and letting  $\varepsilon \rightarrow 0$  yields

$$\lim_{s \rightarrow \infty} \frac{V(s)}{s} \leq \log \eta.$$

**Proposition 5.** *If  $\lim_{s \rightarrow \infty} V(s) = -\infty$  then*

$$\max_{0 < s < \infty} V(s) = \log \beta.$$

By Proposition 3,  $V(s) \rightarrow -\infty$  as  $s \rightarrow 0+$ , and by Proposition 2,  $V(s)$  is continuous in  $s$ . Consequently, if  $V(s) \rightarrow -\infty$  as  $s \rightarrow \infty$  then  $V(s)$  attains its maximum value  $v_{\max}$ . Moreover, since  $V$  is concave, the hypothesis that  $V \rightarrow -\infty$  as  $s \rightarrow \infty$  implies that there exist constants  $a < 0$  and  $0 < s_* < \infty$  such that

$$(17) \quad V(s) \leq as - as_* + v_{\max} - 1 \quad \text{if } s_* \leq s < \infty$$

Let  $x$  be any vertex of  $\mathcal{T}$ . If  $x$  is infected for a first time at time  $\tau$ , then conditional on the history of the contact process up to time  $\tau$  the probability that  $x$  will remain infected until time  $\tau + 1$  is at least  $\exp\{-1\}$ , because the death rate is 1. On this event,  $x$  will remain infected at the first *integer* time after  $\tau$ . Hence,

$$\begin{aligned}
 (18) \quad u_n &\leq \exp\{1\} \sum_{m=1}^{\infty} u_n(m) \\
 &= \exp\{1\} \sum_{m=1}^{\infty} \exp\{nV(m/n)\} \\
 &= \exp\{1\} \left( \sum_{m \leq ns_*} + \sum_{m > ns_*} \right).
 \end{aligned}$$

By (17), the second sum (over  $m > ns_*$ ) is dominated by a convergent geometric series whose first term is no larger than  $\exp\{n(v_{\max} - 1)\}$ . On the other hand, the first sum (over  $m \leq ns_*$ ) contains at most  $ns_* + 1$  terms, none larger than  $\exp\{nv_{\max}\}$ . Thus,

$$\begin{aligned}
 \sum_{m \leq ns_*} &\leq \exp\{nv_{\max}\}(ns_* + 1), \\
 \sum_{m > ns_*} &\leq \exp\{nv_{\max} - n\} / (1 - \exp\{-a\})
 \end{aligned}$$

and so the dominant term is the first of these sums. Taking  $n$ th roots in (18) and letting  $n \rightarrow \infty$  gives  $\log \beta \leq v_{\max}$ . Proposition 1 implies the reverse inequality.

### 3. THE GROWTH PROFILE AND LIGGETT'S $\varphi$ -FUNCTION

In this section we prove Theorem 5, thus making the connection between the growth profile  $V$  and Liggett's  $\varphi$ -function. Recall [10] that  $\varphi$  is defined by

$$\varphi(z) = \lim_{t \rightarrow \infty} (E w_z(A_t))^{1/t} \lim_{t \rightarrow \infty} \left( E \sum_{x \in A_t} z^{L(x)} \right)^{1/t},$$

where  $L(x)$  is the *depth* of the vertex  $x$ . The definition of the depth function  $L$  depends on an arrangement of the vertices of  $\mathcal{T}$  in *levels*  $(\mathcal{L}_n)_n \in \mathbb{Z}$  so that (i) distinct levels  $\mathcal{L}_n$  and  $\mathcal{L}_m$  do not intersect; (ii) every vertex  $y \in \mathcal{L}_n$  has exactly  $d$  neighbors in  $\mathcal{L}_{n+1}$  and exactly one neighbor in  $\mathcal{L}_{n-1}$ ; and (iii) the root vertex  $e$  is in level  $\mathcal{L}_0$ . The *depth*  $L(x)$  of a vertex  $x$  is then defined to be the index  $n$  of the level  $\mathcal{L}_n$  in which  $x$  lies. Observe that there are infinitely many distinct arrangements of the vertices in levels, but that the expectations in the definition of  $\varphi(z)$  do not depend on which arrangement is used, by the isotropy of the contact process.

**Lemma 1.** *Define  $M_\nu^k$  to be the number of vertices of  $\mathcal{T}$  at distance  $k$  from the root  $e$  and at depth  $\nu$ . Then*

$$(19) \quad M_\nu^k = 1 \quad \text{if } \nu = -k;$$

$$(20) \quad M_\nu^k = d^k \quad \text{if } \nu = k;$$

$$M_\nu^k = (d-1)d^{j-1} \leq d^j \quad \text{if } \nu = -k + 2j \text{ and } -k < \nu < k;$$

$$M_\nu^k = 0 \quad \text{otherwise.}$$

Let  $x$  be a vertex at distance  $n$  from the root  $e$ . The geodesic segment from  $e$  to  $x$  makes  $j$  steps “up” and  $n - j$  steps “down” for some integer  $0 \leq j \leq n$ , with the  $j$  up steps preceding the  $n - j$  down steps (otherwise the path would retrace some of its steps, contradicting the supposition that it is a geodesic segment). The depth at termination of any such geodesic segment is  $(n - j) - j$ , and the number of distinct such geodesic segments (with  $j$  up steps followed by  $n - j$  down steps) is  $d^n$  if  $j = 0$  and  $(d - 1)d^{n-j-1}$  if  $j \geq 1$ . Since geodesic segments starting at  $e$  are in one-to-one correspondence with their endpoints, the formulas for  $M_\nu^k$  follow.

The proof of Theorem 5 will use the following intermediate characterization of  $\varphi$ .

**Proposition 6.** *For every  $z > 1/\sqrt{d}$ ,*

$$\varphi(z) = \lim_{t \rightarrow \infty} \left( \sum_{n=0}^{\infty} u_n(t) (dz)^n \right)^{\frac{1}{t}}.$$

For any finite set  $A$  of vertices and any  $z \in \mathbb{R}$ , define  $w_z(A) = \sum_{x \in A} z^{L(x)}$ . Then  $\varphi(z) = \lim_{t \rightarrow \infty} (Ew_z(A_t))^{1/t}$ , where  $A_t$  denotes the set of infected vertices at time  $t$ . Arranging vertices by distance from the root gives

$$Ew_z(A_t) = \sum_{k=0}^{\infty} \sum_{j=0}^k z^{-k+2j} M_{-k+2j}^k u_k(t).$$

Since  $M_k^k = d^k$ , it follows directly that

$$Ew_z(A_t) \geq \sum_{k=0}^{\infty} z^k d^k u_k(t).$$

On the other hand, Lemma 1 and the hypothesis  $z^2 d > 1$  imply that

$$\begin{aligned} Ew_z(A_t) &\leq \sum_{k=0}^{\infty} \sum_{j=0}^k z^k d^k u_k(t) z^{-2j} d^{-j} \\ &\leq \sum_{k=0}^{\infty} z^k d^k u_k(t) (1 - (z^2 d)^{-1})^{-1} \end{aligned}$$

Since the limit  $\lim_{t \rightarrow \infty} (Ew_z(A_t))^{1/t}$  exists and equals  $\varphi(z)$  (see [10]), the desired result follows from the last two displayed inequalities.

Proof of Theorem 5 Fix  $z > 1/\sqrt{d}$ , set  $V_z(s) = V(s) + \log(dz)$ , and define

$$\sigma = \sup_{s>0} V_z(s)/s = \sup_{s>0} (V(s) + \log(dz))/s.$$

By Propositions 3 and 4 and Theorems 1 and 2,  $V_z(s)/s$  converges to  $-\infty$  as  $s \rightarrow 0+$  and  $V_z(s)/s$  converges to  $\log \eta < 0$  as  $s \rightarrow \infty$ . Consequently,  $\sigma < \infty$ .

By definition of  $\sigma$  and the intermediate value theorem, for all  $\sigma_* < \sigma$  sufficiently close to  $\sigma$ , the line of slope  $\sigma_*$  through the origin must intersect the graph of  $V$ . Thus, there exists

$s_* \in (0, \infty)$  such that  $V_z(s_*) = \sigma_*$ . Now

$$\begin{aligned} \left( \sum_{n=0}^{\infty} u_n(t) d^n z^n \right)^{\frac{1}{t}} &\geq \left( u_{[t/s_*]}(t) (dz)^{[t/s_*]} \right)^{\frac{1}{t}} \\ &\rightarrow (U(s_*) dz)^{\frac{1}{s_*}} \\ &= \exp\{\sigma_*\} \end{aligned}$$

as  $t \rightarrow \infty$ . This implies, by Proposition 6, that  $\varphi(z) \geq e^{\sigma_*}$ . Since  $\sigma_* < \sigma$  was arbitrary, it follows that

$$\varphi(z) \geq e^{\sigma}.$$

The proof of the reverse inequality is similar to the proof of Proposition 5. First note that by Proposition 3,  $V(s) \rightarrow -\infty$  as  $s \rightarrow 0+$ , so for any  $z > 0$  there exists  $\delta \in (0, 1)$  sufficiently small that for all  $s \in (0, \delta)$ ,

$$V(s) + \log(dz) < \min(-1, \sigma - 1)$$

Now

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(t) d^n z^n &\leq \sum_{n=0}^{\infty} U(t/n)^n d^n z^n \\ &= \sum_{n=0}^{\infty} \exp\{n(V(t/n) + \log(dz))\} \\ &= \sum_{n \leq \delta t} + \sum_{n > \delta t}. \end{aligned}$$

There are at most  $\delta t$  terms in the first sum, none larger than  $e^{\sigma t}$ , since  $\sigma t \geq V(t) + \log(dz)$  by definition of  $\sigma$ . By our choice of  $\delta$ , the terms of the second are dominated by the terms of a geometric series with ratio  $\leq e^{-1}$  and with initial term no larger than  $e^{t(\sigma-1)/\delta} < e^{t(\sigma-1)}$ . Hence, by Proposition 6,

$$\varphi(z) = \limsup_{t \rightarrow \infty} \left( \sum_{n=0}^{\infty} u_n(t) d^n z^n \right)^{\frac{1}{t}} \leq e^{\sigma}.$$

Proof of Corollary 1 By Theorems 1 and 2, if  $\beta < 1/\sqrt{d}$  then  $\lim_{s \rightarrow \infty} V(s) = -\infty$ . Thus, Theorem 5 implies that for every  $z > 1/\sqrt{d}$  the value of  $\varphi(z)$  is given by equation (13). By Proposition 5, the function  $V(s)$  attains its maximum value of  $\log \beta$  at some  $s = s_* \in (0, \infty)$ . When  $z = 1/(d\beta)$ ,

$$\max_{s > 0} (V(s) + \log(dz)) = (V(s_*) + \log(dz)) = 0,$$

so by (13),  $\varphi(z) = 1$ .

#### 4. THE GROWTH PROFILE AT $\infty$

**Proposition 7.** *If  $\beta < 1/\sqrt{d}$  and  $\lambda < \lambda_2$  then  $\eta < 1$ .*

It suffices to prove that if  $\beta < 1/\sqrt{d}$  then there exists a value of  $z > 0$  such that  $\varphi(z) < 1$ , because for any  $z > 0$ ,

$$\eta = \lim_{t \rightarrow \infty} (P\{e \in A_t\})^{\frac{1}{t}} \leq \lim_{t \rightarrow \infty} (Ew_z(A_t))^{\frac{1}{t}} = \varphi(z)$$

According to Proposition 1.19(e) of [10], if  $\lambda < \lambda_2$  and if  $\varphi(z_*) \leq 1$  for some  $z_* > 1/\sqrt{d}$  then  $\varphi(z) < 1$  for every  $z \in [1/\sqrt{d}, z_*)$ ; hence it suffices to prove that if  $\lambda < \lambda_2$  and  $\beta < 1/\sqrt{d}$  then for some  $z > 1/\sqrt{d}$ ,

$$(21) \quad \varphi(z) = \lim_{t \rightarrow \infty} (Ew_z(A_t))^{1/t} \leq 1.$$

Define  $A_*$  to be the (random) set consisting of all vertices that are *ever* infected. Obviously  $A_t \subset A_*$  for every  $t < \infty$ , and so it is also the case that  $Ew_z(A_t) \leq Ew_z(A_*)$ . We will show that if  $\beta < 1/\sqrt{d}$  and  $z\beta < 1/d$  then  $Ew_z(A_*) < \infty$ . It will then follow that equation (21) holds for all  $z \in [1/\sqrt{d}, 1/d\beta)$ . Summing over vertices at fixed distances from the root  $e$  as in the proof of Proposition 6 gives

$$\begin{aligned} Ew_z(A_*) &= \sum_{k=0}^{\infty} \sum_{j=0}^k z^{-k+2j} M_{-k+2j}^k u_k \\ &\leq \sum_{k=0}^{\infty} \sum_{j=0}^k z^k d^k \beta^k z^{-2j} d^{-j} \\ &\leq \sum_{k=0}^{\infty} z^k d^k \beta^k (1 - (z^2 d)^{-1})^{-1} \\ &= (1 - z\beta d)^{-1} (1 - (z^2 d)^{-1})^{-1} \end{aligned}$$

The remainder of this section is devoted to the proof of the following proposition, which will complete the roof of Theorem 2. The argument is largely borrowed from [7], section 4.

**Proposition 8.** *If  $\beta = 1/\sqrt{d}$  then  $\eta = 1$ .*

For any integer  $n \geq 1$ , define  $F_n$  to be the event that there is an infection trail that begins at the root  $e$  at time  $t = 0$ , reaches a vertex  $x$  at distance  $n$  from  $e$ , and then returns from  $x$  to  $e$ . Since the contact process is, by hypothesis, weakly supercritical,  $P(F_n) \rightarrow 0$  as  $n \rightarrow \infty$  (see [7] section 4). Define

$$\zeta = \limsup_{n \rightarrow \infty} P(F_n)^{\frac{1}{n}}.$$

**Lemma 2.** *If  $\zeta = 1$  then  $\eta = 1$ .*

Let  $H_n$  be the event that  $e \in A_t$  for some  $t \geq n$ , and let  $H'_n$  be the event that  $e \in A_t$  for some *integer*  $t \geq n$ . Since the recovery rate in the contact process is 1, for any  $n$  the conditional probability is at least  $\exp\{-1\}$  that  $e$  remains infected for  $n \leq t \leq n+1$ , given that  $e$  is infected at time  $n$ . Hence,  $P(H_n) \leq \exp\{1\}P(H'_n)$ , and it follows that

$$P(H_n) \leq \exp\{1\} \sum_{m=0}^{\infty} P(e \in A_{n+m}) \leq \exp\{1\} \eta^n / (1 - \eta).$$

Thus, to prove that  $\eta = 1$  it suffices to prove that

$$\limsup_{n \rightarrow \infty} P(H_n)^{1/n} = 1.$$

Fix  $\varepsilon > 0$ , and let  $G_n$  be the event that there is an infection trail starting at  $(e, 0)$  that reaches a vertex at distance  $n$  from  $e$  and then returns to  $e$  by time  $\varepsilon n$ . On the event  $G_n$ , some vertex at distance  $n$  from  $e$  is infected at some time  $t < \varepsilon n$ ; hence, summing over all

such vertices and all integer times  $< \varepsilon n$ , using again the fact that the recovery rate is 1, we obtain that

$$P(G_n) < (d+1)d^{n-1} \left( \exp\{1\} \sum_{m=1}^{\varepsilon n} \exp nV(m/n) + \lambda^n \exp\{-\lambda\}/n! \right)$$

( $\lambda^n e^{-\lambda}/n!$  is an upper bound for the probability that a particular vertex  $x$  at distance  $n$  from  $e$  is infected before time 1). Consequently, by Proposition 3, if  $\varepsilon > 0$  is sufficiently small then

$$\limsup_{n \rightarrow \infty} P(G_n)^{\frac{1}{n}} < 1.$$

Now by hypothesis,  $\limsup_{n \rightarrow \infty} P(F_n)^{1/n} = 1$ . Consequently, if  $\varepsilon > 0$  is sufficiently small then, by the result of the preceding paragraph,  $\limsup_{n \rightarrow \infty} P(F_n \cap G_n^c)^{1/n} = 1$ . But the event  $F_n \cap G_n^c$  is contained in  $H_{[\varepsilon n]}$ , so it follows that

$$\limsup_{n \rightarrow \infty} P(H_{[\varepsilon n]})^{\frac{1}{n}} = 1.$$

Proof of Proposition 8 By Lemma 2, it suffices to show that  $\zeta = 1$ . For this we adapt from [7] (section 3) the notion of a *downward* infection trail. Let  $x, y \in \mathcal{T}$  be vertices such that  $L(x) = m < L(y) = m + k$ , where as earlier  $L(z)$  denotes the *depth* of the vertex  $z$  (the index of its *level*  $\mathcal{L}_{L(z)}$ ). A *downward infection trail* from  $x$  to  $y$  is an infection trail that begins at  $x$ , never enters level  $\mathcal{L}_{m-1}$ , and first reaches  $\mathcal{L}_{m+k}$  at  $y$ , where it terminates. Observe that a downward infection trail is the same as an *outward* infection trail (see section 3 of [7]) provided that the initial vertex  $x$  is at depth  $L(x) > 0$ . By the isotropy of the contact process, the probability that there is a downward infection trail from  $x$  to  $y$  depends only on  $k = d(x, y)$ , so we may define

$$w_k = P\{\exists \text{ downward infection trail } x \rightarrow y\}.$$

By Proposition 1 of [7],  $w_k^{1/k} \rightarrow \beta$  as  $k \rightarrow \infty$ . Thus, the probability that there is a downward infection trail from  $e$  to  $x$  is, in exponential size, just as large as the probability that there is *any* infection trail from  $e$  to  $x$ .

If there is a downward infection trail from an infected site  $x$  at level  $L(x) \geq 0$  (beginning at  $x$  during the *first* infection epoch of  $x$ ) to a site  $x^* \in \mathcal{T}(x)$ , say that  $x^*$  is a *descendant* of  $x$ . Fix an integer  $L \geq 1$  and define  $Y_L$  to be the number of descendants of  $e$  at depth  $L$ . Then by Proposition 2 of [7], there is a Galton-Watson process  $Z_n^L$  with mean offspring number  $d^L w_L$  such that  $Y_{nL} \geq Z_n^L$  for all  $n$ . Since  $w_L^{1/L} \rightarrow \beta = 1/\sqrt{d}$  as  $L \rightarrow \infty$ , if  $L$  is sufficiently large then the Galton-Watson  $Z_n^L$  is supercritical. Hence, for any  $\beta_- < \beta$ , if  $k$  is sufficiently large then

$$(22) \quad p_k \stackrel{\Delta}{=} P\{Y_k > (d\beta_-)^k\} > \varepsilon > 0,$$

where  $\varepsilon > 0$  is the probability that the Galton-Watson process  $Z_n^L$  does not reach extinction.

Consider now the event  $F_n^k$  that there is an infection trail that begins at the root  $e$  at time  $t = 0$ , reaches a vertex  $x \in Y_{nk}$ , and then returns from  $x$  to  $e$ . Clearly,  $P(F_n^k) \leq P(F_{nk})$ , where  $F_{nk}$  is as in Lemma 2 above. The event  $F_n^k$  will occur if there is *just one* descendant  $x$  of  $e$  at depth  $k$  such that (i) there is a descendant  $x^*$  of  $x$  at depth  $kn$  and an infection trail from  $x^*$  to  $x$ , beginning at the instant of initial infection of  $x^*$ ; and (ii) there is a subsequent infection trail from  $x$  to  $e$ . Now conditional on the set  $\mathcal{Y}_k$  of descendants of  $e$  at depth  $k$ , the events (i) for the different  $x^* \in \mathcal{Y}_k$  are mutually independent (since they involve

non-overlapping parts of the percolation structure), and each has probability  $P(F_{n-1}^k)$  (by the isotropy of the contact process). Moreover, given that event (i) occurs for some  $x \in Y_k$ , the conditional probability of event (ii) is at least  $u_k$ . Consequently, if  $r_n = P(F_n^k)$ , then

$$\begin{aligned} r_n &\geq P\{Y_k \geq (d\beta_-)^k\}P(\cup_{x \in Y_k} (i) \& (ii) \mid Y_k \geq (d\beta_-)^k) \\ &\geq p_k(1 - (1 - r_{n-1})^{(d\beta_-)^k})u_k \\ &\geq p_k u_k(1 - \exp\{-(d\beta_-)^k r_{n-1}\}) \end{aligned}$$

For each fixed  $k$ ,  $r_n = P(F_n^k) \rightarrow 0$  as  $n \rightarrow \infty$ , as the contact process is weakly supercritical. This implies that the last exponential displayed above is well-approximated by the first term of its Taylor series around 0. Furthermore, if  $k$  is sufficiently large then  $u_k > \beta_-^k$ , and  $p_k > \varepsilon$  by the last paragraph. Thus, for all large  $n$ ,

$$r_n \geq \varepsilon(d\beta_-^2)^k r_{n-1}/2.$$

It follows that  $\liminf_{n \rightarrow \infty} P(F_n^k)^{1/n} \geq \varepsilon(d\beta_-^2)^k/2$ . Since  $P(F_n^k) \leq P(F_{nk})$ , this implies that  $\limsup_{n \rightarrow \infty} P(F_n)^{1/n} \geq d\beta_-^2$ ; but  $\beta_- < \beta = 1/\sqrt{d}$  was arbitrary, so this proves that

$$\limsup_{n \rightarrow \infty} P(F_n)^{1/n} = \eta = 1.$$

## 5. THE GROWTH PROFILE AND THE SPREAD OF THE INFECTION

**Proof of Theorem 4** When  $\beta < 1/\sqrt{d}$ , by Propositions 7 and 5,  $\max_s V(s) = \log \beta$ . Also,  $\beta \geq -\log d$  because otherwise the expected number of sites *ever* infected would be finite, which would preclude the possibility of weak survival. Consequently, there is at least one solution  $s$  of  $V(s) = -\log d$ . Let  $s_1$  be the smallest solution and  $s_2$  the largest solution. Recall that  $r_t$  and  $R_t$  are the *smallest* and *largest* distances  $d(x, e)$  among the infected sites  $x \in A_t$ , and for any interval  $(a, b)$ ,  $N_t(a, b)$  is the number of vertices  $x \in A_t$  such that  $at < d(x, e) < bt$ .

*Proof of Relations (7)-(8)(Sketch).* We will show that a.s. on the event of survival, for any  $\varepsilon > 0$ ,  $N_t(t/s_1 + \varepsilon t, \infty) = 0$  eventually and  $N_t(0, t/s_1 - \varepsilon t) = 0$  eventually. This will imply that a.s. on survival,  $\limsup R_t/t \leq 1/s_1$  and  $\liminf r_t/t \geq 1/s_2$ . The reverse inequalities  $\liminf R_t/t \geq 1/s_1$  and  $\limsup r_t/t \leq 1/s_2$  will follow from relation (9) proved below.

A routine argument using estimates like those developed in the proof of Proposition 3 shows that it is enough to consider *integer* times  $t$  (the probability that an infection trail moves a distance  $\geq \varepsilon t$  in time 1 decreases more rapidly than any exponential  $e^{-ct}$ ). For a fixed large integer  $t$ , the probability that  $N_t(t/s_1 + \varepsilon t, \infty) > 0$  is smaller than

$$\sum_{n \geq t/s_1 + \varepsilon t} u_n(t) d^n (d+1).$$

But  $u_n(t) \leq \exp\{nV(t/n)\}$ , so by an argument like that used in the proof of Theorem 5 the terms in the above sum are bounded by the terms of a geometric series with sum smaller than  $O(e^{-t\delta})$  for some  $\delta > 0$  depending only on  $\varepsilon$ . Since  $\sum_t \in \mathbb{N} e^{-t\delta} < \infty$ , the Borel-Cantelli Lemma implies that a.s.  $N_t(t/s_1 + \varepsilon t, \infty) = 0$  eventually. A similar argument proves that a.s.  $N_t(0, t/s_1 - \varepsilon t, \infty) = 0$  eventually.

*Proof of Relation (9)(Sketch):* Fix  $s > 0$  such that  $V(s) + \log d > 0$ . The probability  $u_n(ns)$  that a particular vertex at distance  $n$  from the root will be infected at time  $ns$  is no larger

than  $\exp\{nV(s)\}$ . The number of vertices in  $\mathcal{T}$  at distance  $n$  from  $e$  is  $(d+1)d^{n-1}$ . It follows that the expected number of such vertices that are infected at time  $ns$  is no larger than

$$(d+1)d^n e^{nV(s)}.$$

Consequently, by a routine application of the Borel-Cantelli lemma and the Markov inequality,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log N_n(ns) \leq V(s) + \log d.$$

The proof of the reverse direction uses again the notion of a *downward infection trail*, which was introduced in the proof of Proposition 8. Let  $x$  be a vertex at depth  $n > 0$ , and let  $u_n^*(t)$  be the probability that there is a downward infection trail from  $(e, 0)$  to  $(x, t)$ . Then for any  $s > 0$ ,

$$(23) \quad \lim_{n \rightarrow \infty} u_n^*(ns)^{1/n} = e^{V(s)}.$$

This may be proved in much the same manner as Proposition 1 of [7].

The virtue of considering only vertices infected via *downward* infection trails is that the existence (or nonexistence) of these infection trails depends only on the part of the percolation structure above the vertices of the tree in the sector between the initial and terminal vertices. Thus, for any distinct vertices  $x, x', \dots$  at depth  $m$  and any times  $t, s$ , the numbers  $\xi_x(s, t), \xi_{x'}(s, t), \dots$  of vertices  $y, y', \dots$  that are infected at time  $t$  by downward infection trails starting at  $(x, s), (x', s), \dots$ , respectively, are mutually independent. Consequently, if  $N_{nm}(nms)$  denotes the number of vertices at distance  $nm$  that are infected at time  $nms$  (by infection trails starting at  $(e, 0)$ ), then  $N_{nm}(nms)$  dominates a Galton-Watson chain  $Z_n$  whose mean offspring number is

$$d^m u_m^*(ms).$$

(See [7], Proposition 2 for a similar result about time-independent infection trails.) It follows that on the event of non-extinction of this Galton-Watson process,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log N_{nm}(nms) \geq \log d^m u_m^*(ms).$$

By choosing  $m$  large, we can (i) make the event of non-extinction of the G-W process arbitrarily close to the event of non-extinction of the contact process, and (ii) push  $u_m^*(ms)^{1/m}$  close to  $e^{V(s)}$ , by (23). Therefore, almost surely on the event of non-extinction,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log N_n(ns) \geq V(s) + \log d.$$

## 6. STRICT MONOTONICITY OF $\beta$

Fix  $\lambda \in (\lambda_1, \lambda_2]$ , so that the contact process with infection rate  $\lambda$  and recovery rate 1 is weakly supercritical. In this section we will prove the following proposition, which implies Theorem 3.

**Proposition 9.** *If  $\lambda_* < \lambda$  then  $\beta(\lambda_*) < \beta(\lambda)$ .*

We begin the proof by introducing an augmentation of the usual *percolation structure* used in the construction of the contact process. Recall that the percolation structure is a system of independent Poisson processes attached to vertices and ordered pairs of neighboring vertices. For each vertex  $x$  of  $\mathcal{T}$  the Poisson process attached to  $x$  has rate 1, and determines

the recovery times at  $x$ : in particular, at every occurrence time site  $x$  recovers if it is infected. For each ordered pair  $(x, y)$  of neighboring vertices, the Poisson process attached to  $(x, y)$  has rate  $\lambda$ ; the occurrence times are precisely those times when an infection at  $x$  may jump to  $y$ . Occurrences in these Poisson processes are marked on a system of directed rays  $\{x\} \times [0, \infty)$  connected to the vertices  $x$  of  $\mathcal{T}$ , in such a way that (1) at each occurrence time  $t$  of the Poisson process attached to  $(x, y)$  an “infection arrow” is drawn from  $(x, t)$  (the “tail”) to  $(y, t)$  (the “head”) in  $\mathcal{T} \times (0, \infty)$ ; and (2) at each occurrence time  $t$  of the Poisson process attached to  $x$  a “recovery mark”  $*$  is attached to  $(x, t)$ . Note that there are no simultaneous occurrences of infection arrows and/or recovery marks in the percolation structure. The state  $A_t$  of the contact process at time  $t$  may now be defined to be the set of all vertices  $y$  for which there is a (directed) path through the percolation structure (the system of rays and arrows described above) that begins at  $e$ , ends at  $(y, t)$ , and does not pass through any recovery marks  $*$ .

In its augmented form, the percolation structure is provided with a collection of Bernoulli- $p$  random variables  $\xi_\alpha$ , one for each infection arrow  $\alpha$ . These are conditionally independent, given the realization of the percolation structure. The value of  $p$  is chosen so that

$$\lambda_* = \frac{\lambda p}{1 + \lambda q},$$

where  $q = 1 - p$ . A version  $A_t^*$  of the contact process with infection rate  $\lambda p$  and recovery rate  $1 + \lambda q$  may be constructed using the augmented percolation structure by (1) first creating a *modified percolation structure* by changing every infection arrow  $\alpha$  such that  $\xi_\alpha = 0$  to a recovery mark  $*$ ; then (2) defining  $A_t^*$  to be the set of all vertices  $y$  for which there exists a directed path from  $(e, 0)$  to  $(y, t)$  in the *modified* percolation structure that does not pass through any recovery marks  $*$ . This new contact process  $A_t^*$ , with infection rate  $\lambda p$  and recovery rate  $1 + \lambda q$ , is a time-changed version of a contact process with infection rate  $\lambda_*$  and recovery rate 1. Consequently, the value of the parameter  $\beta(\lambda_*)$  satisfies

$$\beta(\lambda_*) = \lim_{n \rightarrow \infty} (u_n^*)^{1/n}$$

where  $u_n^*$  is the probability that the contact process  $A_t^*$  ever infects vertex  $e_n$  at distance  $n$  from  $e$ .

Let  $G_n$  be the event that the contact process  $A_t$  infects vertex  $e_n$  at some finite time  $t$ , and let  $G_n^*$  be the corresponding event for the contact process  $A_t^*$  (thus,  $P(G_n) = u_n$  and  $P(G_n^*) = u_n^*$ ). By construction,  $G_n^* \subset G_n$ , because every infection arrow in the modified percolation structure occurs also in the unmodified percolation structure, and every recovery mark  $*$  in the unmodified percolation structure is retained in the modified percolation structure. On the event  $G_n$ , there is at least one directed path through the unmodified percolation structure that leads from  $(e, 0)$  to  $e_n \times (0, \infty)$  (and in fact there may be many overlapping such paths). Call an infection arrow  $\alpha$  in the unmodified percolation structure *essential* for the event  $G_n$  if (1) event  $G_n$  occurs, and (2) changing  $\alpha$  from an infection arrow to a recovery mark would destroy all directed paths from  $(e, 0)$  to  $e_n \times (0, \infty)$ . Define  $N_n$  to be the number of essential arrows for the event  $G_n$  when  $G_n$  occurs, and  $G_n = 0$  when  $G_n$  does not occur. Since modification of any one of the essential arrows would disconnect  $(e, 0)$  from  $e_n \times (0, \infty)$ , in order that event  $G_n^*$  occur it is necessary that  $N_n \geq 1$  and that  $\xi_\alpha = 1$  for *every* essential arrow  $\alpha$ . The conditional probability of this, given the realization of the unmodified percolation structure, is  $p^{N_n}$ ; hence,

$$u_n^* = P(G_n^*) = E p^{N_n} 1_{G_n}.$$

If it were the case that  $N_n \geq Cn$  on  $G_n$  for some  $C > 0$  then it would follow that  $u_n^* \leq u_n p^{Cn}$ , and consequently that  $\beta(\lambda_*) \leq \beta(\lambda) p^C$ . Alternatively, if for some  $\rho < 1$  it were the case that  $P(N_n \leq Cn | G_n) \leq \rho^n$  then it would follow that  $u_n^* \leq u_n(p^{Cn} + \rho^n)$ , and again  $\beta(\lambda_*) < \beta(\lambda)$ . Thus, to prove Proposition 9 it suffices to prove the following lemma.

**Lemma 3.** *There exist constants  $C < \infty$  and  $0 < \rho < 1$  such that for all  $n$  sufficiently large*

$$P(N_n \leq Cn | G_n) \leq \rho^n$$

Lemma 3 is a statement that refers only to the *unmodified* percolation structure. The remaining arguments make no further reference to the augmented or modified percolation structures. Henceforth, the term *directed path* will mean a directed path in the (unmodified) percolation structure that does not pass through any recovery marks  $*$ .

The proof of Lemma 3 is based on a continuous-time adaptation of (a part of) MENSHIKOV'S proof of the uniqueness of the critical point for bond percolation on the integer lattices  $\mathbb{Z}^d$ ,  $d \geq 2$  (see [12]; also [4], section 3.1 for a detailed exposition). This machinery has also been used in the context of the contact process on  $\mathbb{Z}^d$  by BEZUIDENHOUT & GRIMMETT [1]; the argument here is simpler, however. The key tool is a continuous-time version of the BKF inequality, as stated in section 2.2 of [1], which provides a bound for the probability of *disjoint occurrence* of two events. Recall [1] that for any two events  $F, G$ , the event  $F \circ G$ , the event that  $F$  and  $G$  occur *disjointly*, is defined to be the set of all realizations  $\omega$  of the percolation structure such that there exist disjoint measurable subsets  $\Gamma, \Delta$  of  $\mathcal{T} \times [0, \infty)$  such that the cylinders  $\omega_\Gamma$  and  $\omega_\Delta$  are contained in  $F$  and  $G$ , respectively. (The cylinder  $\omega_\Gamma$  is defined to be the set of all realizations of the percolation structure that agree with  $\omega$  in  $\Gamma$ .) For any sets  $A, B$  of vertices, define  $H(A, B)$  to be the event that there is a directed path from  $A \times \{0\}$  to  $B \times (0, \infty)$ . Then the event

$$H(A_1, B_1) \circ (H(A_2, B_2) \cap H(A_3, B_3)^c)$$

is the event that there exist *nonoverlapping* directed paths from  $A_1 \times \{0\}$  to  $B_1 \times (0, \infty)$  and from  $A_2 \times \{0\}$  to  $B_2 \times (0, \infty)$  and *no* directed path from  $A_3 \times \{0\}$  to  $B_3 \times (0, \infty)$ .

**Lemma 4.** (*BKF Inequality*)

$$P(H(A_1, B_1) \circ (H(A_2, B_2) \cap H(A_3, B_3)^c)) \leq P(H(A_1, B_1))P(H(A_2, B_2) \cap H(A_3, B_3)^c)$$

In addition to the BKF inequality, we will need the following elementary consequence of MENGER'S Lemma (see [2], section ).

**Lemma 5.** *Let  $H$  be the event that there is a directed path  $\gamma$  from  $(x, t)$  to  $\{z\} \times (0, \infty)$ , and let  $K \subset H$  be the event that the only essential arrow for  $H$  is the first infection arrow  $\alpha$  leading out of  $x$  after time  $t$ . Let  $\nu$  be the occurrence time of  $\alpha$ . Then on the event  $K$ , there exists a directed path  $\gamma'$  from  $(y, \nu)$  to  $\{z\} \times (0, \infty)$  that does not intersect  $\alpha$  except possibly at its initial and terminal points.*

It is possible that this second path  $\gamma'$  might be the trivial path that begins and ends at  $(y, \nu)$ ; in this case, it must be the case that  $y = z$ . In every other case the path  $\gamma'$  must be nontrivial, meeting  $\gamma$  only at the point of termination on  $\{z\} \times (0, \infty)$ .

**Proof of Lemma 3** If the event  $G_n$  occurs then there is at least one essential arrow for  $G_n$ , namely, the first infection arrow emanating from the root line  $e \times (0, \infty)$ . The essential arrows may be ordered in sequence  $\alpha_1, \alpha_2, \dots, \alpha_{N_n}$  according to the times  $\nu_1, \nu_2, \dots, \nu_{N_n}$  of their occurrence; note that the spatial locations of the essential arrows are not necessarily ordered in any nice way (for instance, by distance from  $e$ ), nor are the times  $\nu_j$  stopping times for the contact process.

For each integer  $m \geq 1$ , define  $\tau_m$  to be the time of the  $m$ th infection arrow  $\alpha_m^*$  in the percolation cluster attached to the root  $(e, 0)$  (the set of all  $(x, t)$  connected to  $(e, 0)$  by a directed path), or  $\tau_m = \infty$  if there are fewer than  $m$  such arrows. Observe that each  $\tau_m$  is a stopping time for the contact process. Moreover, the times  $\nu_j$  of the essential arrows must occur in the (increasing) sequence  $\tau_1, \tau_2, \dots$ . Suppose that  $\tau_m < \infty$ , and let  $u_m$  and  $v_m$  be the tail and head, respectively, of the infection arrow  $\alpha_m^*$ . Note that there is a directed path from  $(e, 0)$  to  $(u_m, \tau_m)$ , by definition of  $\tau_m$ , and that therefore there is also a directed path to  $(v_m, \tau_m)$ . In order that this infection arrow be an essential arrow for  $G_n$  (i.e., that  $\alpha_m^*$  occurs in the sequence  $\alpha_1, \alpha_2, \dots, \alpha_{N_n}$ ), it is necessary and sufficient that

- (E1) there is a directed path to  $\{e_n\} \times (0, \infty)$  from  $(u_m, \tau_m)$  or from  $(v_m, \tau_m)$ ; and
- (E2) there is *no* directed path to  $\{e_n\} \times (0, \infty)$  from  $(A_{\tau_m} - \{u_m, v_m\}) \times \{\tau_m\}$ .

Furthermore, by Lemma 5, given that  $\alpha_m^*$  is essential for  $G_n$ , in order that the *next* essential arrow has tail  $w$  it is necessary that

- (E3) there is a directed path from  $\{u_m, v_m\} \times \{\tau_m\}$  to  $\{w\} \times (0, \infty)$  that does not intersect the directed path guaranteed by (E1), except possibly at the endpoints;

and in order that  $\alpha_m^*$  be the *last* essential arrow for  $G_n$  (i.e., that  $\alpha_m^* = \alpha_{N_n}$ ) it is necessary that

- (E3') there is a directed path from  $\{u_m, v_m\} \times \{\tau_m\}$  to  $\{e_n\} \times (0, \infty)$  that does not intersect the directed path guaranteed by (E1), except possibly at the endpoints.

Assume that the vertices  $e_1, e_2, \dots$  lie in succession along a geodesic ray  $\gamma_*$  beginning at  $e_0 = e$ , so that  $d(e_j, e_{j+1}) = 1$  for every  $j \geq 0$ . Let  $F_m$  be the event that the arrow  $\alpha_m^*$  is essential for  $G_n$  (i.e., the event that (E1) and (E2) are true), and for  $k \geq 1$  let  $D_{m,k}$  be the event that each of the geodesic segments from  $u_m$  and  $v_m$  to the tail  $w$  of the next essential arrow (or  $e_n$  if there is no next essential arrow) intersects  $\gamma_*$  in a segment of length  $\geq k$ . Then by (E3)-(E3'), on the event  $F_m \cap D_{m,k}$  there is a directed path from  $\{u_m, v_m\} \times \{\tau_m\}$  that travels a distance  $\geq k$  on  $\gamma_*$  and does not intersect the directed path guaranteed by (E1). Consequently, by Lemma 4 and the Strong Markov Property, if  $\mathcal{F}_m$  denotes the  $\sigma$ -algebra generated by the history of the contact process up to time  $\tau_m$  then

$$P(D_{m,k} \mid \mathcal{F}_m \wedge F_m)1_{F_m} \leq 2u_k \leq 2\beta^k.$$

(Compare with Lemma 3.12 of [4].) In words: the conditional distribution of the distance to the next essential arrow is stochastically dominated by a geometric distribution. It follows that the sum of the distances between the first  $m$  essential arrows is stochastically dominated by the sum of  $m$  i.i.d. random variables with geometrically decaying tail, and Lemma 3 clearly follows.

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