# RANDOM WALKS ON INFINITE FREE PRODUCTS AND INFINITE ALGEBRAIC SYSTEMS OF GENERATING FUNCTIONS

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Abstract. The return probabilities of certain random walks on infinite free products of finite groups are shown to obey a Local Limit Theorem of the same type as for nearest-neighbor random walks on finite free products. The analysis is based on an infinite-dimensional extension of a technique for studying finite algebraic systems of generating functions introduced by the author in [12] and [13].

## 1. Introduction

1.1. **The security guard's problem.** A security guard patrols an infinite hallway with an infinite sequence of doors, all of which are initially open. He carries with him a suitcase containing infinitely many keylocks, each with a color $\kappa \in \mathbb{N}$, the colors occurring with relative frequencies $\{p_\kappa\}_{\kappa \in \mathbb{N}}$. At each step of his patrol he selects at random from his suitcase a lock and matching key. If at least one door is locked, and if the color $\kappa$ of the pair he selects from his suitcase matches that of the *last* lock affixed to a door, then he uses the key to unlock that lock, thereby opening the door, and discards the key and both of the locks. Otherwise, he moves to the next unlocked door in the hallway, attaches the lock, and discards the key.

**Problem:** *What is the probability that, after $n$ steps, no doors are locked?*

More generally, suppose that instead of keylocks the guard carries combination locks of different colors. Now, whenever he selects a color $\kappa$ that matches the color of the last combination lock attached to a door, he (randomly) spins the dial of that lock: if it returns the dial to the zero position, the lock unlocks, the guard removes it, and opens the door; otherwise he draws again from his suitcase. The problem is the same.

1.2. **Random walks on infinite free products.** The security guard's random patrol and its generalization are both instances of random walks on *infinite free products* of finite groups. For any sequence $\Gamma_1, \Gamma_2, \dots$ of finite groups, the free product

$$\Gamma = \Gamma_1 * \Gamma_2 * \cdots \tag{1.1}$$

is defined as follows. The elements of the group $\Gamma$ are the finite words with letters in the alphabet $\cup_i(\Gamma_i \setminus \{1\})$ such that adjacent letters are always from distinct groups $\Gamma_i, \Gamma_j$. Multiplication in $\Gamma$ is done by concatenation followed by reduction. Reduction is possible in $(a_1 a_2 \dots a_m)(b_1 b_2 \dots b_n)$ if and only if the letters $a_m$ and $b_1$ are from the same group $\Gamma_i$. In

*Date*: August 31, 2002.
1991 *Mathematics Subject Classification.* Primary 60F05 Secondary 60F99.
*Key words and phrases.* random walk, free product, local limit theorem, singularity analysis, Perron-Frobenius theory, compact operator.
Supported by NSF grant DMS-0071970.

this case, reduction consists of replacing the pair $a_m b_1$ of adjacent letters by the single letter $c \in \Gamma_i$ that represents the product $c = a_m b_1$ in the group $\Gamma_i$ provided $c \neq 1$; if $c = 1$ then reduction consists of deleting the pair $a_m b_1$ and then repeating the reduction step on the concatenation $(a_1 a_2 \dots a_{m-1})(b_2 b_3 \cdots b_n)$. Note that the group identity in $\Gamma$ is the empty word $\emptyset$.

We shall consider only *quasi-nearest-neighbor* random walks on free products $\Gamma$. A right random walk on $\Gamma$ is quasi-nearest-neighbor if the step distribution is supported by the set of words of length 0 or 1. Thus, there is a probability distribution $p_i$ on $\mathbb{N} \cup \{\emptyset\}$ and, for each $i \in \mathbb{N}$, a probability distribution $q_x = q_x^i$ on $\Gamma_i$ satisfying $q_1 = 0$ such that the steps $\xi_1, \xi_2, \dots$ of the random walk are i.i.d. with common distribution

$$P\{\xi_n = x\} = p_i q_x \quad \text{for each } x \in \Gamma_i \setminus \{1\} \qquad \text{and} \qquad P\{\xi_n = \emptyset\} = p_\emptyset. \tag{1.2}$$

The location of the random walker after $n$ steps is

$$S_n = \xi_1 \xi_2 \dots \xi_n. \tag{1.3}$$

To avoid uninteresting complications resulting from periodicity and reducibility, I shall assume without further notification that $p_i > 0$ for every index $i$, that for every $i$ the probability distribution $q_x$ on $\Gamma_i$ is not supported by any proper subgroup of $\Gamma_i$, and that the random walk is aperiodic, that is, that the set of positive integers $n$ such that $P\{S_n = \emptyset\} > 0$ is not contained in any proper subgroup of the integers. This implies, by a standard argument, that there exists a positive integer $n$ such that $P\{S_k = \emptyset\} > 0$ for $k = n$ and $k = n + 1$. The assumption that $p_i > 0$ and that $q_x^i$ is not supported by a proper subgroup of $\Gamma_i$ guarantees that the random walk is irreducible, that is, for any two elements $u, v \in \Gamma$ there is a positive probability path from $u$ to $v$.

1.3. **Local Limit Theorem.** The main result of the paper is a local limit theorem describing the asymptotic behavior of the return probabilities of a quasi-nearest-neighbor random walk on an infinite free product.

**Theorem 1.** *Let $\Gamma = \Gamma_1 * \Gamma_2 * \cdots$ be an infinite free product of finite groups $\Gamma_i$, and let $S_n$ be a quasi-nearest-neighbor random walk whose transition probabilities obey the aperiodicity and irreducibility requirements set out above. Then there exist constants $1 < R < \infty$ and $0 < C < \infty$ such that as $n \to \infty$,*

$$P\{S_n = \emptyset\} \sim C R^{-n} n^{-3/2}. \tag{1.4}$$

For quasi-nearest-neighbor random walks on *finite* free products, the corresponding theorem follows from the main result of [13], and in the special case where each $\Gamma_i = \mathbb{Z}_2$ from the results of [5] (see also [14] for the radially symmetric case). In these special cases the Green's function of the random walk (defined in section 2 below) is algebraic (see [1]), and in the special case studied in [5] an explicit functional equation for the Green's function is available. For random walks on infinite free products this is no longer true. The primary technical innovation here, developed in sections 3 and 4 below, is a set of techniques for studying the leading singularities of non-algebraic Green's functions in cases where these functions are embedded in certain infinite algebraic systems of generating functions.

It is somewhat remarkable that the local limit theorem holds in such generality. No requirements on either the distribution $p_i$, nor on the groups $\Gamma_i$, nor on the probability distributions $q_x^i$ are necessary, other than the minimal requirements of aperiodicity and irreducibility. In addition, Theorem 1 remains valid if some of the factors $\Gamma_i = \mathbb{Z}$, so long as each probability distributions $q_x^i$ has bounded support. Although the skeleton of the argument for this generalization is essentially the same as for the case where all the factor groups are finite, I shall for ease of exposition consider only the case of finite factors.

## 2. The Green's Function

Theorem 1 will be deduced from an analysis of the *Green's function* of the random walk. The Green's function of a random walk on a discrete group is defined to be the generating function of the return probabilities:

$$G(z) = \sum_{n=0}^{\infty} P\{S_n = \emptyset\} z^n. \tag{2.1}$$

In this section, I shall establish some fundamental properties of the Green's function and introduce a denumerable family of auxiliary generating functions to which the Green's function is related by a system of algebraic functional equations. In section 3 below, I will show how such systems may be analyzed to determine the character of the lead singularity of its component functions.

### 2.1. **First-Passage Generating Functions.**

Let $\Gamma$ be a discrete group with identity $\emptyset$, and let $S_n = \xi_1 \xi_2 \dots \xi_n$ be an irreducible right random walk on $\Gamma$. For each element $x \in \Gamma$, define

$$G_x(z) = \sum_{n=0}^{\infty} P\{S_n = x\} z^n \qquad \text{and} \tag{2.2}$$

$$F_x(z) = \sum_{n=1}^{\infty} P\{\tau_x = n\} z^n, \tag{2.3}$$

where

$$\tau_x = \min\{n \geq 0 \,:\, S_n = x\}. \tag{2.4}$$

Note that $G = G_\emptyset$. Also, if the random walk is irreducible (that is, for any two states $x, y$ there is at least one positive probability path leading from $x$ to $y$) then the sums of the series (2.2) and (2.3) are strictly positive (possibly $+\infty$) for all positive arguments $z$. Because the coefficients of the power series defining the functions $G_x$ and $F_x$ are probabilities, each has radius of convergence at least one; moreover, the first-passage generating functions $F_x(z)$ are bounded in modulus by 1 for all $|z| \leq 1$. In fact, all of the the functions $G_x(z)$ have common radius of convergence $R$, as we shall see in section 2.2 below. A fundamental theorem of Kesten [11] asserts that for any irreducible random walk on a nonamenable group, the radius of convergence of the Green's function is strictly greater than 1; since the free products defined in section 1.2 are nonamenable, it will follow that the common radius of convergence of the power series $G_x(z)$ is strictly greater than one.

The functions $G_x(z)$ can be expressed in terms of the first-passage generating functions $F_x(z)$ by an application of the Markov property. Conditioning on (i) the first step of the random walk, and then (ii) the value of the first-passage time $\tau_x$, one obtains the relations

$$G(z) = 1 + p_\emptyset z G(z) + \sum_{x \neq \emptyset} p_x z F_{x^{-1}}(z) G(z) \qquad \text{and} \tag{2.5}$$

$$G_x(z) = F_x(z) G(z) \;\; \forall\, x \neq \emptyset. \tag{2.6}$$

The first of these may be solved for $G(z)$:

$$\boxed{G(z) = \left\{ 1 - p_{\emptyset} z - \sum_{x \neq \emptyset} p_x z F_{x^{-1}}(z) \right\}^{-1}} \tag{2.7}$$

2.2. **Harnack Inequalities.** That the power series defining the functions $G_x(z)$ all have the same radius of convergence follows from a system of Harnack–type inequalities. These inequalities derive from the irreducibility of the random walk. In particular, for any two elements $x, y \in \Gamma$ there is a positive probability path leading from $x$ to $y$. Let $m$ be the length of such a path and $c_m(x, y) > 0$ its probability; then for every $n \geq 0$,

$$P\{S_{n+m} = y\} \geq c_m(x, y) P\{S_n = x\}. \tag{2.8}$$

Consequently, for every positive argument $s$ of the Green's functions,

$$G_y(s) \geq c_m(x, y) s^m G_x(s). \tag{2.9}$$

It follows that the radius of convergence of $G_x(z)$ is at least that of $G_y(z)$. Thus, all of the generating functions $G_x(z)$ have the same radius of convergence $R$. Observe that this argument is valid for irreducible random walk on any discrete group. Notice also that if the random walk is irreducible, so that the probability of return to the group identity after (say) $m$ steps is positive, then $R < \infty$, because $P\{S_{nm} = \emptyset\} \geq \varepsilon^n$ for every $n \geq 1$, where $\varepsilon > 0$ is a lower bound for the probability of return after $m$ steps.

**Proposition 2.1.** *For any irreducible random walk on a nonamenable discrete group,*

$$G(R) < \infty \tag{2.10}$$

*Proof.* The proof follows the same line as in the case of a finitely generated free group – see, for instance, [8]. In brief, the argument is as follows. If $G(R) = \infty$ then by the Harnack inequalities (2.9), the ratios $G_x(s)/G(s)$ remain bounded, and bounded away from zero, as $s \to R-$. Hence, by a diagonal argument, there is a sequence $s_n \to R-$ such that

$$\varphi_x = \lim_{n \to \infty} \frac{G_x(s_n)}{G(s_n)}$$

exists. The Harnack inequalities guarantee that the function $\varphi$ is everywhere positive. Furthermore, by equations (2.6), the function $\varphi$ is $\Gamma-$invariant, that is, the ratios $\varphi_{xy}/\varphi_x$ depend only on $y$. Most important, the function $x \mapsto \varphi_x$ is $R-$harmonic, that is, for every $x \in \Gamma$,

$$\varphi_x = R \sum_y \varphi_y P\{\xi_1 = y^{-1} x\}.$$

This follows from the hypothesis that $G(s) \to \infty$ as $s \to R-$, since by the Markov property the Green's function satisfies the relations

$$G_x(z) = \delta_{0,x} + z \sum_y G_y(z) P\{\xi_1 = y^{-1} x\}.$$

Since $\varphi$ is $\Gamma-$invariant and $R-$harmonic, the Doob $h-$transform of the transition probability kernel of the random walk $S_n$ by the ratios $R\varphi_x/\varphi_y$ is $\Gamma-$invariant. The $h-$transform is the transition kernel defined by

$$q(x, y) := RP\{\xi_1 = y^{-1} x\} \frac{\varphi_x}{\varphi_y} = q(\emptyset, x^{-1} y).$$

The iterates of this transition probability kernel satisfy

$$q^{(n)}(\emptyset, \emptyset) = R^n P\{S_n = \emptyset\}.$$

Since $\varphi_x$ is $\Gamma-$invariant, the kernel $q(x, y)$ is the transition probability kernel of a right random walk on $\Gamma$. The hypothesis that $G(R) = \infty$ and the relation between $q(x, y)$ and the transition kernel of the original random walk now implies that the random walk with transition probability kernel $q$ is recurrent. But this contradicts Kesten's theorem: there can be no recurrent irreducible random walk on $\Gamma$. □

**Corollary 2.2.** *For an irreducible random walk on a nonamenable discrete group,*

$$p_\emptyset + \sum_{x \neq \emptyset} p_x F_{x^{-1}}(R) < 1/R. \tag{2.11}$$

*Furthermore, by (2.7), each $F_x(z)$ has radius of convergence at least $R$, and for at least some $x$ the function $F_x(z)$ is singular at $z = R$.*

2.3. **Behavior off the real axis.**

**Proposition 2.3.** *For any* aperiodic*, irreducible random walk on a nonamenable discrete group, the Green's function $G(z)$ is regular at every point on the circle of convergence $|z| = R$* except *$z = R$.*

See [2] for the proof.

2.4. **The Lagrangian System.** Thus far the arguments have made no use of the particular structure of the group $\Gamma$, except for Proposition 2.1, which requires nonamenability. Assume now that $\Gamma$ is an infinite free product, and that $S_n$ is a quasi-nearest-neighbor random walk on $\Gamma$. Under this assumption, the first-passage generating functions, which by equation (2.5) determine the Green's function, are themselves interrelated by a system of functional equations that derive from the Markov property. Because the random walk is quasi-nearest-neighbor, the only values of $x$ that occur in the relation (2.7) are words with a single letter; consequently, we shall consider only these. The one-letter words $x \in \Gamma$ may be classified by the group $\Gamma_i$ to which the letter belongs: if $x \in \Gamma_i$ then we shall write

$$F_{i;x}(z) = F_x(z). \tag{2.12}$$

For those random paths that ultimately visit $x \in \Gamma_i$ there are three possibilities for the first step: $S_1$ could be $\emptyset$, it could be an element of $\Gamma_i$, or it could be an element of $\Gamma_j$ for some $j \neq i$. In the last case, the path must first return to $\emptyset$ before visiting $\Gamma_i$, and so we may condition on the first time at which this happens. This leads to the equations

(2.13)

$$F_{i;x}(z) = z \left\{ p_i q_x + p_\emptyset F_{i;x}(z) + \sum_{y \in \Gamma_i \setminus \{x\}} p_i q_y F_{i;y^{-1}x}(z) + \sum_{j \neq i} \sum_{y \in \Gamma_j} p_j q_y F_{j;y^{-1}}(z) F_{i;x}(z) \right\}.$$

We shall refer to this system as the *Lagrangian system* of the random walk. Observe that these equations may be iterated, by successive re-substitutions on the right sides. If the random walk is irreducible, as we shall assume henceforth, then for any two indices $(i; x), (j; y)$ some iterate of the equation for $F_{i;x}(z)$ includes on the right side a term in which $F_{j;y}(z)$ occurs as a factor with a positive coefficient. Consequently, all of the functions $F_{i;x}(z)$ have the same radius of convergence $R$.

Using the relation (2.7), one can rewrite the equation (2.13) in a form that relates $F_{i;x}$ directly to the Green's function $G$ and the first-passage generating functions $F_{i;y}$ indexed by those pairs $(i;y)$ such that $y$ is an element of the same factor group as $x$:

(2.14)
$$F_{i;x}(z) = z\left\{p_i q_x + \frac{F_{i;x}(z)G(z) - F_{i;x}(z)}{G(z)} + \sum_{y\neq x} p_i q_y F_{i;y^{-1}x}(z) - \sum_{y\in\Gamma_i} p_i q_y F_{i;y^{-1}}(z) F_{i;x}(z)\right\}.$$

Unlike the equations (2.13), these contain terms with negative coefficients.

2.5. **Summability of the Green's Functions.** The analysis of the Lagrangian system (2.13) will require that the vector function $F(z) = (F_{i;x}(z))$ take values in a suitable Banach space. The following *a priori* bounds ensure that $F(z)$ remains in both $\ell^1$ and $\ell^\infty$ for all $|z| \leq R$.

**Lemma 2.4.**

$$\sum_i \sum_{x\in\Gamma_i\setminus\{1\}} G_{i;x}(R) < \infty \tag{2.15}$$

*Proof.* Denote by $A_i$ the set of one-letter words $x$ such that the sole letter of $x$ is an element of $\Gamma_i \setminus \{1\}$, and consider the event that $S_n \in A_i$. This event requires that the random walk visit $A_i$ for the first time at some time $m \leq n$, and for this to occur it must be the case that the increment $\xi_m \in A_i$, since the random walk is quasi-nearest-neighbor. It is possible, but not necessary, that the random walk revisits the group identity $\emptyset$ at some time $l$ between $m+1$ and $n$, and in this case, the increment $\xi_l$ must be an element of $A_i$; the random walk must then return to $A_i$ from $\emptyset$ at some time prior to $n$, which necessitates another increment in $A_i$. In between successive visits to $A_i$ (and $\emptyset$) the random walk must make excursions into the set of words of length $\geq 2$ beginning with a letter from $A_i$ (respectively, the set of words of length $\geq 1$ that do *not* begin with a letter from $A_i$). Thus,

$$\sum_{n=1}^{\infty} P\{S_n \in A_i\}R^n \leq \sum_{k=1}^{\infty} p_i^k (RG(R))^{k+1} = \frac{p_i R^2 G(R)^2}{1 - p_i RG(R)} \tag{2.16}$$

the last provided $i$ is sufficiently large that $p_i RG(R) < 1$. (Recall that, by Proposition 2.1, the Green's function is finite at its radius of convergence $R$.) Since $\sum_i p_i = 1$, it follows that for sufficiently large $m$,

$$\sum_{i\geq m} \sum_{x\in\Gamma_i\setminus\{1\}} G_{i;x}(R) < \infty. \tag{2.17}$$

Now the difference between this sum and the sum in the statement of the lemma consists of only finitely many terms $G_{i;x}(R)$. Since by Proposition 2.1 and Corollary 2.2 these additional terms are all finite, the assertion of the lemma follows. □

**Corollary 2.5.**

$$\sum_i \sum_{x\in\Gamma_i\setminus\{1\}} F_{i;x}(R) < \infty \tag{2.18}$$

*Proof.* The first-passage generating functions $F_{i;x}(s)$ have nonnegative coefficients that are dominated by those of the corresponding Green's functions $G_{i;x}(s)$. □

## 3. Lagrangian Functional Equations in Banach Spaces

3.1. **A theorem of Flajolet and Odlyzko.** The strategy for proving the local limit theorem (1.4) will be to extract it from an analysis of the Green's function in a neighborhood of the singularity $z = R$. The cornerstone of this analysis is the following Tauberian theorem of Flajolet and Odlyzko ([4], Corollary 2).

**Transfer Theorem .** *Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R$. Suppose that $G$ has an analytic continuation to the* Pac-Man *domain*

$$\Delta_{\rho,\phi} = \{z : |z| < \rho \quad \text{and} \quad |\arg(z - R)| > \phi\}, \tag{3.1}$$

*where $\rho > R$ and $\phi < \pi/2$, and suppose that as $z \to R$ in $\Delta_{\rho,\phi}$,*

$$G(z) - C \sim K(R - z)^{\alpha} \tag{3.2}$$

*for some $K \neq 0$ and $\alpha \notin \{0, 1, 2, \dots\}$. Then as $n \to \infty$,*

$$a_n \sim \frac{K}{\Gamma(-\alpha) R^n n^{\alpha+1}}. \tag{3.3}$$

3.2. **Lagrangian Systems.** To deduce the Local Limit Theorem from the Transfer Theorem, one must show that the Green's function satisfies all of the hypotheses of the the Transfer Theorem, with $\alpha = 1/2$. For this, I shall first outline a general theory for identifying the singularity type of a Banach space-valued analytic function $F(z)$ when this function satisfies a functional equation of the form

$$F(z) = zQ(F(z)), \tag{3.4}$$

where $Q$ is a holomorphic mapping of the Banach space to itself such that $Q(0) \neq 0$. (A mapping $Q : B \to B$ is said to be *holomorphic* if it is infinitely differentiable [in the sense of [17], section 4.5], and if for every analytic function $F : \mathbb{C}^m \to B$ the composition $Q(F)$ is analytic.) Much of this theory extends, with very little change, to the more general class of functional equations

$$F(z) = Q(z, F(z));$$

however, in the interest of simplicity I shall restrict attention to systems of the more special type (3.4). That the system (2.13) satisfies the hypotheses required by the general theory will be verified in 4 below.

Functional equations of the type (3.4) for *scalar*-valued generating functions arise frequently in combinatorial applications (see, for instance [15]), and in such instances the solution $F(z)$ is directly related to the link function $Q$ by *Lagrange's identity.* For this reason I shall refer to systems of type (3.4) as *Lagrangian* systems. For *finite* Lagrangian systems, there is also an identity, due to I. J. Good [6], relating the coefficients of $F(z)$ to those of $Q(w)$, but this is not so useful for asymptotic calculations as is the Lagrange identity. In recent years several authors [12], [13], [3], [16] have developed techniques for analyzing the singularity type of a vector-valued function satisfying a functional equation of Lagrangian type where the coefficients of the link function $Q(w)$ are nonnegative, as is commonly the case in combinatorial or probabilistic applications. The technique outlined in this section is an infinite-dimensional version of the techniques employed in [12] and [13].

3.3. **Analytic continuation of a Banach space-valued function.** Let $B$ be a (complex) Banach space, and let $Q : B \to B$ be a holomorphic mapping. Under what conditions will the functional equation (3.4) have an analytic solution $F(z)$ in a neighborhood of the complex plane containing the origin $z = 0$? Clearly, the equation admits the solution $F(0) = 0$ at $z = 0$. By the Implicit Function Theorem ([17], ch. 4) for Banach-valued functions, this solution admits an analytic continuation to a neighborhood of $z = 0$, as the linearized system

$$dF = Q(F)\,dz + z\frac{\partial Q}{\partial F}\,dF \tag{3.5}$$

is solvable for $dF$ in terms of $dz$ when $z = 0$. (Here and throughout the paper $\partial Q/\partial F$ (or $\partial Q/\partial x$) will denote the Jacobian operator of the mapping $Q$.) Furthermore, analytic continuation of the solution $F(z)$ is possible along any curve in the complex plane starting at $z = 0$ on which the linear operator

$$I - z\frac{\partial Q}{\partial F} \tag{3.6}$$

remains invertible. This will be the case as long as the spectral radius of the operator

$$z\mathcal{L}(z) := z\left(\frac{\partial Q}{\partial F}\right)_{F(z)} \tag{3.7}$$

remains less than one. Thus, singular points of the analytic function $F(z)$ solving (3.4) can only occur at those points where the spectral radius of $z\mathcal{L}(z)$ attains or exceeds the value 1.

3.4. **Lyapunov-Schmidt reduction.** Unfortunately, the spectral radius of a continuous, operator-valued function need not be itself continuous (although it must be at least lower semi-continuous), and so the singular points of the solution $F(z)$ of (3.4) may not in general be located by searching for the points $z$ where the spectrum of $z\mathcal{L}(z)$ includes the value 1. However, in certain problems the spectral radius of $\mathcal{L}(s)$ will for positive $s$ be an isolated eigenvalue of finite multiplicity, and in such cases the spectral radius of $\mathcal{L}(z)$ will vary continuously with $z$ near the positive axis $s > 0$. When this happens, the singular behavior of $F(z)$ at $z = R$ will essentially be determined by a finite-dimensional section, and classical methods of algebraic geometry (the Weierstrass preparation theorem, Newton diagrams, Puiseux expansions, etc.) may be used. This program is known in nonlinear analysis as *Lyapunov-Schmidt reduction*; it is commonly used in bifurcation theory (see, for instance, [17], chapter 8).

**Hypothesis 1.** *There exists $R \in (0, \infty)$ such that*

*(a) the spectral radius of $s\mathcal{L}(s)$ is less than 1 for all $s \in [0, R)$;*
*(b) $\lim_{z\to R} F(z) = F(R)$ exists and is finite;*
*(c) 1 is an isolated eigenvalue of $R\mathcal{L}(R)$ with finite multiplicity.*

Condition (a) guarantees that $F(z)$ has a unique analytic continuation along the line segment $[0, R)$, and condition (b) implies that the Jacobian operator $\mathcal{L}(z)$ has a limit $\mathcal{L}(R)$ as $z \to R$ (from inside the disk). Condition (c) implies that the Banach space $B$ may be decomposed as a direct sum

$$B = V \oplus W, \tag{3.8}$$

where $V$ is the space of eigenvectors of $R\mathcal{L}(R)$ with eigenvalue 1, and that there is a projection operator $P_V : B \to V$ with range $V$ that commutes with $\mathcal{L}(R)$, given by

$$P_V = \frac{1}{2\pi i} \oint (\zeta - R\mathcal{L}(R))^{-1} \, d\zeta, \tag{3.9}$$

where the contour integral extends over a circle in the complex plane surrounding the point $\zeta = 1$ that contains no other points of the spectrum of $R\mathcal{L}(R)$ (see [10], Theorem 6.17). It then follows that

$$P_W = I - P_V \tag{3.10}$$

is a projection operator with range $W$ that also commutes with $R\mathcal{L}(R)$. That 1 is an isolated point of the spectrum of $R\mathcal{L}(R)$ implies that 1 is *not* in the spectrum of the restriction to $W$ of $P_W R\mathcal{L}(R)$, equivalently, $I - P_W R\mathcal{L}(R)$ is invertible on $W$.

**Proposition 3.1.** *Assume that Hypothesis 1 holds, and let $P_V$ be the projection operator defined by (3.9). Then in some neighborhood of $z = R$, the $V-$valued function $P_V F(z)$ satisfies a functional equation of the form*

$$P_V F(z) = z Q_V(z, P_V F(z)), \tag{3.11}$$

*where $Q_V(z, v)$ is a function of* $\dim V + 1$ *complex variables that is holomorphic in a neighborhood of $z = R$, $v = P_V F(R)$.*

*Proof.* Consider the mapping $K : \mathbb{C} \times V \times W \to W$ defined by $K(z, v, w) = w - z P_W Q(v + w)$. This is holomorphic, and takes the value 0 at the point $\omega := (R, P_V F(R), P_W F(R))$. I claim that there is a holomorphic mapping $W(z, v)$, valued in $W$ and defined in a neighborhood of $z = R$, $v = P_V F(R)$, such that for all $z, v$ in this neighborhood,

$$W(z, v) = z P_W Q(v + W(z, v)),$$

and such that all zeros of $K(z, v, w)$ near $\omega$ are of the form $(z, v, W(z, v))$. This follows from the Implicit Function Theorem for the mapping $K$, because the Jacobian of the mapping $w \mapsto K(z, v, w)$ is

$$I - z P_W (\partial Q / \partial F)_{v+w}$$

and this is nonsingular at $z = R$, $v = P_V F(R)$, since $P_W R\mathcal{L}(R) = R\mathcal{L}(R) P_W$ and $I - P_W R\mathcal{L}(R)$ is invertible on $W$. Now consider the functional equation (3.4). Applying the projections $P_V$ and $P_W$ to both sides, and using the relation $I = P_V + P_W$, one obtains

$$\begin{aligned} P_V F(z) &= z P_V Q(P_V F(z) + P_W F(z)) \qquad \text{and} \\ P_W F(z) &= z P_W Q(P_V F(z) + P_W F(z)). \end{aligned}$$

The solution to the second of these equations must be at $P_W F(z) = W(z, P_V F(z))$, and so the first may be rewritten in the form

$$P_V F(z) = z P_V Q(P_V F(z) + W(z, P_V F(z))).$$

Since the function $W(z, v)$ is holomorphic in its arguments, this is the desired functional equation. $\square$

3.5. **Simple eigenvalues and square-root singularities.** Hypothesis 1 requires only that the spectrum of the Jacobian operator $\mathcal{L}(R)$ have an isolated lead eigenvalue of finite multiplicity. If the lead eigenvalue is simple, as will often be the case, the conclusions of Proposition 3.1 can be substantially strengthened.

**Hypothesis 2.** *There exists $R \in (0, \infty)$ such that*

*(a) the spectral radius of $s\mathcal{L}(s)$ is less than 1 for all $s \in [0, R)$;*
*(b) $\lim_{z \to R} F(z) = F(R) \neq 0$ exists and is nonzero and finite;*
*(c) 1 is an isolated eigenvalue of $R\mathcal{L}(R)$ with multiplicity 1.*
*(d) The projection $P_V F(R)$ of $F(R)$ on the 1−eigenspace $V$ is* nonzero.

When Hypothesis 2 holds, the projection $P_V$ defined by (3.9) will have a one-dimensional range, and so there will exist nonzero elements $h \in B$ and $\nu \in B^*$ (here $B^*$ denotes the Banach space dual to $B$) such that for all $\varphi \in B$,

$$P_V \varphi = \langle \nu, \varphi \rangle h. \tag{3.12}$$

Note that since $P_V$ is a projection and $h$ is in the range of $P_V$,

$$\langle \nu, h \rangle = 1. \tag{3.13}$$

**Proposition 3.2.** *Assume that Hypothesis 2 holds, and let $h$ and $\nu$ be such that (3.12) holds. Define*

$$v(z) = \langle \nu, F(z) \rangle. \tag{3.14}$$

*Then there exist an integer $m \geq 1$ and a function $A(\zeta)$ holomorphic in a neighborhood of $\zeta = 0$, satisfying $A(0) = 0$, such that in some neighborhood of $z = R$,*

$$v(R) - v(z) = A((R - z)^{1/m}). \tag{3.15}$$

**Remark:** The representation (3.15) implies that the function $v(z)$ has an analytic continuation to a slit disk centered at $z = R$ of the form

$$\{z : |z - R| < \varepsilon \quad \text{and} \quad |\arg(z - R)| > 0\} \tag{3.16}$$

and that its asymptotic behavior near $z = R$ is of the form (3.2) for some rational $\alpha$.

*Proof.* Set $u(\xi) = v(R) - v(R - \xi)$. By Proposition 3.1, the function $P_V F(z) = v(z)h$ satisfies a functional equation (3.11) in a neighborhood of $z = R$. This may be rewritten in $\xi$ and $u$ as a functional equation for $u(\xi)$ of the form

$$K(\xi, u(\xi)) = 0 \tag{3.17}$$

where $K(\xi, u)$ is a function of two variables $(\xi, u)$ that is holomorphic in a neighborhood of the origin in $\mathbb{C}^2$, and $K(0, 0) = 0$. We may assume that the power series expansion of the function $K(\xi, u)$ contains a term $au^d$, where $a \neq 0$, that is not divisible by $\xi$, for if this were not the case then $K$ could be replaced by $K/\xi^m$ for some $m$. (The power series for $K$ must include terms divisible by $u$, because the functional equation (3.17) derives from (3.11), which holds for $z < R$ near $R$.) Thus, by the Weierstrass Preparation Theorem ([7], Chapter 1) there exists a Weierstrass polynomial

$$\Phi(\xi, u) = u^d + \sum_{j=1}^{d} a_j(\xi) u^{d-j} \tag{3.18}$$

such that in some neighborhood of $(0,0) \in \mathbb{C}^2$ the zero sets of $K$ and $\Phi$ coincide. (NOTE: The definition of a Weierstrass polynomial requires that each of the coefficients $a_j(\xi)$ be holomorphic in $\xi$.)

Since the ring of holomorphic functions near $(0,0)$ is a unique factorization domain ([7], Chapter 1), the polynomial $\Phi$ may be factored into irreducible Weierstrass polynomials:

$$\Phi(\xi, u) = \prod_{i=1}^{r} \Phi_i(\xi, u). \tag{3.19}$$

Here each $\Phi_i$ is irreducible in the ring of locally holomorphic functions; this means that in some neighborhood of the origin, $\Phi_i(\xi, u)$ is, for each fixed $\xi \neq 0$, irreducible in the ring $\mathbb{C}[u]$ of polynomials in the variable $u$. Consequently, the derivative $\partial\Phi_i/\partial u$ cannot have a zero in common with $\Phi_i$ (except when $\xi = 0$), and so by the Implicit Function Theorem, for each $\xi \neq 0$ and each $u$ such that $\Phi_i(\xi, u) = 0$, the equation $\Phi_i(\xi', u') = 0$ defines a branch of an analytic function $u(\xi')$ near $\xi' = \xi$ and $u' = u$. For at least one of the indices $i$ (say $i = 1$), one of the branches of the analytic function $u(\xi)$ defined by $\Phi_i = 0$ coincides with the function $u(\xi) := v(R) - v(R - \xi)$ for $\xi > 0$ small.

The result (3.15) now follows by a standard argument in the theory of algebraic functions of a single variable (see, for instance, [9], vol. 2, section 12.2). If the branch $u(\xi)$ is followed around a small contour $\xi \in \gamma$ surrounding 0, then after a finite number $m$ of circuits $u(\xi)$ will return to the original branch (because for each $\xi$ the polynomial $\Phi_1(\xi, u)$ has only $m$ roots, where $m$ is the degree of $\Phi_1$ in the variable $u$). Consequently, $u$ is an analytic function of $\xi^m$ in a neighborhood of $\xi = 0$ (observe that $\xi = 0$ is a removable singularity because all roots of $\Phi_1(\xi, u) = 0$ approach zero as $\xi \to 0$), and the representation (3.15) follows. □

Let $\mathcal{H}(v, w)$ be the Hessian (second differential) form of the mapping $Q$ at the point $F(R)$, that is, the symmetric bilinear form $\mathcal{H} : B \times B \to B$ defined by

$$\mathcal{H}(v, v) = \left( \frac{d^2}{d\varepsilon^2} Q(F(R) + \varepsilon v) \right)_{\varepsilon=0}. \tag{3.20}$$

**Theorem 2.** *Let $F(z)$ be the solution of the Lagrangian equation (3.4), and assume that Hypothesis 2 holds. Assume that the Hessian $\mathcal{H}(h, h)$ has a nonzero projection on the subspace $V$, that is,*

$$\langle \nu, \mathcal{H}(h, h) \rangle \neq 0. \tag{3.21}$$

*Then the function $v(z) := \langle \nu, F(z) \rangle$ has a square-root singularity at $z = R$, that is, for some nonzero constant $C$,*

$$v(R) - v(z) \sim C\sqrt{R - z} \tag{3.22}$$

*as $z \to R$ in a slit domain (3.16). Furthermore, the function $P_W F(z)$ satisfies*

$$P_W F(R) - P_W F(z) = O(|R - z|) \tag{3.23}$$

*near $z = R$.*

**Remark.** Theorem 2 shows that under Hypothesis 2 and (3.21), the function $v(z)$ satisfies most of the hypotheses of the Flajolet-Odlyzko Transfer Theorem, with $\alpha = 1/2$: the only hypothesis that requires separate verification is that $v$ has no singularity on the circle $|z| = R$ except that at $z = R$. Puiseux expansions for other linear functionals of $F(z)$ can be deduced from (3.22)and (3.23).

*Proof of Theorem 2.* Proposition 3.2 implies that $v(z)$ has a Puiseux expansion in powers of $(R-z)^\alpha$ for some positive rational number $\alpha$. Since $P_W F(z) = W(z, v(z)h)$ where $W(z,v)$ is the holomorphic function constructed in the proof of Proposition 3.1, it follows that $P_W F(z)$ also has a Puiseux expansion in powers of $(R-z)^\alpha$. We must show that $\alpha$ is a multiple of $1/2^k$ for some $k \geq 1$, and that the first nonzero terms in the expansions are as advertised.

Write

$$W(z) = P_W F(z), \qquad \Delta z = R - z,$$
$$V(z) = P_V F(z) = v(z)h \qquad \Delta F = F(R) - F(z),$$

etc. Recall that $v(R) \neq 0$, by (d) of Hypothesis 2. The functional equation (3.4) and Taylor's theorem ([17], section 4.6) imply that

$$\Delta F = (F(R)/R)\Delta z + R\mathcal{L}(R)\Delta F - \frac{1}{2}\mathcal{H}(\Delta F, \Delta F) + \mathcal{R}_3 \tag{3.24}$$

where the remainder $\mathcal{R}_3$ is $o(\Delta z + \|\Delta F\|^2)$ as $\Delta z \to 0$. Apply the projections $P_V, P_W$ to both sides and use the bilinearity of the Hessian form to obtain

$$\Delta W = (W(R)/R)\Delta z + P_W R\mathcal{L}(R)\Delta W - P_W \mathcal{H}(\Delta V, \Delta V)/2 + \mathcal{R}_W \quad \text{and} \tag{3.25}$$

$$\Delta V = (V(R)/R)\Delta z + P_V R\mathcal{L}(R)\Delta V - P_V \mathcal{H}(\Delta V, \Delta V)/2 + \mathcal{R}_V \tag{3.26}$$

where $\mathcal{R}_W, \mathcal{R}_V = O(\|\Delta V\|\|\Delta W\| + \|\Delta W\|^2) + o(\Delta z)$. Since $P_V$ is the projection onto the $1-$eigenspace of $R\mathcal{L}(R)$, the second term on the right side of (3.26) cancels the left side, and so the equation reduces to

$$(\Delta v)^2 \langle \nu, \mathcal{H}(h,h)\rangle = 2(v(R)/R)\Delta z + \mathcal{R}'_V \tag{3.27}$$

where $\mathcal{R}_V = \mathcal{R}'_V h$. Recall that the operator $P_W - P_W R\mathcal{L}(R)$ is invertible on the Banach space $W$, with inverse (say) $\mathcal{M}$, so equation (3.25) may be rewritten as

$$\Delta W = \mathcal{M}(W(R)/R)\Delta z - (\Delta v)^2 \mathcal{M}(\mathcal{H}(h,h)) + \mathcal{R}_W. \tag{3.28}$$

There are now two possibilities: (a) $\|\Delta W\| = o(|\Delta v|)$, or (b) there is a sequence of points $z_n \to R$ along which $\|\Delta w\| \geq \varepsilon|\Delta v|$ for some $\varepsilon > 0$. I claim that possibility (b) cannot occur: If so, equation (3.28) would imply that $\|\Delta W\| = O(\Delta z)$; this would imply that the remainder term in equation (3.27) is $o(\Delta z)$; and this would make (3.27) impossible, since $v(R) \neq 0$ and $\langle \nu, \mathcal{H}(h,h)\rangle \neq 0$. Thus, (a) must hold. But (a) implies that the remainder term $\mathcal{R}'_V$ is of smaller order of magnitude than the other terms in equation (3.27). It follows that $(\Delta v)^2 \sim C\Delta z$ for some nonzero constant $C$. This implies that the leading nonconstant term in the Puiseux expansion of $v(z)$ is $C\sqrt{R-z}$ for some nonzero $C$. That $P_W F(z)$ has Puiseux expansion (3.23) with first nonzero term proportional to $(R-z)$ now follows from (3.10) together with (3.28). □

3.6. **Positivity.** In certain combinatorial applications, including that discussed in section 4 below, the function $F(z)$ and the link mapping $Q$ in the system (3.4) will be defined by power series with nonnegative coefficients. When this is the case, the Jacobian operators $\mathcal{L}(s)$ will, for $s \geq 0$, be positive operators, and so methods from the Perron-Frobenius theory of positive operators may be bought to bear on the problem of verifying the hypotheses of Theorem 2. In other problems, only the link function $Q$ may be known *a priori* to be defined by power series with nonnegative coefficients. In this section, I will show that in such circumstances the theory of positive operators may still be applicable.

*Assume* that the (complex) Banach space $B$ is *ordered*, that is ([17], Chapter 7), there is a closed, nonempty set $B_+ \neq \{0\}$ (called an *order cone*) such that

(a) If $x, y \in B_+$ then $ax + by \in B_+$ for all nonnegative scalars $a, b$; and
(b) If $x \in B_+$ and $-x \in B_+$ then $x = 0$.

In most applications, $B$ will be a space of functions or sequences and $B_+$ the subset of nonnegative functions. The order cone $B_+$ induces an order cone $B_+^*$ on the dual space $B^*$: the order cone $B_+^*$ is defined to be the set of all $\nu \in B^*$ such that $\langle \nu, f \rangle \geq 0$ for all $f \in B_+$. The order cones $B_+$ and $B_+^*$ induce partial orders on the Banach spaces $B, B^*$ in the obvious way: $f \leq g$ if $g - f \in B_+$, and similarly for $B^*$. Elements of $B_+$ and $B_+^*$ will be called *nonnegative*, or *positive* if they are distinct from the zero vector; elements of the *interiors* of $B_+$ and $B_+^*$ will be called *strictly positive*. Note that if $h \in B$ and $\nu \in B^*$ are strictly positive, then

$$\begin{aligned} \langle \mu, h \rangle &> 0 \qquad \text{for all} \quad \mu \in B_+^*, \qquad\qquad \text{and} \\ \langle \nu, f \rangle &> 0 \qquad \text{for all} \quad f \in B_+. \end{aligned}$$

An entire holomorphic mapping $Q : B \to B$ will be called *nonnegative* if for every $B-$valued analytic function $F(z_1, z_2, \ldots, z_m)$ of $m$ variables whose power series coefficients are nonnegative, the function $Q(F(z_1, z_2, \ldots, z_m))$ has a power series with nonnegative coefficients. When the mapping $Q$ is given in the form of a convergent series, as in (2.13), this is equivalent to requiring that the coefficients in the series are nonnegative. Observe that if $Q : B \to B$ is nonnegative, and if $F, G$ are $B-$valued functions with nonnegative power series coefficients such that the coefficients of $F$ are dominated by those of $G$, then the power series coefficients of $Q(F(z))$ are dominated by those of $Q(G(z))$ (this follows from the fact that the function of two variables $Q(F(z) + \zeta(G(z) - F(z)))$ has nonnegative coefficients). Moreover, if $Q$ is nonnegative, and if $F, G$ are $B-$valued analytic functions in one variable whose power series coefficients are nonnegative, then the function

$$z \mapsto \left( \frac{\partial Q}{\partial x} \right)_{F(z)} G(z) \tag{3.29}$$

is nonnegative (since the function $Q(F(z)+\zeta G(z))$ has nonnegative power series coefficients).

**Proposition 3.3.** *If the holomorphic mapping $Q : B \to B$ is nonnegative and such that $Q(0) > 0$, then the unique analytic $B-$valued solution $F(z)$ to (3.4) near $z = 0$ has nonnegative power series coefficients.*

**Note.** In problems where $F(z)$ is defined by a power series with nonnegative coefficients, such as that considered in section 4 below, Proposition 3.3 will be superfluous.

*Proof.* For the purposes of this proof, $F \leq G$ means that the power series coefficients of $F(z)$ are dominated by those of $G(z)$. Define a series of approximate solutions to (3.4) by setting $F_0(z) = 0$ and for each $n \geq 0$,

$$F_{n+1}(z) = zQ(F_n(z)). \tag{3.30}$$

Because the mapping $Q$ is holomorphic, its Jacobian $\partial Q / \partial x$ is uniformly bounded in norm for $\|x\| \leq C$, for any $C < \infty$; consequently, the mapping which sends a function $H(z)$ to the function $zQ(H(z))$ is contractive for small $z$ and $H$. Therefore, the sequence of functions $F_n$ defined by (3.30) is uniformly norm convergent for $|z| \leq \varepsilon$, provided $\varepsilon > 0$ is sufficiently small, and the limit function $F(z)$ is the solution to (3.4).

Since the mapping $Q$ is nonnegative, $F_1 \geq F_0 = 0$, and hence, by induction, $F_{n+1} \geq F_n$ for every $n \geq 0$. In particular, each $F_n(z)$ has nonnegative power series coefficients. Since the functions $F_n(z)$ converge uniformly in norm to $F(z)$, the Cauchy Integral Formula implies that $F(z)$ has nonnegative power series coefficients. $\square$

*Assume henceforth* that the holomorphic mapping $Q : B \to B$ is nonnegative, and let $F(z)$ be the unique $B-$valued solution to the Lagrangian system (3.4).

**Corollary 3.4.** *If the spectral radius of $s\mathcal{L}(s)$ is less than 1 for all $s \in [0, R)$, then for each $s \in [0, R)$ the operator $\mathcal{L}(s)$ is nonnegative, and for $0 < s_1 < s_2 \leq R$,*

$$\mathcal{L}(s_1) \leq \mathcal{L}(s_2). \tag{3.31}$$

*Proof.* Since by Proposition 3.3 the function $F(z)$ has nonnegative power series coefficients, the Jacobian $\mathcal{L}(s)$ must be a nonnegative operator at all $s > 0$ such that $F$ admits an analytic continuation along the line segment $[0, s]$. The hypothesis that the spectral radius of $s\mathcal{L}(s)$ is less than 1 for $s < R$ assures that $F$ has an analytic continuation on $[0, R)$. The monotonicity in $s$ of $\mathcal{L}(s)$ follows from the fact that the function defined by (3.29), with $G$ =constant, has nonnegative coefficients. $\square$

Finally, the existence of an order structure in the Banach space $B$ can sometimes facilitate the verification of conditions (b) of Hypotheses 1 and 2, respectively.

**Proposition 3.5.** *Assume that the ordered Banach space $B$ has the following property: Every nondecreasing, norm-bounded sequence $x_n \in B_+$ converges in $B$. Assume further that for every continuous function $\varphi : [0, \infty) \to B_+$ such that $\lim_{s\to\infty} \|\varphi(s)\| = \infty$,*

$$\lim_{s\to\infty} \text{spectral radius} \left(\frac{\partial Q}{\partial x}\right)_{\varphi(s)} = \infty. \tag{3.32}$$

*Define $R = \sup\{s : \text{spectral radius}\,(s\mathcal{L}(s)) < 1\}$. Then $R < \infty$ and $\lim_{s\to R-} F(s) := F(R)$ exists and is finite.*

**Remark.** Not all Banach spaces have the property required: for instance $\ell^1$ has it, but $\ell^\infty$ does not.

*Proof.* By assumption, $Q(0) \neq 0$, and so by (3.5) the derivative $dF/dz$ is nonzero, and hence positive, at $z = 0$. Since the coefficients in the power series of $F(z)$ are nonnegative, it follows that the growth of the function $F(s)$ along the positive real axis is at least linear. Hence, if $R = \infty$ then as $s \to R-$ the norm of $F(s)$ becomes unbounded, and so by hypothesis the spectral radius of $\mathcal{L}(s)$ also becomes unbounded. Consequently, for some finite $s$ the spectral radius of $s\mathcal{L}(s)$ must exceed 1, contradicting the assumption that $R = \infty$.

Since the power series coefficients of $F(z)$ are nonnegative, $F(s)$ is nondecreasing in $B_+$ for $s \in [0, R)$. Therefore, if $\|F(s)\|$ remains bounded as $s \to R-$ then $\lim_{s\to R-} F(s)$ must exist in $B$. Suppose then that $\|F(s)\|$ does *not* remain bounded as $s \to R-$. Let $\gamma : [0, \infty] \to [0, R]$ be an increasing homeomorphism, and define $\varphi(s) = F(\gamma(s))$. Then $\lim_{s\to\infty} \|\varphi(s)\| = \infty$, so as $s \to R-$ the spectral radius of $(\partial Q/\partial x)_{F(s)}$ becomes unbounded. But this contradicts the definition of $R$. $\square$

## 4. Proof of the Local Limit Theorem

4.1. **Holomorphic character of the link function.** The machinery developed in section 3 requires that the function $F(z)$ take values in a Banach space $B$, that the link function $Q : B \to B$ be holomorphic, and that the conditions of Hypothesis 2 be satisfied. Of these, the most critical is that 1 should be an isolated, simple eigenvalue of the Jacobian operator $R\mathcal{L}(R)$, and the proof of this will occupy most of the argument.

Recall that, by Corollary 2.5, the function $F(z) = (F_{i;x}(z))$ takes values in the Banach space $B$ of bounded sequences with limit 0 for all $|z| \leq R$. The Lagrangian system (2.13) has the special form

$$F(z) = z(M(F(z)) + N(F(z)) \times F(z)) \tag{4.1}$$

where $M, N : B \to B$ are bounded linear operators and the operation $\times$ is coordinatewise multiplication. Now any linear operator $L : B \to B$ is clearly holomorphic, and by the product rule ([17], sec. 4.3) the coordinatewise product of holomorphic mappings is holomorphic, so the implied link function $Q$ for the system (2.13) is holomorphic on $B$.

4.2. **Spectrum of the Jacobian Operator $\mathcal{L}(s)$.** Recall from equation (3.7) that the operator $\mathcal{L}(s)$ is the Jacobian $\partial Q/\partial x$ evaluated at $x = F(s)$, where $Q$ is the link function in the Lagrangian system (3.4). In our case the Lagrangian system specializes to (2.13); the function $F(z)$ has coordinates $F_{i;x}(z)$, indexed by $i \in \mathbb{N}$ and $x \in \Gamma_i \setminus \{1\}$, and by Corollary 2.5 the function $F(z)$ lies in $B$ for all $|z| \leq R$. Thus, by (2.13), the Jacobian in matrix form has entries

$$\begin{aligned} \mathcal{L}(s)_{(i;x),(j;y)} &= p_j q_{y^{-1}} F_{i;x}(s) && \text{for} \quad j \neq y; \\ &= p_i q_{xy^{-1}} && \text{for} \quad j = i \quad \text{and} \quad y \neq x; \\ &= p_\emptyset + \sum_{j \neq i} \sum_{y \in \Gamma_j \setminus \{1\}} p_j q_y F_{j;y^{-1}}(s) && \text{for} \quad j = i \quad \text{and} \quad y = x. \end{aligned} \tag{4.2}$$

This is clearly a positive operator for $s \geq 0$, as all entries in the matrix are nonnegative. Moreover, since the functions $F_{i;x}(s)$ are nondecreasing in $s$, the matrices $\mathcal{L}(s)$ increase with $s$ (this also follows from Corollary 3.4), as do their norms and spectral radii.

The key to the spectral analysis of the Jacobian operator is that it decomposes as the sum of a compact operator and a scalar multiple of the identity. To see this, recall that by equation (2.7),

$$\frac{G(z) - 1}{zG(z)} = p_\emptyset + \sum_j \sum_{y \in \Gamma_j \setminus \{1\}} p_j q_{y^{-1}} F_{j;y}(z) \tag{4.3}$$

where $G(z)$ is the Green's function; note for future reference that $(G(s) - 1)/sG(s)$ is nondecreasing in $s$ for $0 \leq s \leq R$, and that

$$\frac{G(R) - 1}{G(R)} < \frac{1}{R}, \tag{4.4}$$

by Corollary 2.2. Equation (4.3) implies that the diagonal terms of $\mathcal{L}(z)$ may be rewritten as

$$\mathcal{L}(z)_{(i;x),(i;x)} = \frac{G(z) - 1}{zG(z)} - \sum_{y \in \Gamma_i} p_i q_y F_{i;y^{-1}}(z). \tag{4.5}$$

Therefore,

$$\boxed{\mathcal{L}(s) = \mathcal{K}(s) + \mathcal{N}(s) + ((G(s) - 1)/sG(s))I} \tag{4.6}$$

where $\mathcal{K}(s)$ has entries

$$\mathcal{K}(s)_{(i;x),(j;y)} = p_j q_{y^{-1}} F_{i;x}(s) \tag{4.7}$$

and $\mathcal{N}$ is block diagonal, with nonzero entries only in the $((i;x),(i;y))$ positions, given by

$$\begin{aligned} \mathcal{N}(s)_{(i;x),(i;y)} &= p_i q_{xy^{-1}} - p_i q_{y^{-1}} F_{i;x}(s) && \text{for} \quad y \in \Gamma_i \setminus \{x, 1\} \\ &= -p_i q_{x^{-1}} F_{i;x}(s) - \sum_{y \in \Gamma_i} p_i q_y F_{i;y^{-1}}(s) && \text{for} \quad y = x. \end{aligned} \tag{4.8}$$

**Lemma 4.1.** *Let $R$ be the common radius of convergence of the Green's function $G(z)$ and the first-passage generating functions $F_{i;x}(z)$. For each $s \in [0, R]$, the operators $\mathcal{K}(s)$ and $\mathcal{N}(s)$ are compact.*

*Proof.* The operator $\mathcal{K}(s)$ has rank one and so is trivially compact. The operator $\mathcal{N}(s)$ is block diagonal, with nonzero entries only in the $((i;x),(i;y))$ entries, where $x, y \in \Gamma_i$. Moreover, because $y \mapsto q_y = q^i_y$ is for each index $i$ a probability distribution on the elements $y \in \Gamma_i \setminus \{1\}$, and because the functions $F_{i;y}$ are uniformly bounded for $0 < s \leq R$, by Corollary 2.5, the absolute values of the entries in row $(i;x)$ have sum that is $O(p_i)$. Since $p_i \to 0$, it follows that $\mathcal{N}(s)$ is compact, because $\mathcal{N}(s)$ maps the unit ball of $B$ (the set of sequences $z_{i;x}$ with entries bounded by 1 in absolute value) to a compact set, consisting of those sequences $\{\zeta_{i;x}\}$ whose entries satisfy $|\zeta_{i;x}| \leq Cp_i$. □

**Corollary 4.2.** *For each $s \in [0, R]$, the operator $\mathcal{L}(s)$ and its adjoint $\mathcal{L}(s)^*$ have purely discrete spectrum. Each element of the spectrum not equal to $(G(s)-1)/sG(s)$ is an eigenvalue of finite multiplicity (the same for both $\mathcal{L}(s)$ and $\mathcal{L}(s)^*$), and $(G(s) - 1)/sG(s)$ is the only possible accumulation point of the spectrum. The spectral radius of $\mathcal{L}(s)$ varies continuously with $s$ for $0 \leq s \leq R$. There is a unique $\sigma \in (0, R]$ such that the spectral radius of $\sigma\mathcal{L}(\sigma)$ is 1, and at least for $s > \sigma - \varepsilon$, for some $\varepsilon > 0$, the spectral radius of $\mathcal{L}(s)$ is an eigenvalue of finite multiplicity.*

**Note:** Later it will be shown that $\sigma = R$.

*Proof.* The Riesz-Schauder theory of compact operators (see, for example, [10], Ch. 4) asserts that the spectrum of a compact operator consists of at most countably many isolated eigenvalues of finite multiplicity (the same multiplicity for both the operator and its adjoint) that accumulate only at 0. (For a compact operator on an infinite-dimensional Banach space, 0 must also be an element of the spectrum). Since addition of a scalar multiple of the identity to an operator merely shifts its spectrum, the statements about the spectrum of a particular $\mathcal{L}(s)$ follow.

To see that the spectral radius $\rho(s)$ of $\mathcal{L}(s)$ must vary continuously with $s$, observe that either $\rho(s) = (G(s) - 1)/sG(s)$ or $\rho(s)$ is an isolated eigenvalue of finite multiplicity. Isolated eigenvalues of finite multiplicity must vary continuously ([10], Chapter 6), but so does the Green's function; hence, the spectral radius is continuous in $s$. The spectral radius of $s\mathcal{L}(s)$ cannot remain bounded away from 1 for $s \in [0, R]$, because if so then by the Implicit Function Theorem, the function $F(z)$ would have an analytic continuation to a neighborhood of $z = R$. Thus, by the Intermediate Value Theorem of calculus, for some $\sigma \in [0, R]$ the spectral radius of $\sigma\mathcal{L}(\sigma)$ attains the value 1.

Recall now that The function $s \mapsto (G(s) - 1)/sG(s)$ is nondecreasing and is bounded above by $1/R^2$, by (4.4). Consequently, when the spectral radius of $s\mathcal{L}(s)$ exceeds $1/R$ it must coincide with an eigenvalue of finite multiplicity. Since the spectral radius of $s\mathcal{L}(s)$ is nondecreasing in $s$, the last assertion of the corollary follows. □

### 4.3. **Simplicity of the Lead Eigenvalue.**

**Proposition 4.3.** *The eigenvalue* 1 *of the operator* $T = \sigma\mathcal{L}(\sigma)$ *has multiplicity one. Furthermore, there are* strictly *positive right and left eigenvectors* $h$ *and* $\nu$*.*

The proof will consist of several lemmas. By Corollary 4.2, 1 is an eigenvalue of $T$ with finite multiplicity, and no elements of the spectrum have modulus greater than one. *A priori* it is possible that 1 is not the only element of the spectrum on the unit circle; however, Corollary 4.2 implies that the spectrum of $T$ has only finitely many points on the unit circle, and all are eigenvalues of finite multiplicity.

**Lemma 4.4.** *There exist positive right and left eigenvectors* $h$ *and* $\nu$ *for* $T$ *with eigenvalue* 1*.*

*Proof.* Because the operator $T$ is positive, if $T\varphi = \varphi$ then $T|\varphi| \geq |\varphi|$. Since $V$ contains nonzero elements, it follows that there is a nonnegative vector $g$ such that $Tg \geq g$. By the positivity of $T$, the sequence $g_n := T^n g$ is nondecreasing, and so in particular each $g_n$ is positive. If it could be shown that the sequence $g_n$ converges in norm, then the limit $h$ would be a positive eigenvector of $T$ with eigenvalue 1.

**Claim:** The sequence $g_n$ converges in $B-$norm.

*Proof.* Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the eigenvalues of modulus 1, with $\lambda_1 = 1$, and let $V_1, V_2, \ldots, V_r$ be the corresponding eigenspaces. Define projections $P_1, P_2, \ldots, P_r$ by

$$P_j = \frac{1}{2\pi i} \oint_{\gamma_j} (\zeta I - T)^{-1} \, d\zeta,$$

where $\gamma_j$ is a circle surrounding $\lambda_j$ that encircles no other points of the spectrum. Each $P_j$ is a projection whose range is the eigenspace $V_j$ (see [10], Theorem 6.17). The projections $P_j$ commute with each other and with $T$ (since they are linear combinations of powers of $T$). The range $U$ of complementary projection

$$P_U = I - P_1 - P_2 - \cdots - P_r$$

is an invariant subspace of $T$, and the spectral radius of $TP_U$ is strictly less than one. Now

$$\begin{aligned} T^n g &= \sum_{j=1}^{r} T^n P_j g + T^n P_U g \\ &= \sum_{j=1}^{r} \lambda^n P_j g + T^n P_U g. \end{aligned}$$

Since each $\lambda_j$ has absolute value 1, any increasing subsequence of the integers has a subsequence along which $\lambda_j^n$ converges; consequently, along any such subsequence, $T^n g$ converges in norm. The limit is of necessity a positive element of $B$. Finally, there can be only one possible limit, because since $g_n \leq g_{n+1}$ for all $n$, if there were two subsequential limits $\psi_A$ and $\psi_B$ then $\psi_A \leq \psi_B$ and $\psi_B \leq \psi_A$. □

A similar argument shows that there is a positive left eigenvector. □

The following lemma will imply that positive eigenvectors of $T$ and $T^*$ are *strictly* positive, and this in turn will imply that the lead eigenspaces are one-dimensional.

**Lemma 4.5.** *Let $\mathcal{M}_m$ be the finite section of $\mathcal{L}(s)$ indexed by pairs $((i;x),(j;y))$ such that $1 \leq i,j \leq m$. Then for each $m = 3,4,\dots$ and every $s \in (0,R]$, the matrix $\mathcal{M} = \mathcal{M}_m$ is a Perron-Frobenius matrix, that is, some positive power of $\mathcal{M}$ has strictly positive entries.*

*Proof.* Consider first the case where $q^i_x = q_x > 0$ for every pair $(i;x)$. In this case, for every pair $((i;x),(j;y))$ such that $i \neq j$ the corresponding entry of $\mathcal{M}$ is positive, by (4.2) and the fact that $F_{i;x}(s) > 0$. Consequently, the square of the matrix $\mathcal{M}$ has (strictly) positive entries; similarly (since $m \geq 3$) *every* power $\mathcal{M}^k$ such that $k \geq 2$ has all positive entries.

Now consider the general case. By the same argument as above, for every power $k \geq 2$ and for every $(j;y^{-1})$ such that $q_y > 0$, the $(i;x),(j;y^{-1})$ entry of $\mathcal{M}^k$ is positive. Now for $y_* \neq y^{-1}$ the $((j;y_*),(j;y^{-1}))$ entry of $\mathcal{M}$ is $p_j q_{y_* y}$, by (4.2). By hypothesis, for each index $j$ the probability distribution $q^j$ is irreducible, that is, for any two elements $y, y' \in \Gamma_j$ there is a positive probability path (for the random walk on $\Gamma_j$ with step distribution $q^j$) from $y$ to $y'$. Since there are only finitely many indices $j \leq m$, and since the groups $\Gamma_j$ are all finite, the lengths of these positive probability paths are bounded above, by (say) $r$. It follows that the $r+2$ power of $\mathcal{M}$ has all entries positive. □

**Corollary 4.6.** *Positive eigenvectors of $T$ and $T^*$ are* strictly *positive.*

*Proof.* Suppose that $Th = h$ is a positive eigenvector. Since $h$ is not identically 0, some entry, say the $(j;y)$ entry, is strictly positive. By Lemma 4.5, for each $m \geq 1$ some positive power $\mathcal{M}^n_m$ of the finite section $\mathcal{M}_m$ of $T$ has all entries positive; consequently, if $m \geq j$ then all entries of $T^n h$ indexed by $(i;x)$, where $i \leq m$, are positive. Since $m \geq j$ is arbitrary, it follows that $h$ is strictly positive. A similar argument shows that positive eigenvectors of $T^*$ must be strictly positive. □

*Proof of Proposition 4.3.* Denote by $V$ and $V^*$ the $1-$eigenspaces of the operators $T$ and $T^*$. By Corollary 4.2, $V$ and $V^*$ are finite-dimensional subspaces of $B$ and $B^*$ of the same dimension. By Corollary 4.6, there are strictly positive right and left eigenvectors $h \in V$ and $\nu \in V^*$.

**Claim 1.** $\varphi, \psi \in V \implies |\varphi \vee \psi| \in V$.

*Proof.* Since the operator $T$ is positive,

$$T|\varphi \vee \psi| \geq |T\varphi| \vee |T\psi| = |\varphi| \vee |\psi|. \tag{4.9}$$

To see that the inequality is actually an equality, apply the positive linear functional $\nu$ to both sides:

$$\langle \nu, |\varphi \vee \psi| \rangle = \langle T^*\nu, |\varphi \vee \psi| \rangle = \langle \nu, T|\varphi \vee \psi| \rangle \geq \langle \nu, |\varphi \vee \psi| \rangle.$$

Since $\nu$ is *strictly* positive, it must be that equality holds in (4.9). □

**Claim 2.** *Let $\varphi \in V$ be real-valued. Then $\varphi$ is strictly positive, or strictly negative, or identically* 0.

*Proof.* Suppose first that there are indices $\alpha = (i;x)$ and $\beta = (j;y)$ such that $\varphi(\alpha) > 0 > \varphi(\beta)$. By the irreducibility of $T$ (Lemma 4.5), there exists and integer $n \geq 1$ such that the $(\alpha, \beta)$ entry of $T^n$ is positive. Now since $\varphi \in V$ it is an eigenvector of $T^n$ with eigenvalue 1; by Claim 1, the positive part of $\varphi$ is also an eigenvector with eigenvalue 1. Thus,

$$\varphi(\alpha) = \sum_{\gamma \in J} T^n_{\alpha,\gamma} \varphi(\gamma) \qquad \text{and}$$

$$\varphi(\alpha) = \sum_{\gamma \in J_+} T^n_{\alpha,\gamma} \varphi(\gamma)$$

where $J_+$ denotes the set of indices $\gamma$ such that $\varphi(\gamma) \geq 0$ and $J$ the set of *all* indices $\gamma$. The two sums can be equal only if $\sum_J$ contains no negative terms. But $n$ was chosen so that there would be at least one negative term, namely, the term $\gamma = \beta$. This proves that either $\varphi \geq 0$ or $\varphi \leq 0$.

A similar argument shows that if $\varphi \in V$ is nonnegative, then either $\varphi \equiv 0$ or $\varphi$ is strictly positive. □

Let $\varphi \in V$ be any right eigenvector of $T$ with eigenvalue 1; I will show that $\varphi$ is a scalar multiple of $h$. Without loss of generality, assume that $\varphi$ is real-valued (because if $\varphi \in V$ then its real and imaginary parts are both in $V$, as $T$ is positive) and positive (by Claim 2). Define $a \geq 0$ to be the supremum of all nonnegative real numbers $b$ such that $b\varphi \leq h$. If the vector $a\varphi$ is not identically equal to $h$ then there is an index $\alpha = (i;x)$ such that $a\varphi(\alpha) < h(\alpha)$, and so for $b > a$ sufficiently near $a$ it must be the case that $b\varphi(\alpha) < h(\alpha)$. But then $h - b\varphi$ is a real-valued element of $V$ with both negative and positive entries, contradicting Claim 2. □

## 4.4. Singularity of $F$ at $z = \sigma$.

**Lemma 4.7.** *Let $\mathcal{H}$ be the Hessian of the link mapping $Q$ associated with the Lagrangian system (2.13), and let $h$ and $\nu$ be the positive eigenvectors of $T = \sigma\mathcal{L}(\sigma)$ and $T^*$ with eigenvalue* 1*. Then*

$$\langle \nu, \mathcal{H}(h,h) \rangle > 0. \tag{4.10}$$

*Proof.* The Hessian operator is the matrix of second partial derivatives of the link mapping $Q$. The first partial derivatives are given in equations (4.2); from these equations it is apparent that the nonzero second partials are

$$\frac{\partial^2 Q_{i;x}}{\partial F_{i;x} \partial F_{j;y}} = p_j q_{y^{-1}} \qquad \text{for} \quad j \neq i \tag{4.11}$$

Hence,

$$\mathcal{H}(h,h)_{i;x} = \sum_{j \neq i} \sum_{y \in \Gamma_i \setminus \{1\}} h_{i;x} h_{j;y} p_j q_{y^{-1}}. \tag{4.12}$$

Since the eigenvector $h$ has strictly positive entries, this sum is strictly positive for all $i, x$. Since $\nu$ is positive, (4.10) follows. □

**Corollary 4.8.** *For every positive $\mu \in B^*$, the function $\langle \mu, F(z) \rangle$ has a square-root singularity at $z = \sigma$. Consequently,*

$$\sigma = R. \tag{4.13}$$

*Proof.* Proposition 4.3 and Lemma 4.7 imply that the hypotheses of Theorem 2 are satisfied. Consequently, the function $\langle \nu, F(z) \rangle$ has a square-root singularity at $s = \sigma$. But the common radius of convergence of the power series $F_{i;x}(z)$ is $R$; therefore, it must be that $\sigma = R$. Consequently, For any positive $\mu \in B^*$,

$$\langle \mu, F(z) \rangle = \langle \mu, h \rangle \langle \nu, F(z) \rangle + \langle \mu, P_W F(z) \rangle.$$

By Theorem 2, the projection $P_W F(z)$ has a Puiseux series around $z = R$ whose first term is linear in $R - z$, and by Proposition 4.3, the lead eigenvectors $\nu$ and $h$ are *strictly* positive. Thus, $\langle \mu, h \rangle > 0$, and so the Puiseux expansion of $\langle \mu, F(z) \rangle$ has as its first nonconstant term a nonzero multiple of $\sqrt{R - z}$. □

## 4.5. **Singularity of the Green's function at $z = R$.**

**Proposition 4.9.** *Let $G(z)$ be the Green's function of an aperiodic, irreducible, quasi-nearest-neighbor random walk on a countable free product of finite groups, and let $R$ be its radius of convergence. Then $G(z)$ has a square-root singularity at $z = R$, that is, it has a Puiseux expansion in a neighborhood of $z = R$ of the form*

$$G(z) - G(R) = \sum_{n=2^k}^{\infty} g_n (R - z)^{n/2^{k+1}} \tag{4.14}$$

*whose first nonconstant term is $g_{2^k} \sqrt{R - z}$, with $g_{2^k} < 0$.*

*Proof.* The fundamental relation (2.7) exhibits $G$ as a meromorphic function of a positive linear combination $\mathcal{F}$ of the first-passage generating functions $F_{i;x}$. By Proposition 2.1, $G(z)$ is finite at $z = R$, and so near $z = R$ the function $G(z)$ is in fact an *analytic* function of $\mathcal{F}(z)$. By Corollary 4.8, every positive linear functional of $F(z)$ has a square-root singularity at $z = R$; consequently, the same is true of $G$. □

4.6. **Proof of Theorem 1.** By Proposition 2.3 the Green's function $G(z)$ has no singularity on the circle of convergence except at $z = R$, and by the preceding proposition at $z = R$ the function $G(z)$ admits the expansion (4.14). Therefore, by the Flajolet-Odlyzko Transfer Theorem, the coefficients of the power series defining $G(z)$ satisfy the asymptotic relation (1.4).

□

Acknowledgment. The author thanks Walter Lalley for his encouragement.

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