Nash Equilibria for Quadratic Voting

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Abstract

A group of N individuals must choose between two collective alternatives. Under Quadratic Voting (QV), agents buy votes in favor of their preferred alternative from a clearing house, paying the square of the number of votes purchased, and the sum of all votes purchased determines the outcome. Each agent is assumed to have a private value that determines her utility; these values are assigned by simple random sampling from a probability distribution F with a smooth density on a compact interval $[\underline{u}, \overline{u}]$. Under these assumptions, the structure of the Bayes-Nash equilibrium is described when N is large. The results imply that the quadratic voting mechanism is asymptotically efficient.

Keywords: social choice, collective decisions, large markets, costly voting, vote trading, Bayes-Nash equilibrium.

1 Introduction and Main Results

Consider a binary collective-decision problem in which a group of N individuals must choose between two alternatives. Each individual i has a privately known value u_i that determines her willingness to pay for one alternative over the other; positive values indicate affinity for outcome $+1$, negative values for outcome -1. *Quadratic Voting* (*QV*) is a simple and detail-free mechanism designed to maximize

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utilitarian efficiency in this setting.[1] In this system, individuals buy votes (either negative or positive, depending on which alternative is favored) from a clearing house, paying the square of the number of votes purchased. The sum of all votes purchased then determines the outcome. The utility (payoff) of the outcome to an individual with value u is $+u$ if outcome $+$ is adopted, but $-u$ if outcome $-$ is adopted.[2]

The heuristic rationale for QV is quite simple. The marginal benefit to a voter of an additional vote is her value multiplied by her *marginal pivotality* (roughly, the perceived probability that an additional unit of vote will sway the decision). She maximizes utility by equating this marginal utility to the linear marginal cost of a vote. Therefore, if voters share the same marginal pivotality, they will buy votes in proportion to their values, thus bringing about utilitarian efficiency. Furthermore, the quadratic cost function is the *unique* cost function with this property. This argument is explained in further detail in Subsection 1.3.

Variations of this rationale have been used to justify quadratic mechanisms in a number of related collective-decision-making problems. However, to our knowledge, this heuristic rationale has never been translated into a rigorous argument for efficiency in the sort of non-cooperative, incomplete information game theoretic model in which mechanisms for allocating private goods have been studied at least since the work of Myerson [11]. In fact, as we will show, in the modified setting of quadratic voting that we will consider the crucial *ansatz* of this rationale – that in equilibrium all voters will have the same marginal pivotality – is false. As a result, formal equilibrium analysis is a far more subtle task than the heuristic argument of the previous paragraph might suggest. Nevertheless, we will show that for voters with values in the "bulk" of the distribution, the marginal pivotality is approximately constant.[3]

To our knowledge the use of quadratic pricing for collective decision-making was first suggested by Groves and Ledyard [6], who proposed it as a Nash implementation of the optimal level of continuous public goods under complete information that avoids the fragility of previously suggested efficient mechanisms. Hylland and Zeckhauser [8] provided the first variant of the heuristic rationale above to uniquely justify quadratic pricing mechanism and proposed an iterative

[1]Clearly many other objectives are possible for this problem, and many involve distributional considerations. However, we focus on a utilitarian objective because it is the one most extensively studied in the literature [2, 5].

[2]Our results will apply to a modified version of the problem in which the utility is "smoothed" in such a way that each voter's utility is a continuous function of the vote total. See section 1.1 for details.

[3]Theorems of Kahn et al. [9] and Al-Najjar et al. [1] imply marginal pivotality must converge to zero as $N \to \infty$. Our results will show that, with probability approaching 1, the *ratio* between the marginal pivotalities of two randomly chosen voters will be close to 1.

procedure that they conjectured would converge to [6]'s complete information optimum in the presence of private information. In a previous version of this paper, Weyl [13] first proposed the use of QV for binary collective decision problems, and conjectured that it would lead to asymptotically efficient decisions in the environment we consider based on an (independently discovered) extension of [8]'s heuristic rationale. Goeree and Zhang [4] independently suggested using a detail-based, approximately direct variant of QV in the special case where values are sampled from zero-mean normal distributions, and derived an equilibrium in the case $N = 2$.

1.1 Model Assumptions

We consider an independent symmetric private values environment with N voters $i = 1, \ldots, N$. Each voter i is characterized by a value, u_i; these values are drawn independently from a continuous probability distribution F with C^∞, strictly positive density f supported by a finite closed interval $[\underline{u}, \overline{u}]$.[4] Each individual knows her own value, but not the values of any of the other $N-1$ voters; however, the sampling distribution F is known to all. For normalization, we assume the numeraire has been scaled so that $\min(-\underline{u}, \overline{u}) \geq 1$. We denote by μ, σ^2, and μ_3, respectively, respectively the mean, variance, and raw third moments of u under F.

We consider a variant of the payoff described above, in which the utility of the outcome is "smoothed". [5] Each voter i chooses a number of votes $v_i \in \mathbb{R}$ to buy, and pays v_i^2 dollars for these. The payoff to voter i is then

$$\Psi(V)u_i, \quad \text{where } V = \sum_{i=1}^{N} v_i \tag{1}$$

is the vote total and $\Psi : \mathbb{R} \to [-1, 1]$ is an odd, nondecreasing, C^∞ function such that for some $\delta > 0$,

[4]The assumption that the density f is positive at the endpoints $\underline{u}, \overline{u}$ is of critical importance for our main results, as "extremists" play a crucial role in the Bayes-Nash equilibria for the game. Our methods would extend to densities f that vanish at one or both of the endpoints, but the nature of the Bayes-Nash equilibria changes in these cases.

[5]Although both the discrete binary choice set-up of [13] and the continuous public goods model of [8] helped inspire this model, it differs from both. Consequently, our results have no direct implications for those models. It differs from [13]'s model in that the outcome is smoothed rather than jumping discontinuously at 0. It differs in a variety of respects from [8]'s, notably in that utility is linear in the common and bounded outcome, whereas [8] assume strictly concave preferences with heterogeneous ideal points and an outcome that may take values in the full real space. [8] also consider a multidimensional issue space with no access to transfers and an iterative procedure to converge to this outcome, none of which feature in our model.

(M1) $\Psi(x) = \text{sgn}(x)$ for all $|x| \geq \delta$;
(M2) $\psi(x) := \Psi'(x) > 0$ for all $x \in (\delta, \delta)$;
(M3) $\psi'(x) > 0$ for all $x \in (-\delta, 0)$; and
(M4) $\psi(x)$ has a unique inflection point $x = \iota$ in $(-\delta, 0)$, such that
 (M4a) $\psi'(x)$ is strictly increasing in $[-\delta, \iota]$, and
 (M4b) $\psi'(x)$ is strictly decreasing in $[\iota, 0]$.

We shall refer to Ψ as the *payoff function*, because it determines the quantity by which values u_i are multiplied to obtain the allocative component of each individual's utility.[6] Conditional on the values $\{v_i\}$, individual i earns expected utility

$$\Psi(V)u_i - v_i^2. \tag{2}$$

Thus, in a *type-symmetric, pure-strategy Bayes-Nash equilibrium*[7], a voter with value u will maximize

$$E\left[u\Psi\left(S_n + v\right)\right] - v^2, \tag{3}$$

where $n = N - 1$ and $S_n := \sum_{i=1}^{n} v_i$ is the *one-out vote total*, the sum of all votes cast by all but a single individual. For brevity, we shall refer to *type-symmetric, pure-strategy Bayes-Nash equilibria* as *Nash equilibria*.

We define the *expected inefficiency* as

$$EI \equiv \frac{1}{2} - \frac{E\left[U\Psi(V)\right]}{2E\left[|U|\right]} \in [0, 1],$$

where $U \equiv \sum_i u_i$. This measure is the unique negative monotone linear functional of aggregate utility realized $U\Psi(V)$ that is normalized to lie in the unit interval.

1.2 Existence of Equilibria

Proposition 1. *For any $N > 1$ a monotone increasing, pure-strategy Nash Equilibrium v exists.*

This result follows directly from [12], Theorem 4.5. All of [12]'s conditions can easily be checked, so we highlight only the less obvious ones. Continuity of payoffs

[6]The assumptions on the payoff function Ψ are primarily for mathematical convenience. However, there are some circumstances where a smoothing of the payoff for vote totals near 0 might be natural: for instance, (i) in some close elections, it might be necessary for the winning side to form a coalition with some of the losers to form a functioning majority; or (ii) for vote totals near 0, a recount might be necessary, leading to the possibility that the winning side might be overturned.

[7]A type-symmetric, pure-strategy Bayes-Nash equilibrium is a function $v(u)$ such that, if all players use the rule $u \mapsto v(u)$ for buying votes then for each value of u, any player with value u could not improve her expected utility by defecting from the strategy.

as functions of the actions v_i follows from the continuity and boundedness of Ψ. Type-conditional utility is only bounded from above, not below, but boundedness from below can easily be restored by simply deleting for each value type u votes of magnitude greater $\sqrt{2\,|u|}$. The existence of a monotone best-response follows from the obvious super-modularity of payoffs in value and votes.

Although Nash equilibria always exist, they need not be unique. Indeed, we will show that in some circumstances (cf. Theorem 3) Nash equilibria have points u_* of discontinuity; at any such point, there are at least two distinct pure-strategy Nash equilibria, one with $v(u_*) = v(u_*+)$, the other with $v(u_*) = v(u_*-)$. We conjecture, however, that at least when N is large, non-uniqueness of Nash equilibria can only occur for this trivial reason: in particular, we conjecture that if v_1 and v_2 are distinct Nash equilibria then $v_1(u) = v_2(u)$ for all but at most one value u.

1.3 Rationale for QV

Formally differentiating expression (3) with respect to v (see Appendix B for a formal proof) yields the following first-order condition for maximization:

$$uE\left[\psi\left(S_n + v\right)\right] = 2v \implies v(u) = \underbrace{\frac{E\left[\psi\left(S_n + v(u)\right)\right]}{2}}_{\text{marginal pivotality}} u. \tag{4}$$

The marginal benefit of an additional unit of vote is thus twice the individual's value multiplied by the influence this extra vote has on the chance the alternative is adopted, the vote's *marginal pivotality*. The marginal cost of a vote is twice the number of votes already purchased.

When the number N of voters is large, most would reason that their votes $v(u)$ will have a negligible effect on the vote total $S_n + v(u)$. Taking this logic to an extreme, if voters acted as if marginal pivotality p were constant across the population, then an individual with value u would buy $v(u) = pu$ votes. This voting strategy would imply $V = p\sum_i u_i$; that is, the vote total would be exactly proportional to the sum of the values, and consequently the expected inefficiency would be 0. Clearly, this argument holds only for a quadratic cost function, because only quadratic functions have linear derivatives.

Our main results will show, however, that the marginal pivotality is not constant; in fact, when the mean μ of the sampling distribution F is non-zero the marginal pivotality can have large jump discontinuities in the tail of the distribution. Thus, voters do not buy votes strictly in proportion to their values, and so in general the vote total will not always be a scalar multiple of the aggregate value $\sum_i u_i$.

Nevertheless, as our results will show, quadratic voting is asymptotically efficient, in the sense that the expected inefficiency converges to 0 as $N \to \infty$.

Although it is perhaps obvious, we emphasize that one-person-one-vote is in many cases *not* efficient. Such inefficiency will occur, for instance, if the distribution F has positive mean μ but attaches probability $q < 1/2$ to the interval $[0, \overline{u}]$, because by the law of large numbers, when N is large,

$$\frac{1}{N}\sum_{i=1}^{N} U_i \approx \mu > 0$$

but

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{U_i \geq 0\}} &\approx q \quad \text{and} \\ \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{U_i < 0\}} &\approx 1 - q,\end{aligned}$$

so under one-person-one-vote the vote total would, with high probability, be near $(-1 + 2q)N < 0$.

1.4 Main Results

Our main results concern the structure of equilibria in the game described in the previous section when the number N of agents is large, and the implications for the efficiency of QV.

1.4.1 Characterization of equilibrium in the zero mean case

The structure of a Nash equilibrium differs radically depending on whether $\mu = 0$ or $\mu \neq 0$. The case $\mu = 0$, although non-generic, is of particular interest because in some elections – for instance, when two candidates are vying for an elected office – the alternatives may be tailored so that an approximate population balance is achieved [10].

Theorem 1. *For any sampling distribution F with mean $\mu = 0$ that satisfies the hypotheses above, there exist constants $\epsilon_N \to 0$ such that in any Nash equilibrium, $v(u)$ is C^∞ and strictly increasing on $[\underline{u}, \overline{u}]$ and satisfies the following approximate proportionality rule:*

$$\left|\frac{v(u)}{p_N u} - 1\right| \leq \epsilon_N \quad \textit{where} \quad p_N = \frac{1}{2^{\frac{3}{4}}\sqrt{\sigma}\sqrt[4]{\pi(N-1)}}. \tag{5}$$

Furthermore, there exist constants $\alpha_N, \beta_N \to 0$ such that in any equilibrium the vote total $V = V_N$ and expected inefficiency satisfy

$$|E[V]| \leq \alpha_N \sqrt{\mathrm{var}(V)} \quad \text{and} \tag{6}$$

$$EI < \beta_N. \tag{7}$$

The proof will be given in section 7.

Thus, in any equilibrium, agents buy votes *approximately* in proportion to their values u_i. Given this fact, it is not difficult to understand why the number of votes a typical voter buys should be of order $N^{-1/4}$. If the vote function $v(u)$ in a Nash equilibrium follows a proportionality rule $v(u) \approx \beta u$, the constant β must be the consensus marginal pivotality. On the other hand, by the local limit theorem of probability (see [3], ch. XVI), if $\beta = CN^{-\alpha}$ for some constants $C \neq 0$ and $\alpha \in \mathbb{R}$, the chance that $V \in [-\delta, \delta]$ would be of order $N^{\alpha - \frac{1}{2}}$, and so α must be $1/4$.

Although the relation (5) asserts the ratio $v(u)/u$ is approximately constant, it is not *exactly* constant: in fact, $v(u)$ is a genuinely nonlinear function of u. Thus, even though $E[U] = 0$, it need not be the case that $E[V] = 0$. To establish the asymptotic efficiency assertion (7), we must establish assertion (6), namely, that the non-linearities vanish rapidly enough that the bias created by non-linearity is smaller than the sampling variation in u. This will require a rather subtle application of the *Edgeworth expansion* (cf. [3], ch. XVI) of the distribution of S_n. If it were the case that $E[V] = 0$, and if the distribution of S_n were exactly normal, a standard Taylor expansion and the $N^{-1/4}$ decay of $v(u)/u$ could be used directly to show that non-linearities vanish with N^{-1} even relative to the leading term of $v(u)/u$. A detailed analysis of this argument leads us to conjecture that, under the hypotheses of Theorem 1, the inefficiency of QV decays like $\mu_3^2/(16\sigma^6 N)$.

1.4.2 Characterization of equilibrium in the non-zero mean case

When μ is not zero, the nature of equilibria can be quite different: in particular, if the payoff function Ψ is sufficiently sharp (i.e., the support of its derivative ψ is sufficiently small) then for all large N, every Nash equilibrium has a large discontinuity in the extreme tail of the sampling distribution. Nevertheless, in all cases the quadratic voting mechanism is asymptotically efficient, as the following theorem shows.

Theorem 2. *Assume that the sampling distribution F has mean $\mu > 0$ and that F and Ψ satisfy the hypotheses laid out in section 1.1 above. Then there exist constants $\beta_N \to 0$ such that in any Nash equilibrium $v(u)$,*

$$EI < \beta_N. \tag{8}$$

Furthermore, there exist constants $\xi \geq \delta$ and $\beta > 0$ depending on the sampling distribution F and the payoff function Ψ but not on N such that in any equilibrium $v(u)$, for any $\epsilon > 0$,

$$\sup_{\underline{u}+\beta N^{-3/2} \leq u \leq \overline{u}} \left| v(u) - \left(\frac{\xi}{\mu N}\right) u \right| < \epsilon_N / N \quad \textit{and hence} \tag{9}$$

$$P\{|V_N - \xi| > \epsilon\} \leq \epsilon'_N, \tag{10}$$

where $\epsilon_N, \epsilon'_N \to 0$ are constants that depend only on the sample size N, and not on the particular equilibrium.

This theorem allows for two cases. In the first, where $\xi = \delta$, the vote total is near δ with high probability for large N. This case occurs for large δ and thus relatively smooth payoff functions. In the second, $\xi > \delta$, so that with high probability the vote total is outside $[-\delta, \delta]$ for large N. This case arises when $\delta > 0$ is small (see Proposition 2 below). In both cases, the approximate proportionality rule (9) holds except possibly in the extreme lower tail of the value distribution F.

To see how the dichotomy arises, suppose that for some $\xi \geq \delta$ there were a value $w \in (-\delta, \delta)$ such that

$$(1 - \Psi(w))\, |\underline{u}| > (\xi - w)^2; \tag{11}$$

then an agent with value u near the lower extreme $\underline{u}$, knowing that with high probability the one-out vote total S_n is near ξ, would find it worthwhile to buy $-\xi + w$ votes and thus single-handedly move the vote total to w. Consequently, there can be no equilibrium in which S_n concentrates strictly below ξ if such a w exists, as this would lead a large number of individuals to act as extremists, contradicting the concentration of the vote total. Therefore, in any equilibrium the voters with positive values u_j must buy enough votes to guarantee that the vote total concentrates at or above ξ. The minimal value $\xi \geq \delta$ below which there is no advantage to "extremist" behavior in the extreme lower tail thus determines the equilibrium behavior (9). This will be at $\xi = \delta$ *unless* there is a solution to the following problem.

Optimization Problem. Determine $\xi > \delta$ and a matching real number $w \in [-\delta, \delta]$ such that

$$\begin{aligned} &(1 - \Psi(w))\, |\underline{u}| = (\xi - w)^2 \quad \text{and} \\ &(1 - \Psi(w'))\, |\underline{u}| < (\xi - w')^2 \quad \text{for all } w' \in [-\delta, \delta] \setminus \{w\} \end{aligned} \tag{12}$$

Proposition 2. *If $\delta < 1/\sqrt{2}$ then there exists a unique pair $\xi > \delta$ and $w \in (-\delta, \delta)$ that satisfy the Optimization Problem* (12).

The proof will be given in Appendix E. When the Optimization Problem has a solution, Nash equilibria take a rather interesting form in which extremists must appear, but with vanishing probability, as the following theorem shows.

Theorem 3. *Assume that the sampling distribution F has mean $\mu > 0$ and that F and Ψ satisfy the hypotheses above. Assume further that the Optimization Problem* (12) *has a unique solution (ξ, w) such that $\xi > \delta$. Then there exists a constant $\zeta > 0$ depending on F such that for any $\epsilon > 0$ and any Nash equilibrium $v(u)$, when N is sufficiently large, there exists $u_* \in [\underline{u}.\overline{u}]$ such that*

(i) $v(u)$ *has a jump discontinuity at* $u = u_*$*;*
(ii) $v(u)$ *is continuous and continuously differentiable for all* $u \in (u_*, \overline{u}]$*;*
(iii) the approximate proportionality rule (9) *holds for all* $u \in [u_*, \overline{u}]$*;*
(iv) $|v(u) + \xi - w| < \epsilon$ *for* $u \in [\underline{u}, u_*)$*; and*
(v) $|u_* - (\underline{u} + \zeta N^{-2})| < \epsilon N^{-2}$.

Theorems 2 and 3 will be proved in section 6. Theorem 3 asserts that any Nash equilibrium has a single large discontinuity near $\underline{u} + \zeta N^{-2}$; it does not preclude the possibility of other discontinuities in the interval $[\underline{u}, u_*)$, but (iv) implies that if these occur, the jumps must be small.

Theorem 2 implies that an agent with value u will buy approximately $\xi\mu^{-1}u/N$ votes unless u is in the extreme lower tail of F. Theorem 3 implies that when a solution to the Optimization Problem exists, such exceptional agents occur only with probability $\approx \zeta N^{-1} f(\underline{u})$; consequently, by the law of large numbers, with probability $\approx 1 - \zeta N^{-1} f(\underline{u})$ the vote total will be very near ξ. If, on the other hand, the sample contains an agent with value less than u_* then this agent will buy approximately $w - \xi \approx -\sqrt{|\underline{u}|}$ votes, enough to move the overall vote total close to w. Agents of the first type will be called *moderates*, and agents of the second kind *extreme contrarians* or *extremists* for short. Because the tail region in which extremists reside has $F-$probability on the order N^{-2}, the sample of agents will contain an extremist with probability only on the order N^{-1}, and will contain two or more extremists with probability on the order N^{-2}. Given that the sample contains no extremists, the conditional probability that $|V - \xi| > \epsilon$ is $O(e^{-\varrho n})$ for some $\varrho > 0$, by standard large deviations estimates, and so the event that $V < 0$ essentially coincides with the event that the sample contains an extremist.

Why does equilibrium take the somewhat counter-intuitive form described in Theorem 3? Following is a brief heuristic explanation. For an agent with value u in the "bulk" of the sampling distribution F, there is very little information about the vote total V in the agent's value, and so for most such agents the marginal pivotality $\frac{1}{2} E\psi\left(S_n + v(u)\right)$ will be approximately $\frac{1}{2} E\psi(V)$. Consequently, $v(u)$ will be approximately linear in u except possibly in the extreme tails of the distribution, and so by the law of large numbers, the vote total will, with high probability, be near $\frac{1}{2}N\mu E\psi(V)$.

Because $\mu > 0$, agents with negative values will, with high probability, be on the losing side of the election. However, if $\frac{1}{2}N\mu E\psi(V)$ were small, then an agent with even moderately negative value could increase her expected utility by buying a large number of (negative) votes; since many voters with negative values would find it beneficial to adopt such a strategy, the vote total would, with high probability, be negative, in contradiction to the fact that it must be concentrated near $\frac{1}{2}N\mu E\psi(V)$.Therefore, $NE\psi(V)\mu$ must remain bounded away from 0.

On the other hand, if $\frac{1}{2}N\mu E\psi(V)$ were too large, then no individual could profitably act as an extremist, so except with exponentially small probability the vote total V would be bounded away from $[-\delta, \delta]$. But this would force $\mathbb{E}\psi(V)$ to be exponentially small, which is impossible. Thus, the aggregate number of votes must concentrate near a constant value, and so most voters must buy on the order of $1/N$ votes. For this scenario to occur, $E\psi(V)$ must decay as $\frac{1}{N}$. But the primary contribution to this expectation must come from the event in which an extremist exists, and so the probability of this event must decay as $\frac{1}{N}$.

1.5 Plan of the paper

The remainder of the paper will be devoted to the proofs of Theorems 1–3 and Proposition 2. Because essentially nothing (other than monotonicity) is known *a priori* about the nature of Nash equilibria, information must be teased out in steps, each relying on the previous steps. We begin in section 2 by collecting some relatively easy consequences of monotonicity and the first-order necessary condition (13), including a useful necessary condition (section 2.3) for discontinuities of Nash equilibria, which will ultimately be used to prove that these can occur only in the extreme tails of the distribution F. In section 3, a weak form of the approximate proportionality rule will be proved for agents in the bulk of the distribution F. Using this weak approximate proportionality rule, we will, in section 4, use an anti-concentration inequality for sums of i.i.d. random variables to derive bounds for Nash equilibria. We will then be able to deduce, in section 5, that approximate proportionality holds except in the extreme tails of F. The proofs of Theorems 2–3 will be given in section 6, and the proof of Theorem 1 in section 7.

1.6 Notation

The symbols Ψ, ψ, δ, F, f, μ, σ^2, ξ, ζ, ι, $\underline{u}, \overline{u}$ will be reserved for the functions and constants specified in Subsection 3.1 of the text, and the letters N, n will be used only for the sample size and sample size minus one. The symbols $\alpha, \beta, \gamma, \epsilon, \varrho$ and $C, C', \cdots$ will be used for generic constants whose values might change from one lemma to the next. Because many of the arguments will involve the values of the equilibrium vote function v at points near one of the endpoints $\underline{u}, \overline{u}$, we will use the following shorthand notation, for any $0 < \epsilon < 1$:

$$\overline{u}_\epsilon = \overline{u} - \epsilon \quad \text{and} \quad \underline{u}_\epsilon = \underline{u} + \epsilon.$$

2 Nash Equilibria: Basic Properties

2.1 Monotonicity of Nash Equilibria

Proposition 3. *Any pure-strategy Nash equilibrium $v(u)$ is* strictly *increasing in u.*

Monotonicity of Nash equilibria has already been established (cf. Proposition 1); this follows from general results in game theory. Strict monotonicity requires an additional argument. Because this is relatively standard, we relegate the proof to Appendix A.

2.2 Necessary Condition for a Nash Equilibrium

Proposition 4. *If $v(u)$ is a Nash equilibrium then at every $u \in [\underline{u}, \overline{u}]$ the function v satisfies the functional equation*

$$E\psi(S_n + v(u))u = 2v(u). \tag{13}$$

The necessary condition (13) will be of central importance in the analysis to follow. The proof, which is both easy and completely standard, is given in Appendix B. Observe, though, that the proposition depends crucially on the differentiability of the payoff function Ψ; for functions with discontinuities, such as $\Psi = \mathbf{1}_{[0,\infty)} - \mathbf{1}_{(-\infty,0)}$, the argument breaks.

2.3 Discontinuities of Nash Equilibria

Because any Nash equilibrium $v(u)$ is a monotone functions of u, it can have at most countably many discontinuities, all of which are jumps. Clearly, any Nash equilibrium $v(u)$ is continuous at $u = 0$, because for very small $|u|$ an agent with value u would never pay more than $2|u|$ for votes, since this is the maximal change in the agent's utility that could result. The following proposition asserts that there is a lower bound on the magnitude $|v(u)|$ of a Nash equilibrium near any point of discontinuity.

Proposition 5. *There exists $\Delta > 0$ such that for all sufficiently large n, at any point u_* of discontinuity of a Nash equilibrium,*

$$\limsup_{u \to u_*} |v(u)| \geq \Delta. \tag{14}$$

By the monotonicity of v, the limsup must be the limit of $v(u)$ as either $u \downarrow u_*$ or as $u \uparrow u_*$, depending on whether u_* is negative or positive. Thus, at any point u_* of discontinuity, $|v(u)|$ must jump to at least Δ as u passes through the value u_*. *A priori*, we have no information about the size of a Nash equilibrium; however, in sections 3 and 4 we will show that large values of $|v(u)|$ can only occur in the extreme tails of the sampling distribution F (in particular, within distance $O(n^{-3/2})$ of one of the endpoints $\underline{u}, \overline{u}$).

The proof of Proposition 5 will require two auxiliary lemmas, both of which will be of use later in the paper.

Lemma 6. *Let $v(u)$ be a Nash equilibrium. If v is discontinuous at $u \in (\underline{u}, \overline{u})$ then*

$$E\psi'(\tilde{v} + S_n)u = 2 \tag{15}$$

for some $\tilde{v} \in [v_-, v_+]$, where v_- and v_+ are the left and right limits of $v(u')$ as $u' \to u$.

Proof. The necessary condition (13) holds at all u' in a neighborhood of u, so by monotonicity of v and continuity of ψ, Equation (13) must hold when $v(u)$ is replaced by either of $v_\pm$, that is,

$$\begin{aligned} 2v_+ &= E\psi(v_+ + S_n)u \quad \text{and} \\ 2v_- &= E\psi(v_- + S_n)u. \end{aligned}$$

Subtracting the second equation from the first and using the differentiability of ψ we obtain

$$2v_+ - 2v_- = uE\int_{v_-}^{v_+} \psi'(t + S_n)\, dt = u\int_{v_-}^{v_+} E\psi'(t + S_n)\, dt.$$

The lemma now follows from the mean value theorem of calculus. □

Lemma 7. *For any $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that for any Nash equilibrium $v(u)$, any $\tilde{v} \in \mathbb{R}$, any $u \in [\underline{u}, \overline{u}]$, and all n,*

$$\begin{aligned} E|\psi'(\tilde{v} + S_n)u| \geq \alpha \quad &\Longrightarrow \quad E\psi(\tilde{v} + S_n)|u| \geq \beta \;\; \textit{and} \\ E|\psi''(\tilde{v} + S_n)u| \geq \alpha \quad &\Longrightarrow \quad E\psi(\tilde{v} + S_n)|u| \geq \beta. \end{aligned} \tag{16}$$

Proof. Recall that $\psi/2$ is a C^∞ probability density with support $[-\delta, \delta]$, and that ψ is *strictly* positive in the open interval $(-\delta, \delta)$. Consequently, on any interval $J \subset (-\delta, \delta)$ where $|\psi'|$ (or $|\psi''|$) is bounded below by a positive number, so is ψ.

Fix $\epsilon > 0$ so small that $\epsilon \max(|\underline{u}|, \overline{u}) < \alpha/2$. In order that $E|\psi'(\tilde{v} + S_n)u| \geq \alpha$, it must be the case that the event $\{|\psi'(\tilde{v} + S_n)| \geq \epsilon\}$ contributes at least $\alpha/2$ to the expectation; hence,

$$P\{|\psi'(\tilde{v} + S_n)| \geq \epsilon\} \geq \frac{\alpha}{2\|\psi'\|_\infty \max(|\underline{u}|, \overline{u}}.$$

But on this event the random variable $\psi(\tilde{v} + S_n)$ is bounded below by a positive number $\eta = \eta_\epsilon$, so it follows that

$$E\psi(\tilde{v} + S_n)|u| \geq \frac{\eta\alpha}{2\|\psi'\|_\infty}.$$

A similar argument proves the corresponding result for ψ''. $\square$

Proof of Proposition 5. Without loss of generality, we can assume that the point u_* of discontinuity is positive, because if necessary we can exchange the roles of positive and negative voters. By Lemma 7, equation (15) implies that

$$E\psi(\tilde{v} + S_n)u_* \geq \beta = \beta(2) > 0$$

for some $\tilde{v} \in [v_-, v_+]$, where v_-, v_+ are the left and right limits of $v(u)$ at u_* and $\beta(\alpha) > 0$ is the constant in (16). Consequently, either

$$E\psi(v_- + S_n)u_* \geq \beta/2 \quad \text{or}$$
$$(E\psi(\tilde{v} + S_n) - E\psi(v_- + S_n))u_* \geq \beta/2.$$

In the former case, we must have $\lim_{u \to u_*-} 2v(u) \geq \beta/2$, by the necessary condition (13). In the latter case,

$$\begin{aligned}
\beta/2 &\leq (E\psi(\tilde{v} + S_n) - E\psi(S_n))u_* \\
&= u_* E \int_{v_-}^{\tilde{v}} \psi'(v + S_n)\, dv \\
&\leq \overline{u}\|\psi'\|_\infty(\tilde{v} - v_-),
\end{aligned}$$

which implies that $v_+ \geq \tilde{v} \geq \tilde{v} - v_- \geq \beta/(4\||\psi'|\|_\infty \overline{u})$. $\square$

2.4 Smoothness

Because Nash equilibria are monotone, by Lemma 1, they are necessarily differentiable almost everywhere. The following proposition gives more precise information about points of non-differentiability. Together with the results of sections 3 and 4 below, this proposition will imply that Nash equilibria must be smooth except in the extreme tails of F.

Proposition 8. *A Nash equilibrium $v(u)$ is C^∞ at every $u \in (\underline{u}, \overline{u})$ except those u at which $v(u)$ is discontinuous or at which*

$$E\psi'(v(u) + S_n)u = 2. \tag{17}$$

At any point $u \in (\underline{u}, \overline{u})$ where $v(u)$ is differentiable, the first and second derivatives are determined by the equations

$$2v'(u) = E\psi(v(u) + S_n) + E\psi'(v(u) + S_n)v'(u)u, \tag{18}$$
$$2v''(u) = E\psi'(v(u) + S_n)(2v'(u) + v''(u)u) + E\psi''(v(u) + S_n)v'(u)^2u. \tag{19}$$

Proof. The necessary condition (13) can be written as $H(v(u), u) = 0$ where H is the function of two variables defined by

$$H(v, u) = E\psi(v + S_n)u - 2v.$$

Since ψ is by hypothesis C^∞ and has compact support, H is also C^∞, with first partial derivative

$$\frac{\partial H}{\partial v} = E\psi'(v + S_n)u - 2$$

Hence, the implicit function theorem of calculus implies that every point $(u, v(u))$ where equation (17) does not hold has an open neighborhood in the plane in which the solution set of the equation $H(v, u) = 0$ defines a C^∞ function $v(u)$. The necessary condition (13) implies that this function must coincide locally with the Nash equilibrium unless the equilibrium has a discontinuity at u.

Since the left side of equation (17) is continuous in u, any point u where (17) fails has an open neighborhood in which (17) fails, and hence in which $v(u)$ is differentiable. The equations (18) and (19) for the first two derivatives follow from (13) by the chain rule (differentiation under the expectation is justified because ψ is C^∞ with bounded support). $\square$

The differential equation (18) can be rewritten as

$$v'(u) = \frac{E\psi(v(u) + S_n)}{2 - E\psi'(v(u) + S_n)u}. \tag{20}$$

From this it is evident that the derivative $v'(u)$ could blow up in a neighborhood of a point u where $E\psi'(v(u) + S_n)u = 2$. The following corollary shows that this cannot occur in any region where the function $|v(u)|$ remains suitably small.

Corollary 9. *There exists a constant $\beta > 0$ independent of n and of the particular Nash equilibrium $v(u)$ such that for all $u \in [\underline{u}, -1] \cup [1, \overline{u}]$,*

$$\limsup_{u' \to u} |v(u')| \leq \beta \quad \Longrightarrow \quad |v'(u)| \leq \|\psi\|_\infty. \tag{21}$$

Proof. The constant β can be chosen smaller than the discontinuity threshold Δ (cf. Proposition 5), in which case the inequality $\limsup_{u' \to u} |v(u')| \leq \beta$ implies that $v(\cdot)$ is continuous at u. Consequently, $v(\cdot)$ is differentiable at u unless the denominator in (20) is 0, by Proposition 8.

Assume now that $|u| \geq 1$. By the necessary condition (13), if $|2v(u)| \leq \beta$ then $E\psi(v(u) + S_n) \leq \beta$, and by Lemma 7, if $\beta > 0$ is sufficiently small, then $E|\psi'(v(u) + S_n)u| < 1$, and so the denominator on the right side of (20) is greater than 1. Thus, (20) implies that if $|v(u)| < \beta$ then

$$|v'(u)| \leq E\psi(v(u) + S_n) \leq \|\psi\|_\infty.$$

□

2.5 An *A Priori* Bound on Nash Equilibria

Lemma 10. *For all sufficiently large n, in any Nash equilibrium,*

$$\max_{\underline{u}_{1/n} \leq u \leq \overline{u}_{1/n}} |v(u)| \geq \frac{\delta}{4n}. \tag{22}$$

Proof. By Proposition 3, the maximum occurs at one of the two endpoints $\underline{u}_{1/n}$ or $\overline{u}_{1/n}$. Suppose that the inequality (22) were not true; then an agent with value $u \in [1, \overline{u}_{1/n}]$ would purchase at most $\delta/4n$ votes (in absolute value). On the other hand, by the necessary condition (13), the number of votes $v(u)$ bought by this agent is $u\times$ the agent's marginal pivotality $\frac{1}{2}E\psi(v(u) + S_n)$. Consider the event that all n of the remaining agents have values in $[\underline{u}_{1/n}, \overline{u}_{1/n}]$: the probability of this event is

$$\begin{aligned}\left(1 - \int_{\underline{u}}^{\underline{u}_{1/n}} f(t)\,dt - \int_{\overline{u}_{1/n}}^{\overline{u}} f(t)\,dt\right)^n &\approx (1 - f(\underline{u})/n - f(\overline{u})/n)^n \\ &\approx \exp\{-f(\underline{u}) - f(\overline{u})\} := p > 0,\end{aligned}$$

and on this event, the vote total would not exceed $\delta/4 + \delta/(4n) \leq \delta/2$ in absolute value. Consequently, the marginal pivotality satisfies

$$E\psi(v(u) + S_n) \geq p \min_{x \in [-\delta/2, \delta/2]} |\psi(x)|,$$

and so

$$2|v(u)| \geq |u| p E\psi(v(u) + S_n) \geq p \min_{x \in [-\delta/2, \delta/2]} |\psi(x)|.$$

Since ψ is bounded away from 0 on the interval $[-\delta/2, \delta/2]$, for large n this would contradict our supposition that $\max |v(u)|$ on the interval $u \in [\underline{u}_{1/n}, \overline{u}_{1/n}]$ is below $\delta/(4n)$. □

3 Weak Consensus Estimates

According to Proposition 4, in any Nash equilibrium the number of votes $v(u)$ an agent with utility u purchases is $u\times$ the *marginal pivotality* $\frac{1}{2}E\psi(v(u) + S_n)$. When the sample size $N = n + 1$ is large, the effect of a single voter's contribution $v(u)$ to the vote total $v(u) + S_n$ should be small, and so one expects that an agent with value u should have approximately the same marginal pivotality as an agent with value u'. However, as we will show later, this is not always true: an agent with value u in the extreme tails of the distribution F can, in some circumstances, have a drastically different marginal pivotality than agents with values in the bulk of the distribution.

Nevertheless, agents with values in the bulk of the distribution do have nearly the same marginal pivotalities, as the following lemma shows. This lemma will figure critically in the proof of Lemma 19 (the key anti-concentration estimate) in section 4 below.

Lemma 11. *For any $\epsilon > 0$ and any $0 < \alpha < 1$, if n is sufficiently large then in every Nash equilibrium $v(\cdot)$,*

$$1 - \alpha \leq \frac{E\psi(v(u) + S_n)}{E\psi(v(u') + S_n)} \leq (1 - \alpha)^{-1} \tag{23}$$

for any two values u, u' not within distance ϵ of either $\underline{u}$, or $\overline{u}$, or 0. Furthermore, there exists $C > 0$ such that for any $\beta > 0$, if n is sufficiently large then in any equilibrium, for all $u \in [\underline{u}, \overline{u}] \setminus (-\epsilon, \epsilon)$,

$$\frac{E\psi(v(u) + S_n)}{E\psi(v(\overline{u}_{\beta n^{-3/2}}) + S_n)} \geq \frac{C\beta}{n^{1/2}} \quad \textit{and} \tag{24}$$

$$\frac{E\psi(v(u) + S_n)}{E\psi(v(\underline{u}_{\beta n^{-3/2}}) + S_n)} \geq \frac{C\beta}{n^{1/2}}. \tag{25}$$

Proof. Let $J_1, J_2, \dots, J_k$ be any partition of the interval $[\underline{u}, \overline{u}]$ into non-overlapping Borel sets of positive Lebesgue measure, and for each index i let M_i be the number of agents (in the entire sample of size $N = n+1$) with values in the set J_i. The random vector $(M_1, M_2, \dots, M_k)$ has the multinomial distribution

$$P(M_i = m_i \text{ for each } i \leq k) = \frac{(n+1)!}{m_1! m_2! \cdots m_k!} \prod_{i=1}^{k} p_i^{m_i}$$

where

$$p_i = P(U_1 \in J_i) = \int_{J_i} f(u)\, du.$$

Conditional on the event that $M_i = m_i$ for each $i \leq k$, the sample $\{U_1, U_2, \dots, U_n\}$ has the same distribution as a stratified random sample gotten by choosing m_i elements from the set J_i according to the density $f\mathbf{1}_{J_i}/p_i$ for each $i \leq k$. Consequently, for any choice of index $i \leq k$,

$$E[\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J_i] = \sum_{m_1, m_2, \dots, m_k} \frac{n!}{m_1! m_2! \cdots m_k!} \left(\prod_{i=1}^{k} p_i^{m_i} \right) E_*(m_1, m_2, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots m_k) \tag{26}$$

where

$$E_*(m_1, m_2, \dots, m_k) = E(\psi(S_{n+1}) \,|\, M_i = m_i \;\forall\, i \leq k)$$

for any set of nonnegative integers m_j that sum to $n+1$. Note that the conditional expectations $E_*(m_1, m_2, \dots, m_k)$ are all nonnegative, and are bounded above by $\|\psi\|_\infty$.

The proof of the lemma will be based on systematic exploitation of equation (26). To relate the conditional expectation on the left side of (26) to the marginal pivotality $\frac{1}{2} E\psi(v(u) + S_n)$, we appeal to the monotonicity of $v(\cdot)$. Fix $u \in (\epsilon, \overline{u}]$ and let $J = [u - \alpha u, u]$ and $J' = [u, u + \alpha u] \cap [u, \overline{u}]$, where $\alpha > 0$ is small enough that $J \subset [0, \overline{u}]$; then by (13) and the monotonicity of v,

$$\begin{aligned} (1-\alpha) E\psi(v(u') + S_n) &\leq E\psi(v(u) + S_n) \quad \text{for all } u' \in J \quad \text{and} \\ (1+\alpha) E\psi(v(u'') + S_n) &\geq E\psi(v(u) + S_n) \quad \text{for all } u'' \in J'. \end{aligned}$$

Consequently,

$$\begin{aligned} (1-\alpha) E[\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J] &\leq E\psi(v(u) + S_n) \quad \text{and} \\ (1+\alpha) E[\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J'] &\geq E\psi(v(u) + S_n). \end{aligned} \tag{27}$$

A similar argument shows that for values $u \in [\underline{u}, -\epsilon]$, the inequalities (27) hold with $J = [u, u - \alpha u]$ and $J' = [u + u\alpha, u] \cap [\underline{u}, u]$. Thus, to prove the inequalities (23), (24), and (25), it suffices to prove analogous inequalities for conditional expectations of the form (26).

Proof of inequalities (24) *and* (25). There are two cases for each inequality, depending on whether $u \geq \epsilon$ or $u \leq -\epsilon$; by symmetry (i.e., reversing the roles of *positive* and *negative* values and votes), it is enough to consider only the case $u \geq \epsilon$. Moreover, it suffices to prove only one of the inequalities (24) and (25), specifically, the one for which the denominator on the left side is larger, because the other will then follow trivially. For the sake of exposition, let's assume that

$$E\psi(v(\overline{u}_{\beta n^{-3/2}} + S_n) \geq E\psi(v(\underline{u}_{\beta n^{-3/2}} + S_n);$$

the other case is similar. Under this assumption,

$$E\psi(v(\overline{u}_{\beta n^{-3/2}} + S_n) \geq \gamma n^{-1} \tag{28}$$

for some $\gamma > 0$ independent of n, by the *a priori* bound of Lemma 10.

Fix $\beta > 0$ and $\alpha \in (0, 1)$, and let

$$\begin{aligned} J_1 &= [u - \alpha u, u], \\ J_2 &= [\overline{u}_{\beta n^{-3/2}}, \overline{u}], \quad \text{and} \\ J_3 &= [\underline{u}, \overline{u}] \setminus (J_1 \cup J_2). \end{aligned}$$

By (27), to prove (24) it suffices to show that

$$\frac{E\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J_1]}{E\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J_2]} \geq \frac{C'\beta}{n^{1/2}}, \tag{29}$$

for some constant $C' > 0$ independent of n and β. Now (28), together with inequality (27) and the monotonicity of v, implies that the denominator is of size at least

$$E\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J_2] \geq \gamma' n^{-1};$$

consequently, in proving (29) we can ignore errors that decay exponentially in n.

For brevity, denote the numerator on the left side of (29) by E_1 and the denominator by E_2. Both E_1 and E_2 can be expressed as sums of the form (26) (with $k = 3$). For any triple (m_1, m_2, m_3) with $m_1 \geq 1$ and $m_2 \geq 1$, terms with factor $E_*(m_1, m_2, m_3)$ occur in both sums, but with different coefficients

$$\begin{aligned} &\frac{n!}{(m_1-1)!m_2!m_3!} p_1^{m_1-1} p_2^{m_2} p_3^{m_3} \quad \text{in } E_1, \quad \text{and} \\ &\frac{n!}{m_1!(m_2-1)!m_3!} p_1^{m_1} p_2^{m_2-1} p_3^{m_3} \quad \text{in } E_2, \end{aligned}$$

where p_1, p_2, p_3 are the $F-$probabilities of J_1, J_2, J_3, respectively. The ratio of these coefficients (E_1 to E_2) is

$$\frac{m_1}{m_2} \frac{p_2}{p_1} \sim \frac{m_1 \beta f(\overline{u})}{p_1 m_2 n^{3/2}}.$$

We will show that triples (m_1, m_2, m_3) with either $m_1 \leq np_1/2$ or $m_2 \geq 5$ can be ignored; for all other triples, the coefficient ratio is, for large n, at least

$$\frac{m_1}{m_2}\frac{p_2}{p_1} \geq \frac{\beta f(\overline{u})}{(2 \times 2 \times 5)n^{1/2}}$$

This will prove that inequality (29) holds with $C' = f(\overline{u})/(40n^{1/2})$, provided n is sufficiently large.

Consider first those triples (m_1, m_2, m_3) with $m_1 \leq np_1/2$. In both sums E_1 and E_2, the total contribution of the terms with factors $E_*(m_1, m_2, m_3)$ where $m_1 \leq np_1/2$ is at most $\|\psi\|_\infty$ times the probability that a random sample of size n from the distribution F has fewer than $1 + np_1/2$ elements in the interval J_1. By Hoeffding's inequality (cf. Appendix C), this probability is decays exponentially in n; therefore, the total contribution of these terms to either E_i is negligible compared to E_i when n is large.

Finally, consider those triples with $m_2 \geq 5$. In E_1, these triples correspond to samples with at least 5 elements in the interval J_2, while in E_2 they correspond to samples with at least 4 entries in J_2. The probability that a random sample of size n contains at least k elements in an interval of length $\beta n^{-3/2}$ is no larger than

$$\binom{n}{k}(\beta n^{-3/2}\|f\|_\infty)^k.$$

For $k = 4$ this is $O(n^{-2})$, which is of smaller order of magnitude than E_2 (which is $\geq \gamma'/n$, as noted above); and for $k = 5$ it is $O(n^{-5/2})$, which is of smaller order than E_1, as this (by the argument above) is of size at least $C'E_2/(40n^{1/2})$.

$\square$

Proof of inequality (23). The proof of inequality (23) follows a similar line. Clearly, it is enough to prove the lower bound in (23), because the upper bound will then follow by reversing the roles of u and u'. By inequalities (27), it suffices to show that for any $\alpha > 0$ and any two fixed, non-overlapping intervals J_1, J_2 of positive length contained in $[\underline{u}_\epsilon, -\epsilon] \cup [\epsilon, \overline{u} - \epsilon]$,

$$\frac{E[\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J_1]}{E[\psi(v(U_{n+1}) + S_n) \,|\, U_{n+1} \in J_2]} \geq 1 - \alpha$$

provided n is sufficiently large. In proving this, we can ignore terms of exponentially small size, because (24) and (25), together with Lemma 10, imply that both numerator and denominator are of magnitude at least (constant)$\times n^{-3/2}$.

As in the proof of inequality (24), denote the numerator and denominator by E_1 and E_2, respectively. Each of the conditional expectations E_1, E_2 has a representation (26) with $k = 3$, where J_3 is the complement of $J_1 \cup J_2$ in $[\underline{u}, \overline{u}]$. For any $\gamma > 0$, the contribution to either E_1 or E_2 from terms of (26) for which $|m_i - np_i| \geq n\gamma p_i$, for either $i = 1$ or $i = 2$, is exponentially small, by Hoeffding's inequality (cf. Appendix C), and hence can be ignored. For any triple (m_1, m_2, m_3) with $m_i \geq 1$ and $m_2 \geq 1$, terms with factor $E_*(m_1, m_2, m_3)$ occur in both E_1 and E_2, with the same coefficients as in the proof of (24). The ratio of these coefficients (E_1 to E_2) is

$$\frac{m_1}{m_2}\frac{p_2}{p_1}.$$

Since only those triples with $|m_i - np_i| < n\gamma p_i$ contribute substantially to the expectations, it follows that for large n,

$$\frac{E_1}{E_2} \geq \frac{1+2\gamma}{1-2\gamma}.$$

Clearly, if $\gamma > 0$ is sufficiently small then the lower bound will exceed $1 - \alpha$. $\square$

$\square$

4 Concentration and size constraints

Because the vote total S_n is the sum of independent, identically distributed random variables $v(U_i)$ (albeit with unknown distribution), its distribution is subject to concentration restrictions, such as those imposed by the following lemma.

Lemma 12. *For any $\epsilon > 0$ and any $\beta > 0$ there exists $\gamma < \infty$ such that for all sufficiently large values of n and any Nash equilibrium $v(u)$, if*

$$\|v\|_\infty \geq \epsilon \tag{30}$$

then

$$|v(u)| \leq \frac{\gamma}{\sqrt{n}} \quad \textit{for all } u \in [\underline{u}_\beta, \overline{u}_\beta]. \tag{31}$$

We will deduce Lemma 12 from the following general fact about sums of independent, identically distributed random variables.

Proposition 13. *Fix $\alpha > 0$. For any $\epsilon > 0$ and any $C < \infty$ there exist constants $C' > 0$ and $n' < \infty$ such that the following statement is true: if $n \geq n'$ and $Y_1, Y_2, \dots, Y_n$ are independent, identically distributed random variables such that*

$$E|Y_1 - EY_1|^3 \leq C\mathrm{var}(Y_1)^{3/2} \quad \textit{and} \quad \mathrm{var}(Y_1) \geq C'/n \tag{32}$$

then for every interval $J \subset \mathbb{R}$ of length α or greater,

$$P\left\{\sum_{i=1}^n Y_i \in J\right\} \leq \epsilon|J|/\alpha. \tag{33}$$

The proof of this proposition, a routine exercise in the use of the Berry-Esseen theorem, is relegated to Appendix D.

Proof of Lemma 12. By the monotonicity of Nash equilibria and the necessary condition (13), if a Nash equilibrium $v(u)$ has supremum norm $\|v\|_\infty > \epsilon$ then for at least one of the endpoints $u = \underline{u}$ or $u = \overline{u}$,

$$|u|E\psi(v(u) + S_n) > \epsilon, \tag{34}$$

and so

$$P\{S_n + v(u) \in [-\delta, \delta]\} \geq \frac{2\epsilon}{|u|\|\psi\|_\infty} \geq \epsilon' := \min_{u=\underline{u},\overline{u}} \frac{2\epsilon}{|u|\|\psi\|_\infty}.$$

We will use Proposition 13 to show that such a high concentration of probability in an interval of length $\alpha = 2\delta$ is impossible unless the function v is bounded above by $\gamma/\sqrt{n}$ in the interval $[\underline{u}_\beta, \overline{u}_\beta]$.

We can assume without loss of generality that $\beta > 0$ is sufficiently small that the interval $[-2\beta, 2\beta]$ is contained in the open interval $(\underline{u}_\beta, \overline{u}_\beta)$, because the condition (31) becomes less stringent as β increases. For such β, the intervals $[\underline{u}_\beta, -2\beta]$ and $[2\beta, \overline{u}_\beta]$ both have positive length; hence, since the density f is bounded below, it follows that for suitable constants $C > 0$ and $\frac{1}{2} > p > 0$ there are intervals

$$\begin{aligned} J_+ &= [u_+, \overline{u}_\beta] \subset (2\beta, \overline{u}) \quad \text{and} \\ J_- &= [u_-, \underline{u}_\beta] \subset (\underline{u}, -2\beta), \end{aligned}$$

both with $F-$probability p.

Claim: If U has density f, then for a suitable constant $C' < \infty$ the *conditional* distribution of $v(U)$ given the event $G := \{U \in J_+ \cup J_-\}$ satisfies the moment conditions

$$E(|v(U) - E(v(U)\,|\,G)|^2\,|\,G) \geq \frac{1}{2}\min(v(u_+)^2, v(u_-)^2) \quad \text{and} \tag{35}$$

$$E(|v(U) - E(v(U)\,|\,G)|^3\,|\,G) \leq C' E(|v(U) - E(v(U)\,|\,G)|^2\,|\,G)^{3/2} \tag{36}$$

Proof of the Claim. By Lemma 28, any Nash equilibrium $v(u)$ is nondecreasing, strictly positive for $u > 0$, and strictly negative for $u < 0$. Consequently, $v(U) \geq v(u_+) > 0$ on the event $G \cap \{U > 0\}$, and $v(U) \leq v(u_-) < 0$ on the event $G \cap \{U \leq 0\}$. By construction, the conditional probability that $U > 0$ given the event G is $p/(2p) = 1/2$. Thus,

$$E(|v(U) - E(v(U)\,|\,G)|^2\,|\,G) \geq E(|v(U)|^2 \mathbf{1}_{\{U>0\}}\,|\,G) \geq \frac{1}{2}v(u_+)^2 \quad \text{if } E(v(U)\,|\,G) \leq 0,$$

$$E(|v(U) - E(v(U)\,|\,G)|^2\,|\,G) \geq E(|v(U)|^2 \mathbf{1}_{\{U\leq 0\}}\,|\,G) \geq \frac{1}{2}v(u_-)^2 \quad \text{if } E(v(U)\,|\,G) \geq 0.$$

This proves (35).

Denote by $v_* = \min(v(u_+), |v(u_-)|) > 0$ the minimum value of $|v(u)|$ on the set $u \in J_+ \cup J_-$. By Lemma 11, there is a constant $A > 0$ depending on $\beta > 0$ but not on n such that for any Nash equilibrium,

$$v_* := \min_{u \in J_+ \cup J_-.} |v(u)| \geq A \max_{u \in J_+ \cup J_-} |v(u)|.$$

Clearly, the conditional expectation $E(v(U) \,|\, G)$ falls in the interval $[-v_*, v_*]$, so deviations from the conditional expectation are bounded by $2v_*$. Therefore,

$$E(|v(U) - E(v(U) \,|\, G)|^3 \,|\, G) \leq 8 \max_{u \in J_+ \cup J_-} |v(u)|^3 \leq 8(v_*/A)^3.$$

But (35) implies that the conditional variance of $v(U)$ is at least $\frac{1}{2}v_*^2$, so (36) follows, with $C' = 8 \cdot 2^{3/2} \cdot A^{-2}$. □

Let M be the number of points U_i in the sample $U_1, U_2, \dots, U_n$ that fall in $J_+ \cup J_-$, and let S_n^* be the sum of the votes $v(U_i)$ for those agents i whose values U_i fall in this range. By construction, M has the binomial-$(n, 2p)$ distribution. Moreover, conditional on the event $M = m$ and $S_n - S_n^* = w$, the random variable S_n^* is the sum of m independent random variables Y_i whose common distribution is the conditional distribution of $v(U)$ given the event $G = \{U \in J_+ \cup J_-\}$. By the Claim above, the third moment hypothesis of Proposition 13 is met, so for any $\epsilon' > 0$ there exists $C'' = C''(\epsilon') < \infty$ such that if

$$\max(v(\overline{u}_\beta), |v(\underline{u}_\beta)|) \geq C''/\sqrt{n} \tag{37}$$

then the conditional probability, given $M = m \geq np$ and $S_n - S_n^* = w$, that S_n^* lies in any interval of length 4δ is bounded above by ϵ'. This in turn implies

$$E\psi(S_n + v) \leq \|\psi\|_\infty \left(P\{M \leq np\} + \epsilon'\right) \quad \text{for all } v \in \mathbb{R}.$$

Since $P\{M \leq np\} \to 0$ as $n \to \infty$, by the weak law of large numbers, it then follows that for all sufficiently large n,

$$E\psi(S_n + v) \leq 2\epsilon' \|\psi\|_\infty.$$

For small ϵ', this bound is smaller than $\epsilon / \max(\overline{u}_\beta, |\underline{u}_\beta|)$, so the hypothesis (37) is incompatiable with (34).

□

Lemma 12 implies that for any $\beta > 0$, if n is sufficiently large then for any Nash equilibrium $v(u)$, the absolute value $|v(u)|$ must be small except at values u within distance β of one of the endpoints $\underline{u}, \overline{u}$. The following lemma improves this bound to the extreme tails of the distribution.

Lemma 14. *For any $0 < \epsilon < \infty$ there exists $\alpha > 0$ such that for all sufficiently large n, every Nash equilibrium $v(u)$ satisfies the inequality*

$$|v(u)| \leq \epsilon \tag{38}$$

for all u at distance greater than $\alpha n^{-3/2}$ from both endpoints $\underline{u}, \overline{u}$.

Proof. It suffices, by symmetry, to consider only values $u > 0$. Lemma 12 implies that for any $\epsilon > 0$ there exists $\gamma > 0$ such that if n is sufficiently large and $\|v\|_\infty \geq \epsilon$ then $2|v(u)| \leq \gamma/\sqrt{n}$ for all $u \in [\underline{u}_\epsilon, \overline{u}_\epsilon]$. Hence, by the necessary condition (13),

$$E\psi(v(\overline{u}_\epsilon) + S_n)\overline{u}_\epsilon \leq \frac{\gamma}{\sqrt{n}}.$$

But by Lemma 11 (cf. inequalities (24) and (25)), there exists $C > 0$ such that for any $\alpha > 0$, if n is sufficiently large then for every Nash equilibrium v,

$$\frac{E\psi(v(u) + S_n)}{E\psi(v(\overline{u}_{\alpha n^{-3/2}}) + S_n)} \geq \frac{C\alpha}{\sqrt{n}}$$

for all $u \in [\underline{u}_\epsilon, \overline{u}_\epsilon]$, and in particular for $u = \overline{u}_\epsilon$. The last two displayed inequalities now combine to yield

$$E\psi(v(\overline{u}_{\alpha n^{-3/2}}) + S_n) \leq \frac{\gamma}{C\alpha} \quad \Longrightarrow \quad 2v(\overline{u}_{\alpha n^{-3/2}}) \leq \frac{\gamma\overline{u}}{C\alpha}.$$

Thus, the inequality $2v(u) > \epsilon$ can hold at some $u = u_{\alpha n^{-3/2}}$ only if $\alpha < (\gamma\overline{u})/(C\epsilon)$. □

Proposition 15. *For any Nash equilibrium $v(u)$ there is a maximal nonempty interval $J_{\max}$ containing $u = 0$ in its interior on which $v(u)$ is continuous. For sufficiently large n the endpoints $u_- < 0 < u_+$ of this interval lie within distance $\beta n^{-3/2}$ of $\underline{u}$ and $\overline{u}$, respectively, for some constant $\beta > 0$ not depending on n or on the particular Nash equilibrium. The function $v(u)$ is C^∞ on the interior of $J_{\max}$, and for any $\epsilon > 0$, if n is sufficiently large then*

$$\sup_{u \in J_{\max}} |v(u)| < \epsilon. \tag{39}$$

Consequently, for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that if $n \geq n_\epsilon$ then any Nash equilibrium $v(u)$ with no discontinuities satisfies

$$\|v\|_\infty < \epsilon. \tag{40}$$

Proof. The existence of a nonempty interval of continuity containing $u = 0$ is clear, because any Nash equilibrium is continuous at 0. That the endpoints $u_- < u_+$ of this interval lie within distance $\beta n^{-3/2}$ of the endpoints $\underline{u}, \overline{u}$ follows from Proposition 5 and Lemma 14, because the former asserts that there is no discontinuity on any interval in which $|v(u)| < \Delta$, where $\Delta > 0$ is the discontinuity threshold, and the latter implies that $|v(u)|$ is bounded by Δ on the interval $(\underline{u}_{\beta n^{-3/2}}, \overline{u}_{\beta n^{-3/2}})$, for some $\beta > 0$ independent of n and the particular Nash equilibrium.

It remains to show that $v(u)$ is not only continuous but C^∞ on the interior of J, and that for sufficiently large n the inequality (39) holds. Since $v(u)$ is continuous on J, Proposition 8 implies that $v(u)$ fails to be C^∞ only at points u where the equality (17) holds. But Lemma 7 and the necessary condition (13) imply that if $\epsilon > 0$ is sufficiently small then equation (17) cannot be satisfied at any point where $2|v(u)| < \epsilon$. Thus, to complete the proof it will suffice to show that for any $\epsilon > 0$, if n is sufficiently large then $|v(u)| < 2\epsilon$ on the interval $J_{\max}$.

Assume that $2\epsilon < \Delta$, where Δ is the discontinuity threshold. By Corollary 9, if $\epsilon > 0$ is sufficiently small then $|v'(u)| \leq \|\psi\|_\infty$ for all $u \geq 1$ at which $|v(u)| < \epsilon$. By Proposition 14, the inequality $|v(u)| < \epsilon$ holds for all u at distance more than $\beta n^{-3/2}$ from the endpoints $\underline{u}, \overline{u}$, where $\beta > 0$ does not depend on n or on the particular Nash equilibrium. Now suppose that $v(u) \geq 2\epsilon$ for some $u \in [1, \overline{u}] \cap J_{\max}$, and define

$$u_* := \inf\{u \geq 1 \,:\, v(u) \geq 2\epsilon\}.$$

Since $v(u)$ has no discontinuities in $J_{\max}$, it must be the case that $v(u_*) = 2\epsilon$. But the fundamental theorem of calculus implies

$$\begin{aligned} v(u_*) &= v(\overline{u}_{\beta n^{-3/2}}) + \int_{\overline{u}_{\beta n^{-3/2}}}^{u_*} v'(u)\, du \\ &\leq \epsilon + (u_* - \overline{u}_{\beta n^{-3/2}})\|\psi\|_\infty \\ &\leq \epsilon + \beta n^{-3/2}\|\psi\|_\infty; \end{aligned}$$

thus, if n is sufficiently large then the equality $v(u_*) = 2\epsilon$ is impossible. The same argument shows that for large n there can be no $u_* \in [\underline{u}, -1] \cap J_{\max}$ at which $v(u_*) = -2\epsilon$. $\square$

5 Approximate Proportionality

5.1 The approximate proportionality rule

We have shown in Lemma 14 that any Nash equilibrium $v(u)$ must be small (in magnitude) except in the extreme tails of the distribution (in particular, for all u at distance much more than $n^{-3/2}$ from both endpoints $\underline{u}, \overline{u}$). Because ψ is uniformly continuous, it follows that the marginal pivotality $\frac{1}{2}E\psi(v(u)+S_n)$ cannot differ by very much from $\frac{1}{2}E\psi(S_n)$. Thus, the approximation $2v(u) \approx E\psi(S_n)u$ is valid up to an error of size $\epsilon_n|u|$ where $\epsilon_n \to 0$ as $n \to \infty$. However, as $n \to \infty$ the expectation $E\psi(S_n) \to 0$, and so the error in the approximation above might be considerably larger than the approximation itself. The following proposition makes the stronger assertion that when n is large the relative error in the approximate proportionality rule is small. This extends the range of validity of the weak consensus estimate (23) to the maximal interval $J_{\max}$ of continuity.

Proposition 16. *For all n sufficiently large, in any Nash equilibrium, $E\psi(S_n) > 0$, and for any $\epsilon > 0$ there exists n_ϵ such that for all $n \geq n_\epsilon$ and any Nash equilibrium $v(u)$, if $J_{\max}$ is the maximal interval of continuity of $v(u)$ containing $u = 0$ then*

$$(1+\epsilon)^{-1} \leq \frac{2v(u)}{E\psi(S_n)u} \leq (1+\epsilon) \quad \textit{for all } u \in J_{\max} \setminus \{0\}\,. \tag{41}$$

Proof of Proposition 16. Since ψ is C^∞ and has compact support $[-\delta, \delta]$, the function $v \mapsto E\psi(v+S_n)$ is also C^∞, with derivative $E\psi'(v+S_n)$. By Proposition 15, the function $v(u)$ is C^∞ on the interval $J_{\max}$, and so, by the chain rule, the function $u \mapsto E\psi(v(u)+S_n)$ is C^∞, with derivative $E\psi'(v(u)+S_n)v'(u)$. Hence, by Taylor's theorem, for every $u \in J_{\max}$ there exists $\tilde{u}(u)$ intermediate between 0 and u such that

$$\begin{aligned} 2v(u) = E\psi(v(u)+S_n)u = E\psi(S_n)u + E\psi'(v(\tilde{u}(u))+S_n)v(u)u \quad \Longrightarrow \\ (2 - E\psi'(v(\tilde{u}(u))S_n)u)v(u) = E\psi(S_n)u. \end{aligned} \tag{42}$$

We will prove that for any $\epsilon' > 0$, if n is sufficiently large then in any Nash equilibrium $v(u)$,

$$|E\psi'(v(\tilde{u}(u))+S_n)u| < \epsilon' \quad \text{for all } u \in J_{\max}. \tag{43}$$

This will imply that $E\psi(S_n) \neq 0$ (because otherwise equation (5.1) would imply that $v(u) = 0$ for all $u \in J_{\max}$, contradicting Proposition 3), and the assertion (41) will then follow directly from (5.1).

By Lemma 7, for any $\epsilon' > 0$ there exists $\varrho = \varrho(\epsilon') > 0$ such that if $E\psi(\tilde{v}+S_n)|u| < \varrho$ then $E|\psi'(\tilde{v}+S_n)u| < \epsilon'$. Thus, to prove (43) it suffices to show that for any $\varrho > 0$, if n is sufficiently large then in every Nash equilibrium,

$$E\psi(v(\tilde{u}(u))+S_n)|u| < \varrho \quad \text{for all } u \in J_{\max}. \tag{44}$$

Since $|u|$ is bounded by $\max(|\underline{u}|, \overline{u})$ and $\tilde{u}(u)$ is intermediate between 0 and u, to prove (44) it is enough to show that for any $\varrho' > 0$, if n is sufficiently large then in every Nash equilibrium,

$$E\psi(v(u)+S_n) < \varrho' \quad \text{for all } u \in J_{\max}. \tag{45}$$

Proposition 15 implies that for any $\varrho'' > 0$ the function $2|v(u)|$ is bounded above by ϱ'' on the interval $J_{\max}$, provided n is large. Hence, by the necessary condition (13),

$$E\psi(v(u) + S_n) < \varrho'' \quad \text{for all } u \in J_{\max} \setminus (-1, 1). \tag{46}$$

Thus, inequality (45) will hold outside the interval $(-1, 1)$ if $\varrho'' < \varrho'$. On the other hand, by the fundamental theorem of calculus, for any u in the interval $(-1, 1)$,

$$\begin{aligned} |E\psi(v(u) + S_n) - E\psi(v(-\alpha) + S_n)| &\le \int_{v(-1)}^{v(u)} E|\psi'(v(t) + S_n)|\, dt \\ &\le \|\psi'\|_\infty (v(1) - v(-1)) \\ &\le 2\|\psi'\|_\infty \max_{u \in J_{\max}} |v(u)| \\ &\le \|\psi'\|_\infty \varrho''; \end{aligned}$$

consequently, if $\|\psi'\|_\infty \varrho'' < \varrho'$ then (45) will hold in the interval $(-1, 1)$. □

5.2 Consequences

Proposition 16 implies that any agent with value u in the maximal interval $J_{\max}$ of continuity of a Nash equilibrium $v(u)$ will buy $E\psi(S_n)u$ votes, up to an error of size $o(v(u))$. Henceforth, we will refer to agents with values $u \in J_{\max}$ as *moderates*, and agents with values $u \notin J_{\max}$ as *extremists*. By Proposition 15, there is a constant β such that the endpoints of $J_{\max}$ are within distance $\beta n^{-3/2}$ of the endpoints $\underline{u}, \overline{u}$ of the support of the sampling density f; since f is continuous at the endpoints $\underline{u}, \overline{u}$, it follows that when n is large, the extremist region $[\underline{u}, \overline{u}] \setminus J_{\max}$ has $F-$probability less than $2\beta n^{-3/2}(f(\underline{u}) + f(\overline{u}))$. Consequently, if

$$G = \bigcap_{i=1}^{n} \{U_i \in J_{\max}\} \tag{47}$$

is the event that the all-but-one sample contains no extremists, then for all sufficiently large n,

$$P(G^c) \le 2\beta n^{-1/2}(f(\underline{u}) + f(\overline{u})). \tag{48}$$

In particular, the probability that the sample contains an extremist is vanishingly small as $n \to \infty$.

Given the event G, the conditional distribution of the sample $U_1, U_2, \cdots, U_n$ is the same as the *unconditional* distribution of a random sample of size n from the density

$$f_G(u) := \frac{f(u)\mathbf{1}_{J_{\max}}}{\int_{J_{\max}} f}.$$

Since voters in the moderate range follow the approximate proportionality rule (41), it follows that for large n, conditional on the event G, the votes $v(U_i)$ are independent, identically distributed random variables bounded above and below by $2E\psi(S_n)\underline{u}$ and $2E\psi(S_n)\overline{u}$. Thus, Hoeffding's inequality (cf. Appendix C) implies the following.

Corollary 17. *For all sufficiently large n and any Nash equilibrium $v(u)$,*

$$P(|S_n - E(S_n|G)| \geq t \,|\, G) \leq 2\exp\{-2t^2/(4n[E\psi(S_n)]^2(\overline{u} - \underline{u})^2)\}; \tag{49}$$

and so for any Nash equilibrium with no discontinuities,

$$P(|S_n - ES_n| \geq t) \leq 2\exp\{-2t^2/(4n[E\psi(S_n)]^2(\overline{u} - \underline{u})^2)\}. \tag{50}$$

Proposition 16 also implies uniformity in the normal approximation to the distribution of S_n, because the proportionality rule (41) guarantees that the ratio of the (conditional on G) third moment to the $3/2$ power of the (conditional) variance of $v(U_i)$ is uniformly bounded. Hence, by the Berry-Esseen theorem (cf. Appendix D), we have the following corollary.

Corollary 18. *There exists $\kappa < \infty$ such that for all sufficiently large n and any Nash equilibrium $v(u)$, the vote total S_n satisfies*

$$\sup_{t\in\mathbb{R}} |P(S_n - E(S_n|G) \leq t\sqrt{\operatorname{var}(S_n|G)} \,|\, G) - \Phi(t)| \leq \kappa n^{-1/2},$$

and consequently, since $1 - P(G) = O(n^{-1/2})$, there exists $\kappa' < \infty$ such that

$$\sup_{t\in\mathbb{R}} |P\{S_n - E(S_n|G) \leq t\sqrt{\operatorname{var}(S_n|G)}\} - \Phi(t)| \leq \kappa' n^{-1/2}.$$

Here Φ denotes the standard normal cumulative distribution function.

6 Unbalanced Populations: Proofs of Theorems 2–3

6.1 Concentration of the vote total

The following lemma shows that if the sampling distribution F has positive mean $\mu > 0$ then when n is large the vote total, in any Nash equilibrium, concentrates at a value between δ and $\delta + \sqrt{2|\underline{u}|}$. (Recall that $[-\delta, \delta]$ is the support of the function ψ.)

Lemma 19. *If $\mu > 0$ then for all large n no Nash equilibrium $v(u)$ has a discontinuity at a nonnegative value of u. Moreover, for any $\epsilon > 0$, if n is sufficiently large then in any Nash equilibrium the vote total S_n of the one-out sample must satisfy*

$$\delta - \epsilon \leq ES_n \leq \delta + \epsilon + \sqrt{2\,|\underline{u}|]} \quad \textit{and} \tag{51}$$

$$P\{|S_n - ES_n| > \epsilon\} < \epsilon. \tag{52}$$

In addition, for any $\epsilon > 0$ there exists $\gamma > 0$ such that if n is sufficiently large and $v(u)$ is a Nash equilibrium with no discontinuities, then

$$P\{|S_n - \delta| > \epsilon\} < e^{-\gamma n}. \tag{53}$$

The proof of this will rely on a simple estimate for the expected vote total of the extremists in the out-sample. Denote by N_e the number of extremists among the first n agents, and by S'_n their vote total, formally,

$$N_e = \sum_{i=1}^{n} \mathbf{1}_{J^c_{\max}}(U_i) \quad \text{and} \tag{54}$$

$$S'_n = \sum_{i=1}^{n} v(U_i)\mathbf{1}_{J^c_{\max}}(U_i). \tag{55}$$

Lemma 20. *There exists a constant $C < \infty$ independent of n and of the particular Nash equilibrium such that*

$$E|S'_n| \leq Cn^{-1/2} \quad \textit{and} \quad |ES_n - E(S_n\,|\,G)| \leq Cn^{-1/2}. \tag{56}$$

Proof. Since not even an extremist would ever buy more than $\kappa := \max(\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}})$ votes, the extremist total S'_n is bounded by κN_e. For large n the extremist range $[\underline{u}, \overline{u}] \setminus J_{\max}$ has $F-$probability less than $2\beta n^{-3/2}(f(\overline{u}) + f(\underline{u})) := \alpha n^{-3/2}$; consequently,

$$P\{N_e \geq k\} \leq \frac{n^k}{k!}(\alpha n^{-3/2})^k \leq \frac{\alpha^k}{k! n^{k/2}} \quad \text{for all } k \geq 1. \tag{57}$$

Thus, if n is sufficiently large,

$$E|S'_n| \leq \alpha\kappa \sum_{k\geq 1} n^{-k/2} \leq 2\alpha\kappa n^{-1/2}.$$

This proves the first inequality in (56).

Suppose now that the N_e extremists in the sample were replaced by N_e independently chosen moderates. Denote by S''_n the vote total of these replacements.

Clearly, $|S''_n| \leq \kappa N_e$, since no agent ever buys more than κ votes, and so by the same calculation as above,

$$E|S''_n| \leq 2\alpha\kappa n^{-1/2}.$$

By construction, the sample obtained by replacing the extremists by moderates is a sample of size n chosen from $J_{\max}$, and so $E(S_n - S'_n + S''_n) = E(S_n \,|\, G)$. Consequently, the second inequality in (56) follows from the bounds on $E|S'_n|$ and $E|S''_n|$ obtained above. $\square$

Proof of Lemma 19. By Lemma 20, the difference $ES_n - E(S_n \,|\, G)$ is vanishingly small, and so the expectation ES_n can be replaced by $E(S_n \,|\, G)$ in the inequalities (51), (52), and (53). By Proposition 16, Nash equilibria $v(u)$ obey the approximate proportionality rule (41) in the moderate range, and by inequality (48), the difference between μ and $E(U_i \,|\, U_i \in J_{\max})$ is negligible; consequently, for any $\epsilon > 0$, if n is sufficiently large then

$$nE\psi(S_n)(\mu - \epsilon)(1 - \epsilon) \leq E(S_n \,|\, G) \leq nE\psi(S_n)(\mu + \epsilon)(1 + \epsilon). \tag{58}$$

By Proposition 16, $E\psi(S_n) > 0$, and by hypothesis, $\mu > 0$, so it follows that the lower bound in (58) is positive (provided $\epsilon > 0$ is sufficiently small).

Suppose now that for some $\epsilon' > 0$ there are Nash equilibria for arbitrarily large n such that $E(S_n \,|\, G) < \delta - 2\epsilon'$. Since the constant $\epsilon > 0$ in (58) can be chosen arbitrarily small relative to ϵ', the lower inequality in (58) implies that $nE\psi(S_n)\mu \leq \delta$; thus, in particular, the quantity $nE\psi(S_n)$ remains bounded as $n \to \infty$. Consequently, the Hoeffding bound (49) is exponentially decaying in n:

$$P(|S_n - E(S_n \,|\, G)| \geq \epsilon) \leq 2\exp\left\{-2n\epsilon^2/(4[nE\psi(S_n)]^2(\overline{u} - \underline{u})^2)\right\} \leq 2e^{-\gamma n}$$

for some constant $\gamma = \gamma(\epsilon') > 0$. Since $0 \leq E(S_n \,|\, G) < \delta - 2\epsilon'$, it follows that

$$P(-\epsilon' < S_n < \delta - \epsilon' \,|\, G) \geq 1 - 2e^{-n\gamma}.$$

Now $P(G) \to 1$, by inequality (48), so for large n we have

$$E\psi(S_n) \geq \frac{1}{2} \min_{v \in [-\epsilon', \delta - \epsilon']} \psi(v).$$

But since ψ is bounded away from 0 on the interval$[-\epsilon', \delta - \epsilon']$, this contradicts the fact that $nE\psi(S_n)$ remains bounded as $n \to \infty$. This proves the lower bound in (51).

Next, suppose that the upper bound in (51) were not true, that is, suppose that for some $\epsilon' > 0$ and indefinitely large n there were Nash equilibria for which

$$E(S_n \,|\, G) > \delta + \sqrt{2\,|\underline{u}|} + \epsilon'.$$

Then the event $S_n \leq \delta + \sqrt{2|\underline{u}|}$ would be a large deviation event in (49): in particular, if $\alpha := \epsilon'/(\delta + \sqrt{2|\underline{u}|} + \epsilon')$ then for any $\epsilon > 0$, if n is sufficiently large,

$$|S_n - E(S_n \,|\, G)| \geq \alpha E(S_n \,|\, G) \geq \alpha E\psi(S_n)(\mu - \epsilon)(1 - \epsilon),$$

by inequality (58). Consequently, by Corollary 17, there exists $\gamma = \gamma(\epsilon', \epsilon) > 0$ such that

$$P(S_n \leq \delta + \sqrt{2\,|\underline{u}|} \,|\, G) \leq e^{-\gamma n}.$$

But for all $v \geq -\sqrt{2\,|\underline{u}|}$,

$$\begin{aligned} E\psi(v + S_n) &\leq \|\psi\|_\infty P(S_n \leq \delta + \sqrt{2|\underline{u}|} \,|\, G) + \|\psi\|_\infty P(G^c) \\ &\leq e^{-\gamma n}\|\psi\|_\infty + P(G^c)\|\psi\|_\infty = O(n^{-1/2}). \end{aligned}$$

Hence, by the necessary condition (13), the function $v(u)$ must be vanishingly small (of order no greater than $O(n^{-1/2})$) for all $u \in [\underline{u}, \overline{u}]$, and so by Proposition 5 can have no discontinuities. But then the proportionality rule (41) would hold for *all* $u \in [\underline{u}, \overline{u}]$, and so the Hoeffding bound (50) for Nash equilibria with no discontinuities would give

$$P(S_n \leq \delta + \sqrt{2\,|\underline{u}|}) \leq e^{-\gamma n} \quad \Longrightarrow \quad E\psi(S_n) \leq e^{-\gamma n}\|\psi\|_\infty,$$

which, in view of the approximate proportionality rule (41), contradicts the *a priori* lower bound on Nash equilibria provided by Lemma 10. This proves that for every $\epsilon' > 0$, if n is sufficiently large then for every Nash equilibrium,

$$E(S_n \,|\, G) \leq \delta + \sqrt{2\,|\underline{u}|} + \epsilon',$$

thus establishing the upper bound in (51). A similar argument shows that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium $v(u)$ with no discontinuities,

$$\delta - \epsilon \leq E(S_n \,|\, G) \leq \delta + \epsilon.$$

Because we have now proved that $E(S_n \,|\, G)$ is bounded away from 0 and ∞, it follows as before, by the inequalities (58), that $nE\psi(S_n)$ is bounded away from 0 and ∞. Therefore, by the Hoeffding bound (49), for any $\epsilon > 0$,

$$P(|S_n - E(S_n \,|\, G)| > \epsilon \,|\, G) \leq 2\exp\left\{-2n\epsilon^2/(4[nE\psi(S_n)]^2(\overline{u} - \underline{u})^2)\right\} \leq e^{-\gamma n}$$

for some $\gamma = \gamma(\epsilon) > 0$. This proves (53). Furthermore, since $|ES_n - E(S_n \,|\, G)| = O(n^{-1/2})$, by (56), it follows that

$$P\{|S_n - ES_n| > \epsilon\} \leq P(|S_n - E(S_n \,|\, G)| > 2\epsilon \,|\, G) + P(G^c) = O(n^{-1/2}),$$

proving (52). □

6.2 Identification of the concentration point

Lemma 21. *Assume that $\mu > 0$, and let (ξ, w) be the unique solution of the Optimization Problem (cf. section 1.4.2), if one exists, or let $\xi = \delta$ if not. Then for any $\epsilon > 0$, if n is sufficiently large then in every Nash equilibrium,*

$$\left| \frac{n}{2} E\psi(S_n) - \xi\mu^{-1} \right| < \epsilon. \tag{59}$$

Proof. The lemma is equivalent to the assertion that $|ES_n - \xi| \to 0$, by the proportionality rule (41) and the estimate (48). We will prove this assertion in two steps, by first showing that for any $\epsilon > 0$ if n is sufficiently large the expectation ES_n cannot be smaller than $\xi - 2\epsilon$, and then that it cannot be larger than $\xi + 2\epsilon$.

If $\xi = \delta$ then Lemma 19 implies that $ES_n < \xi - 2\epsilon$ is impossible for large n, so to prove that $ES_n \geq \xi - 2\epsilon$ for large n it suffices to consider the case where $\xi > \delta$. In this case (ξ, w) is the unique solution to the Optimization Problem (12), so for sufficiently small $\epsilon > 0$ there is a $\varrho > 0$ such that

$$(1 - \Psi(w))|u| > (\xi - \epsilon - w)^2 \quad \text{for all } u \in [\underline{u}, \underline{u} + \varrho].$$

By Lemma 19, for any $\epsilon' > 0$, for all sufficiently large n,

$$P\{|S_n - ES_n| \geq \epsilon\} < \epsilon';$$

hence, if it were the case that $ES_n < \xi - 2\epsilon$ then a voter with value $u \in [\underline{u}, \underline{u} + \varrho]$ could, with probability in excess of $1 - \epsilon'$, improve her expected utility payoff from $\approx u$ to $\Psi(w)u$, at a cost less than $(\xi - \epsilon - w)^2$, and so all such voters would defect from the equilibrium strategy. Since $\epsilon' > 0$ can be chosen arbitrarily small relative to ϵ, this is a contradiction, because the approximate proportionality rule (Proposition 16) hold for all $u \in J_{\max}$, and $J_{\max}$ overlaps with the interval $[\underline{u}, u + \varrho]$ when n is large, by Proposition 15. This shows that for all large n, in any equilibrium, $ES_n \geq \xi - 2\epsilon$.

Now suppose that $ES_n > \xi + 2\epsilon$. Then by Lemma 19, the one-out vote total S_n would exceed $\xi + \epsilon$ with probability near 1, for large n. But by (12), for all $w \leq \delta$,

$$(\xi + \epsilon - w)^2 \geq \epsilon^2 + (\xi - w)^2 \geq \epsilon^2 + (1 - \Psi(w))|\underline{u}|,$$

so for a voter with value $u < 0$ the cost of buying enough votes to move the vote total from above $\xi + \epsilon$ to any value $w \in [-\delta, \delta]$ would exceed the increase in expected utility. Hence, for any $\alpha > 0$, if n is large then it would be suboptimal for a voter with negative value u to buy more than α votes. Since this is true, in particular, for $\alpha = \Delta/2$, where Δ is the discontinuity threshold, it follows by Proposition 5 that $v(u)$ would have no discontinuity at a value $u < 0$. Lemma 19 guarantees that there are no discontinuities at values $u \geq 0$, so it follows that $v(u)$ is continuous on the entire interval $[\underline{u}, \overline{u}]$. But then, because $ES_n \geq \xi + 2\epsilon \geq \delta + 2\epsilon$, assertion (53) implies that

$$P\{S_n \leq \delta\} < e^{-\varrho n}$$

which, since ψ has support $[-\delta, \delta]$, implies that

$$E\psi(S_n) \leq \|\psi\|_\infty e^{-\varrho n}.$$

This is impossible, because the proportionality rule (41) would then imply that for some constant $C < \infty$ not depending on n or the particular equilibrium, $\|v\|_\infty \leq Ce^{-\varrho n}$, contradicting Lemma 10. □

6.3 Proof of Theorem 2

Proof of assertion (8). The asymptotic efficiency of quadratic voting in the unbalanced case $\mu > 0$ is a direct and easy consequence of Lemma 19. This implies that for any $\epsilon > 0$, the probability that the vote total $S_N = S_n + v(U_{n+1})$ will fall below $\delta - 2\epsilon$ is less than ϵ for all large n, and so by the continuity of the payoff function Ψ, for any $\epsilon > 0$

$$P\{\Psi(S_N) \leq 1 - \epsilon\} < \epsilon$$

for all sufficiently large N and all Nash equilibria. Moreover, the law of large numbers guarantees that for large N,

$$P\left\{|N^{-1}U_+ - \mu| \geq \epsilon\right\} < \epsilon \quad \text{where} \quad U_+ := \sum_{i=1}^{N} U_i.$$

Because the random variables U_+/N and $\Psi(S_N)$ are bounded, it therefore follows that for any $\epsilon > 0$, if N is sufficiently large then in any equilibrium

$$\left|\frac{E[U\Psi(S_N)]}{2E|U|} - 1\right| < \epsilon.$$

□

Proof of assertions (9)–(10). The second assertion (10) follows immediately from the first, by the law of large numbers for the sequence $U_1, U_2, \ldots$, and the assertion (10) follows directly from the approximate proportionality rule (41) and Lemma 21.

□

6.4 Proof of Theorem 3

Assume now that $\mu > 0$ and that the Optimization Problem (12) has a unique solution (ξ, w) such that $\xi > \delta$. By Lemmas 21 and 19 (in particular, relation (52)), together with the approximate proportionality rule (41), the sum S_n of the one-out sample must concentrate near ξ. But by assertion (53) of Lemma 19, for *continuous* Nash equilibria the sum S_n must concentrate near δ. Therefore, when the Optimization Problem has a unique solution (ξ, w) with $\xi > \delta$, all Nash equilibria must have discontinuities, at least when n is large. Lemma 19 guarantees that there are no discontinuities at nonnegative values u, so by Proposition 15, the discontinuities must occur in an interval $[\underline{u}, \underline{u} + \beta n^{-3/2}]$ of length $O(n^{-3/2})$.

Let $v(u)$ be a Nash equilibrium, and let u_* be the rightmost point of discontinuity of v. By Lemma 5 and the monotonicity of Nash equilibria, $v(u) < -\Delta$ for every $u < u_*$. Obviously, the expected payoff for an agent with value u must exceed the expected payoff under the alternative strategy of buying no votes. The latter expectation is approximately $\underline{u}$, because S_n is highly concentrated near $ES_n > \xi - \epsilon$ and so $E\Psi(S_n) \approx 1$. On the other hand, the expected payoff at $u < u_*$ for an agent playing the Nash strategy is approximately

$$\Psi(\xi - v(u))\,(\underline{u}) - v(u)^2.$$

In order that this an improvement over the alternative strategy of buying no votes, for which the expected payoff is about $-|\underline{u}|$, it must be the case that

$$|v(u)| \approx \xi - w,$$

because by hypothesis, (ξ, w) is the unique pair such that relations (12) hold. This proves assertion (iv) of Theorem 3.

It remains to prove assertion (v) of Theorem 3, that the rightmost discontinuity occurs at $u_* \approx \underline{u} + \zeta n^{-2}$. To accomplish this, we will first argue that the major contribution to the expectation $E\psi(S_n)$ comes from the event $N_e = 1$ that there is precisely 1 extremist in the out-sample. Recall first that $E\psi(S_n) \sim 2\xi/(n\mu)$, by (59); thus the expectation is of order $O(n^{-1})$. Now on the event $N_e = 1$, the vote total of the $n-1$ moderates in the out-sample will concentrate near ξ, by Lemmas 21 and 19, and the single extremist will buy approximately $-\xi + w$ votes, by the argument above, so conditional on the event $N_e = 1$ the vote total S_n will be close to w, with (conditional) probability approaching 1. Consequently, as $n \to \infty$

$$E\psi(S_n)\mathbf{1}_{\{N_e=1\}} \sim \psi(w)P\{N_e = 1\} \sim \psi(w)nf(\underline{u})(u_* - \underline{u}). \tag{60}$$

It follows that $u_* - \underline{u}$ cannot be larger than $O(n^{-2})$.

Next, consider the event $N_e \geq 2$. This event has probability

$$
\begin{aligned}
P\{N_e \geq 2\} &= \sum_{k=2}^{\infty} P\{N_e = k\} \\
&\leq \sum_{k=2}^{\infty} \frac{n^k}{k!}(2f(\underline{u})(u_* - \underline{u}))^k \\
&= O((n(u_* - \underline{u}))^2) = O(n^{-3}),
\end{aligned}
$$

since $u_* - \underline{u} = O(n^{-2})$. Therefore,

$$
E\psi(S_n)\mathbf{1}_{\{N_e \geq 2\}} \leq \|\psi\|_\infty P\{N_e \geq 2\} = O(n^{-3})
$$

is of smaller order of magnitude than $E\psi(S_n)$, which is of size $O(n^{-1})$.

Lastly, consider the event $G = \{N_e = 0\}$. This event has probability converging to 1, by inequality (48), and $|ES_n - E(S_n \,|\, G)| \to 0$, by Lemma 20. Since $ES_n \to \xi$, by Lemma21, it follows that $E(S_n \,|\, G) \to \xi > \delta$. But by Corollary 17, the conditional probability that S_n differs from $E(S_n \,|\, G)$ by more than ϵ decays exponentially with n, so for some $\gamma > 0$,

$$
P(S_n < \delta \,|\, G) \leq e^{-\gamma n}
$$

provided n is sufficiently large. This implies that

$$
E\psi(S_n)\mathbf{1}_{\{N_e = 0\}} = O(e^{-\gamma n}),
$$

which is negligible compared to $E\psi(S_n)$, since this is of magnitude $O(n^{-1})$.

We have now shown that $E\psi(S_n) \sim E\psi(S_n)\mathbf{1}_{\{N_e=1\}}$ as $n \to \infty$. Since $E\psi(S_n) \sim 2\xi/(n\mu)$, by Lemma 21, it now follows by relation (60) that

$$
\psi(w) n f(\underline{u})(u_* - u) \sim \frac{2\xi}{n\mu}.
$$

Thus,

$$
n^2(u_* - \underline{u}) \longrightarrow \frac{2\xi}{n\mu\psi(w)f(\underline{u})} := \zeta.
$$

□

7 Balanced Populations: Proof of Theorem 1

7.1 Continuity of Nash equilibria

Proposition 22. *If $\mu = 0$, then for all sufficiently large values of n no Nash equilibrium $v(u)$ has a discontinuity in $[\underline{u}, \overline{u}]$. Moreover, for any $\epsilon > 0$, if n is sufficiently large every*

Nash equilibrium $v(u)$ satisfies

$$\|v\|_\infty \leq \epsilon. \tag{61}$$

Proof. The size of any discontinuity is bounded below by a positive constant Δ, by Lemma 5, so it suffices to prove the assertion (61). Fix $\epsilon > 0$, and suppose that in some Nash equilibrium there is a value $u \in [\underline{u}, \overline{u}]$ (necessarily in the extremist range $[\underline{u}, \overline{u}] \setminus J_{\max}$, by Proposition 15) such that $|v(u)| \geq \epsilon$; and hence, by the necessary condition (13),

$$E\psi(v(u) + S_n)|u| \geq 2\epsilon. \tag{62}$$

On the other hand, by Proposition 12 there exists $\gamma = \gamma(\epsilon) > 0$ such that if n is sufficiently large then any Nash equilibrium $v(u)$ satisfying $\|v\|_\infty > \epsilon$ must also satisfy $|v(u)| \leq \gamma/\sqrt{n}$ for all u not within distance ϵ of one of the endpoints $\underline{u}, \overline{u}$. Hence, the approximate proportionality relation (41) implies

$$E\psi(S_n) \leq \frac{C}{\sqrt{n}} \tag{63}$$

for a suitable $C = C(\gamma) < \infty$ independent of n. It then follows from the approximate proportionality rule (41) that the inequality $|v(u)| \leq \gamma/\sqrt{n}$ (possibly with a different constant γ) holds for all $u \in J_{\max}$.

We will show that if $\mu = 0$ then inequality (63) is impossible for large n; this will imply that the hypothesis $|v(u)| \geq \epsilon$ for some $u \in [\underline{u}, \overline{u}]$ is untenable. The strategy will be to show that the inequality $|v(u)| \leq \gamma/\sqrt{n}$, which by (62) must hold for all $u \in J_{\max}$, forces concentration of the distribution of S_n in the interval $[-\delta/2, \delta/2]$. This in turn keeps $E\psi(S_n)$ bounded away from 0, contradicting (63).

The necessary condition (13) and Taylor's theorem imply that for any $u \in [\underline{u}, \overline{u}]$,

$$2v(u) = E\psi(S_n)u + E\psi'(S_n)v(u)u + \frac{1}{2}E\psi''(S_n)v(u)^2u + \frac{1}{6}E\psi'''(S_n + v(\tilde{u}))v(u)^3u$$

for some $v(\tilde{u})$ intermediate between 0 and u. By (63), the expectation $E\psi(S_n)$ is of order no larger than $O(n^{-1/2})$; hence, by Lemma 7, there exists $C' < \infty$ such that

$$\max(E\psi(S_n), E|\psi'(S_n)|, E|\psi''(S_n)|) \leq C'/\sqrt{n}.$$

Since $|v(u)| \leq \gamma/\sqrt{n}$ for all $u \in J_{\max}$, and since ψ''' is bounded, it follows that the Taylor expansion can be rewritten as

$$2v(u) = E\psi(S_n)u + \frac{1}{2}E\psi'(S_n)E\psi(S_n)u^2 + R_n(u) \quad \text{for all } u \in J_{\max},$$

where $|R_n)(u)| \leq C'n^{-3/2}$ for a constant $C' < \infty$ independent of n and u. Now the extremist region $[\underline{u}, \overline{u}] \setminus J_{\max}$ has $F-$probability of order $n^{-3/2}$, by Proposition 15, and so the event G^c that the sample has at least one extremist has probability of order no larger than $O(n^{-1/2})$; consequently, since $\mu = EU_i = 0$, the second form of the Taylor expansion implies that

$$|2E(S_n \,|\, G) - \frac{n}{2} E\psi'(S_n) E\psi(S_n)\sigma^2| = O(n^{-1/2}) \quad \text{and} \tag{64}$$

$$|4\text{var}(S_n \,|\, G) - n(E\psi(S_n))^2\sigma^2| = O(n^{-1/2}) \tag{65}$$

Inequality (65) implies that var$(S_n \,|\, G)$ remains bounded, by (63). We will now argue that var$(S_n \,|\, G)$ must also remain bounded away from 0. Suppose not; then for indefinitely large sample sizes n there would be Nash equilibria for which var$(S_n \,|\, G)$, and hence also $n(E\psi(S_n))^2$, approach 0. But then, by Lemma 7 and relation (64), it would also be the case that $E(S_n \,|\, G)$ approaches 0, and so by Chebyshev's inequality, for every $\alpha > 0$

$$P(|S_n| \geq \alpha \,|\, G) \longrightarrow 0.$$

Since $P(G) \to 1$, this would force $E\psi(S_n) \to \psi(0) > 0$, contradicting (63).

Now recall that the Berry-Essen theorem (Corollary 18) implies that for some constant $\kappa < \infty$ independent of n,

$$|P\{S_n - E(S_n|G) \leq t\sqrt{\text{var}(S_n|G)}\} - \Phi(t)| \leq \kappa n^{-1/2} \quad \text{for all } t \in \mathbb{R}.$$

Since the standard normal distribution gives positive probability to any nonempty open interval, and since var$(S_n \,|\, G)$ and $E(S_n \,|\, G)$ remain bounded as $n \to \infty$, with var$(S_n \,|\, G)$ also bounded away from 0, it follows that

$$\liminf_{n\to\infty} P\{S_n \in [-\delta/2, \delta/2]\} > 0.$$

Since ψ is bounded away from 0 on the interval $[-\delta/2, \delta/2]$, this implies that

$$\liminf_{n\to\infty} E\psi(S_n) > 0,$$

contradicting (63).

□

Corollary 23. *If $\mu = 0$ then for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium $v(u)$ the first and second derivatives satisfy*

$$1 - \epsilon \leq \frac{2v'(u)}{E\psi(S_n)} \leq 1 + \epsilon \quad \textit{and} \quad |v''(u)| \leq \epsilon E\psi(S_n). \tag{66}$$

Proof. This follows from the approximate proportionality rule (41) and the formulas (18)–(19) for the derivatives v', v''. The key is that for every $\epsilon > 0$, if n is large then every Nash equilibrium satisfies $\|v\|_\infty < \epsilon$, by Proposition 22; together with equation (41), this implies that $0 < E\psi(S_n) < \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, it follows from Lemma 7 that $E|\psi'(S_n)| < \epsilon$.

Using formula (18) for $v'(u)$ and Taylor's theorem (twice), we obtain

$$2v'(u) = E\psi(S_n) + E\psi'(S_n)v(u) + E\psi'(S_n)v'(u)u \\ + \frac{1}{2}E\psi''(S_n + v(\tilde{u}))v(u)^2 + E\psi''(S_n + v(\hat{u}))v'(u)v(u)u$$

where $\tilde{u}$ and $\hat{u}$ are intermediate between 0 and u. Since $E\psi(S_n)$, $E|\psi'(S_n)|$, and $\|v\|_\infty$ are all smaller than ϵ, where $\epsilon >$ can be taken arbitrarily small, and since $|v(u)| \leq \frac{1}{2}E\psi(S_n)\max(|\underline{u}|, \overline{u})$, all terms in the above equation for $2v'(u)$ are small compared to the term $E\psi(S_n)$. This proves the first set of inequalities in (66).

The inequality for the second derivative in (66) can be proved in similar fashion. Use formula (19) together with Taylor's theorem to write

$$2v''(u) = E\psi'(S_n)(2v'(u) + v''(u)u) + E\psi''(S_n)v'(u)^2u \\ + E\psi''(S_n + v(\tilde{u}))v(u)(2v'(u) + v''(u)u) + E\psi'''(S_n + v(\hat{u}))v(u)v'(u)^2u$$

where $\tilde{u}$ and $\hat{u}$ are intermediate between 0 and u. As above, for large n the quantities $E\psi(S_n)$, $E|\psi'(S_n)|$, and $\|v\|_\infty$ are all smaller than ϵ, where $\epsilon >$ can be taken arbitrarily small. By the result of the preceding paragraph, $2|v'(u)| \leq 2(1+\epsilon)E\psi(S_n)$ for large n. Consequently, all terms in the displayed equation for $2v''(u)$ are small compared to the first, and so

$$2v''(u) = E\psi'(S_n)(2v'(u))(1 + o(1)).$$

This proves that $|v''(u)|$ is small compared to $E|\psi'(S_n)|$ when n is large. $\square$

Because $\|v\|_\infty$ is small for any Nash equilibrium v, the distribution of the vote total S_n cannot be too highly concentrated. This in turn implies the proportionality constant $E\psi(S_n)$ in (41) cannot be too small.

Lemma 24. *For any $C < \infty$ there exists $n_C < \infty$ exists such that for all $n \geq n_C$ and every Nash equilibrium,*

$$nE\psi(S_n) \geq C. \tag{67}$$

Proof. Since $\mu = 0$, the approximate proportionality rule (41) and the necessary condition (13) imply that for any $\epsilon > 0$ and all sufficiently large n,

$$|ES_n| \leq n\epsilon E\psi(S_n)E|U|.$$

Thus, by Hoeffding's inequality (Corollary 17), if $nE\psi(S_n) < C$ then the distribution of S_n must be highly concentrated in a neighborhood of 0. But if this were so we would have, for all large n,

$$E\psi(S_n) \approx \psi(0) > 0,$$

which is a contradiction. □

7.2 Edgeworth expansions

For the analysis of the case $\mu = 0$ refined estimates of the errors in the approximate proportionality rule (41) will be necessary. We derive these from the Edgeworth expansion for the density of a sum of independent, identically distributed random variables (cf. [3], Ch. XVI, sec. 2, Th. 2). The relevant summands here are the random variables $v(U_i)$, and because the function $v(u)$ depends on the particular Nash equilibrium (and hence also on n), we must employ a version of the Edgeworth expansion in which the error is precisely quantified. The following variant of Feller's Theorem 2 (which can be proved in the same manner as in [3]) will suffice for our purposes.

Proposition 25. *Let $Y_1, Y_2, \dots, Y_n$ be independent, identically distributed random variables with mean $EY_1 = 0$, variance $EY_1^2 = 1$, and finite $2r$th moment $E|Y_1|^{2r} = \mu_{2r} \leq m_{2r}$. Assume the distribution of Y_1 has a density $f_1(y)$ whose Fourier transform $\hat{f}_1$ satisfies $|\hat{f}_1(\theta)| \leq g(\theta)$, where g is a C^{2r} function such that $g \in L^\nu$ for some $\nu \geq 1$ and such that for every $\epsilon > 0$,*

$$\sup_{|\theta| \geq \epsilon} g(\theta) < 1. \tag{68}$$

Then for some sequence $\epsilon_n \to 0$ depending only on m_{2r} and on the function g, the density $f_n(y)$ of $\sum_{i=1}^n Y_i/\sqrt{n}$ satisfies

$$\left| f_n(x) - \frac{e^{-x^2/2}}{\sqrt{2\pi n}} \left(1 + \sum_{k=3}^{2r} n^{-(k-2)/2} P_k(x) \right) \right| \leq \frac{\epsilon_n}{n^{r-1}} \tag{69}$$

for all $x \in \mathbb{R}$, where $P_k(x) = C_k H_k(x)$ is a multiple of the kth Hermite polynomial $H_k(x)$, and C_k is a continuous function of the moments $\mu_3, \mu_4, \dots, \mu_k$ of Y_1.

The following lemma ensures that in any Nash equilibrium the sums $S_n = \sum_{i=1}^n v(U_i)$, after suitable renormalization, meet the requirements of Proposition 25.

Lemma 26. *There exist constants $0 < \sigma_1 < \sigma_2 < m_{2r} < \infty$ and a function $g(\theta)$ satisfying the hypotheses of Proposition 25 (with $r = 4$) such that for all sufficiently large n and any Nash equilibrium $v(u)$ the following statement holds. If $w(u) = 2v(u)/E\psi(S_n)$*

(a) $\sigma_1^2 < \text{var}(w(U_i)) < \sigma_2^2$*;*
(b) $E|w(U_i) - Ew(U_i)|^{2r} \leq m_{2r}$*; and*
(c) the random variables $w(U_i)$ *have density* $f_W(w)$ *whose Fourier transform is bounded in absolute value by* g*.*

Proof. These statements are consequences of the proportionality relations (41) and the smoothness of Nash equilibria. By Proposition 22, Nash equilibria are continuous on $[\underline{u}, \overline{u}]$ and for large n satisfy $\|v\|_\infty < \epsilon$, where $\epsilon > 0$ is any small constant. Consequently, by Proposition 16, the proportionality relations (41) hold on the entire interval $[\underline{u}, \overline{u}]$. Because $EU_1 = 0$, it follows that for any $\epsilon > 0$, if n is sufficiently large then $|Ew(U_i)| < \epsilon$, and so assertions (a)–(b) follow routinely from (41).

The existence of the density $f_W(w)$ follows from the smoothness of Nash equilibria and smoothness of the sampling density f, by standard change-of-variables rules of calculus. Any Nash equilibrium $v(u)$ is continuous on the entire interval $[\underline{u}, \overline{u}]$; hence, by Proposition 15, $v(u)$ is C^∞ on $(\underline{u}, \overline{u})$. Thus, if U is a random variable with density $f(u)$ then the random variable $W := 2v(U)/E\psi(S_n)$ has density

$$f_W(w) = f(u)E\psi(S_n)/(2v'(u)), \quad \text{where } w = 2v(u)/E\psi(S_n), \tag{70}$$

at every point w such that $v'(u) \neq 0$. It follows by Corollary 23 that for any $\epsilon > 0$, if n is sufficiently large then

$$(1-\epsilon)f(u) \leq f_W(w) \leq (1+\epsilon)f(u) \quad \text{where } w = 2v(u)/E\psi(S_n).$$

Furthermore, since the functions $f(u)$, $v(u)$, and ψ are all C^∞ with compact support, equation (70) implies that the density $f_W(w)$ is continuously differentiable, with derivative

$$f_W'(w) = \frac{f'(u)(E\psi(S_n))^2}{4v'(u)^2} - \frac{f(u)(E\psi(S_n))^2 v''(u)}{4v'(u)^3}.$$

By Corollary 23 implies that the ratio $(E\psi(S_n))^2/4v'(u)^2$ is bounded above and below by $1 \pm \epsilon$ when n is large, and also that $|v''(u)|$ is small compared to $|v'(u)|$, so it follows that for large n the ratio $|f_W'(w)|/|f'(u)|$ is uniformly close to 1. Thus, in particular,

$$|f_W'(w)| \leq \kappa \tag{71}$$

where $\kappa < \infty$ is a constant that does not depend on either n or on the choice of Nash equilibrium.

The last step is to prove the existence of a dominating function $g(\theta)$ for the Fourier transform of f_W. This will rely on the differentiablility of the density $f_W(w)$ and the inequality (71). We will analyze the Fourier transform in three regions: (i) for values $|\theta| \leq \gamma$, where $\gamma > 0$ is a small fixed constant; (ii) for values $|\theta| \geq K$, where K is a large but fixed constant; and (iii) for $\gamma < |\theta| < K$. Region (i) is easily dealt with, in view of the bounds (a)–(b) on the second and third moments and the estimate $|Ew(U)| < \epsilon'$, as these together with Taylor's theorem imply that for all $|\theta| < 1$,

$$|\hat{f}_W(\theta) - (1 + i\theta Ew(U) - \theta^2 \text{var}(w(U))/2| \leq m_3|\theta|^3.$$

Next consider region (ii), where $|\theta|$ is large. Integration by parts shows that

$$\hat{f}_W(\theta) = \int_{w(\underline{u})}^{w(\overline{u})} f_W(w) e^{i\theta w}\, dw = -\int_{w(\underline{u})}^{w(\overline{u})} \frac{e^{i\theta w}}{i\theta} f'_W(w)\, dw + \frac{e^{i\theta w}}{i\theta} f_W(w)\Big|_{w\underline{u}}^{w(\overline{u})};$$

because $f_W(w)$ is uniformly bounded at $w(\underline{u})$ and $w(\overline{u})$, by (**??**) and (70), and because $|f'_W(w)| \leq \kappa$, by (71), it follows that there is a constant $C < \infty$ such that for all sufficiently large n and all Nash equilibria,

$$|\hat{f}_W(\theta)| \leq C/|\theta| \quad \forall\, \theta \neq 0.$$

Thus, setting $g(\theta) = C/|\theta|$ for all $|\theta| \geq 2C$, we have a uniform bound for the Fourier transforms $\hat{f}_W(\theta)$ in the region (ii).

Finally, to bound $|\hat{f}_W(\theta)|$ in the region (iii) of intermediate $\theta-$values, we use the proportionality rule once again in the form $|w(u) - u| < \epsilon$, valid for all $u \in [\underline{u}, \overline{u}]$. This implies

$$\begin{aligned}
\hat{f}_W(\theta) &= \int_{\underline{u}}^{\overline{u}} e^{i\theta w(u)} f(u)\, du \\
&= \int_{\underline{u}}^{\overline{u}} e^{i\theta u} f(u)\, du + \int_{\underline{u}}^{\overline{u}} (e^{i\theta w(u)} - e^{i\theta u}) f(u)\, du \\
&= \hat{f}_U(\theta) + R(\theta)
\end{aligned}$$

where $|R(\theta)| < \epsilon'$ uniformly for $|\theta| \leq C$ and $\epsilon' \to 0$ as $\epsilon \to 0$. Because $\hat{f}_U$ is the Fourier transform of an absolutely continuous probability density, its absolute value is bounded away from 1 on the complement of $[-\gamma, \gamma]$, for any $\gamma > 0$. Since $\epsilon > 0$ can be made arbitrarily small (cf. Proposition 16), it follows that a continuous, positive function $g(\theta)$ that is bounded away from 1 on $|\theta| \in [\gamma, C]$ exists such that $|\hat{f}_W)\theta| \leq g(\theta)$ for all $|\theta| \in [\gamma, C]$. The extension of g to the whole real line can now be done by smoothly interpolating at the boundaries of regions (i), (ii), and (iii). $\square$

7.3 Proof of Theorem 1

Because the function ψ is smooth and has compact support, differentiation under the expectation in the necessary condition $2v(u) = E\psi(v(u) + S_n)u$ is permissible, and so for every $u \in [-\underline{u}, \overline{u}]$ a $\tilde{v}(u)$ exists intermediate between 0 and $v(u)$ such that

$$2v(u) = E\psi(S_n)u + E\psi'(S_n + \tilde{v}(u))v(u)u. \tag{72}$$

The proof of Theorem 1 will hinge on the use of the Edgeworth expansion (Proposition 25) to approximate each of the two expectations in (72) precisely.

As in Lemma 26, let $w(u) = 2v(u)/E\psi(S_n)$. We have already observed, in the proof of Lemma 26, that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium, $|Ew(U)| < \epsilon$. It therefore follows from the proportionality rule that

$$\left| \frac{4\,\mathrm{var}(v(U))}{(E\psi(S_n))^2\sigma_U^2} - 1 \right| \leq \epsilon \quad \text{and} \quad \left| \frac{E|v(u) - Ev(u)|^k}{(E\psi(S_n))^k E|U|^k} \right| < \epsilon \;\; \forall\, k \leq 8. \tag{73}$$

Moreover, Lemma 26 and Proposition 25 imply the distribution of S_n has a density with an Edgeworth expansion, and so for any continuous function $\varphi : [-\delta, \delta] \to \mathbb{R}$,

$$E\varphi(S_n) = \int_{-\delta}^{\delta} \varphi(x) \frac{e^{-y^2/2}}{\sqrt{2\pi n}\sigma_V} \left(1 + \sum_{k=3}^{m} n^{-(k-2)/2} P_k(y) \right) dx + r_n(\varphi) \tag{74}$$

where

$$\begin{aligned} \sigma_V^2 &:= \mathrm{var}(v(U)), \\ y &= y(x) = (x - ES_n)/\sqrt{\mathrm{var}(S_n)}, \end{aligned}$$

and $P_k(y) = C_k H_3(y)$ is a multiple of the kth Hermite polynomial. The constants C_k depend only on the first k moments of $w(U)$, and consequently are uniformly bounded by constants C_k' not depending on n or on the choice of Nash equilibrium. The error term $r_n(\varphi)$ satisfies

$$|r_n(\varphi)| \leq \frac{\epsilon_n}{n^{(m-2)/2}} \int_{-\delta}^{\delta} \frac{|\varphi(x)|}{\sqrt{2\pi \mathrm{var}(S_n)}}\, dx. \tag{75}$$

In the special case $\varphi = \psi$, (74) and the remainder estimate (75) (with $m = 4$) imply

$$E\psi(S_n) \leq \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x)\, dx + o(n^{-1}\sigma_V^{-1}).$$

Because $4\,\sigma_V^2 \approx (E\psi(S_n))^2\sigma_U^2$ for large n, this implies that for a suitable constant $\kappa < \infty$,

$$E\psi(S_n) \leq \frac{\kappa}{\sqrt[4]{n}}. \tag{76}$$

Claim 27. *There exist constants $\alpha_n \to \infty$ such that in every Nash equilibrium,*

$$|ES_n| \leq \alpha_n^{-1}\sqrt{\mathrm{var}(S_n)} \quad \textit{and} \tag{77}$$

$$\mathrm{var}(S_n) \geq \alpha_n^2. \tag{78}$$

Proof of Theorem 1: Conclusion. Before we begin the proof of the claim, we indicate how it will imply Theorem 1. If (77) and (78) hold, then for every $x \in [-\delta, \delta]$,

$$|y(x)| \leq (1 + 2\delta)/\alpha_n \to 0.$$

Consequently, the dominant term in the Edgeworth expansion (74) for $\varphi = \psi$ (with $m = 4$), is the first, and so for any $\epsilon > 0$, if n is sufficiently large,

$$E\psi(S_n) = \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x)\, dx(1 \pm \epsilon).$$

(Here the notation $(1 \pm \epsilon)$ means the ratio of the two sides is bounded above and below by $(1 \pm \epsilon)$.) Thus $4\,\sigma_V^2 \approx (E\psi(S_n))^2 \sigma_U^2$ implies

$$\sqrt{\pi n/2}\sigma_U (E\psi(S_n))^2 = \int_{-\delta}^{\delta} \psi(x)\, dx(1 \pm \epsilon) = 2 \pm 2\epsilon,$$

proving the assertion (6). □

Proof of Claim 27. First we deal with the remainder term $r_n(\varphi)$ in the Edgeworth expansion (74). By Lemma 24, the expectation $E\psi(S_n)$ is at least C/n for large n, and so by (73) the variance of S_n must be at least C'/n. Consequently, by (75), the remainder term $r_n(\varphi)$ in (74) satisfies

$$|r_n(\varphi)| \leq C'' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-2)/2} \sqrt{\mathrm{var}(S_n)}} \leq C''' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-3)/2}}.$$

Suitable choice of m will make this bound small compared to any desired monomial n^{-A}, and so we may ignore the remainder term in the arguments to follow.

Suppose there were a constant $C < \infty$ such that along some sequence $n \to \infty$ Nash equilibria existed satisfying $\mathrm{var}(S_n) \leq C$. By (73), this would force $C/n \leq E\psi(S_n) \leq C'/\sqrt{n}$, which in turn would imply that

$$C''\mathrm{var}(S_n) \log n \geq |ES_n|^2 \geq C'''\mathrm{var}(S_n) \log n, \tag{79}$$

because otherwise the dominant term in the Edgeworth series for $E\psi(S_n)$ would be either too large or too small asymptotically (along the sequence $n \to \infty$) to match the requirement that $C/n \leq E\psi(S_n) \leq C'/\sqrt{n}$. (Observe that because the ratio $|ES_n|^2/\mathrm{var}(S_n)$ is bounded above by $C'' \log n$, the terms $e^{-y^2/2}P_k(y)$ in the integral (74) are of size at most $(\log n)^A$ for some A depending on m, and so the first term in the Edgeworth series is dominant.) We will show that (79) leads to a contradiction.

Suppose $ES_n > 0$ (the case $ES_n < 0$ is similar). The Taylor expansion (72) for $v(u)$ and the hypothesis $EU = 0$ implies

$$2Ev(U) = E\psi'(S_n + \tilde{v}(U))v(U)U. \tag{80}$$

The Edgeworth expansion (74) for $E\psi'(S_n + \tilde{v}(u))$ together with the independence of S_n and U and the inequalities (79), implies that for any $\epsilon > 0$, if n is sufficiently large then

$$E\psi'(S_n + \tilde{v}(u)) \\ = \frac{1}{\sqrt{2\pi \mathrm{var}(S_n)}} \int_{-\delta}^{\delta} \psi'(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2 / 2\mathrm{var}(S_n)\}\, dx (1 \pm \epsilon). \quad (81)$$

Now because ψ and ψ' have support $[-\delta, \delta]$, integration by parts yields

$$\int_{-\delta}^{\delta} \psi'(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2 / 2\mathrm{var}(S_n)\}\, dx \\ = \int_{-\delta}^{\delta} \psi(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2 / 2\mathrm{var}(S_n)\} \frac{x + \tilde{v}(u) - ES_n}{\mathrm{var}(S_n)}\, dx, \quad (82)$$

and because $x + \tilde{v}(u)$ is of smaller order of magnitude than ES_n, it follows that for large n

$$E\psi'(S_n + \tilde{v}(u)) = -\frac{ES_n}{\mathrm{var}(S_n)} E\psi(S_n)(1 \pm \epsilon). \quad (83)$$

But it now follows from the Taylor series for $2Ev(U_i)$ (by summing over i) that

$$2ES_n = -n\frac{ES_n}{\mathrm{var}(S_n)} E\psi(S_n) Ev(U) U (1 \pm \epsilon), \quad (84)$$

which is a contradiction, because the right side is negative and the left side positive. This proves the assertion (78).

The proof of inequality (77) is similar. Suppose for some $C > 0$ Nash equilibria existed along a sequence $n \to \infty$ for which $ES_n \geq C\sqrt{\mathrm{var}(S_n)}$. In view of (78), this hypothesis implies in particular that $ES_n \to \infty$, and also that $|y(x)| \geq C/2$ for all $x \in [-\delta, \delta]$. Thus, the Edgeworth approximation (81) remains valid, as does the integration by parts identity (82). Because $ES_n \to \infty$, the terms $x + \tilde{v}(u)$ are of smaller order of magnitude that ES_n, and so once again (83) and therefore (84) follow. Again we have a contradiction, because the right side of (84) is negative while the left side diverges to $+\infty$.

□

A Strict Monotonicity of Nash Equilibria

Lemma 28. *If $v(u)$ is a Nash equilibrium then $v(u) \neq 0$ for all $u \neq 0$.*

Proof. Since any Nash equilibrium $v(u)$ is a nondecreasing function of u (by Proposition 1), if $v(u) = 0$ for some $u > 0$ then $v(u') = 0$ for all $u' \in (0, u)$. Because the density $f(u)$ of the value distribution F is strictly positive on $[\underline{u}, \overline{u}]$, it follows that the probability p that every agent in the sample casts vote $V_i = 0$ is strictly positive. But then an agent with utility u could improve her expectation by buying $\varepsilon > 0$ votes, where $\varepsilon \ll u\psi(0)p$, because the expected utility gain would be at least

$$u\Psi(\varepsilon)p \sim u\psi(0)p\varepsilon$$

at a cost of ε^2. Because by hypothesis $\psi(0) > 0$, the expected utility gain would overwhelm the increased vote cost for small $\varepsilon > 0$. □

Corollary 29. *Any Nash equilibrium $v(u)$ is* strictly *increasing on $[\underline{u}, \overline{u}]$.*

Proof. Proposition 1 implies that $v(u)$ is nondecreasing in u, so it suffices to show that $v(\cdot)$ takes distinct values at distinct arguments $u_1 \neq u_2$. By Lemma 28, $v(u) > 0$ for $u > 0$ and $v(u) < 0$ for $u < 0$; hence, by the necessary condition (13), for every $u \neq 0$,

$$E\psi(S_n + v(u)) > 0.$$

Let $u_1 \neq u_2$ be two distinct nonzero values. By the necessary condition (13),

$$\begin{aligned} &E\psi(S_n + v(u_1))u_1 = v(u_1) \quad \text{and} \\ &E\psi(S_n + v(u_2))u_2 = v(u_2); \end{aligned}$$

since $E\psi(S_n + v(u_1)) \neq 0$, it follows that

$$E\psi(S_n + v(u_1))u_2 \neq v(u_1) \quad \Longrightarrow \quad v(u_2) \neq v(u_1).$$

Thus, the function $v(u)$ takes distinct values at distinct arguments u. By Lemma 1, $v(u)$ is nondecreasing in u; therefore, $v(u)$ is strictly increasing. □

B Necessary Condition for a Nash Equilibrium

Proof of Proposition 4. For an agent with value $u > 0$ a best response $v = v(u)$ must maximize expected utility minus vote cost (3), and so for every $\Delta > 0$,

$$\begin{aligned} &E\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\}\, u \leq 2\Delta v + \Delta^2 \quad \text{and} \\ &E\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\}\, u \leq -2\Delta v + \Delta^2. \end{aligned}$$

Dividing by Δ and letting $\Delta \to 0$ yields

$$2v \geq \limsup_{\Delta \to 0+} \frac{1}{\Delta}(E\Psi(S_n + v + \Delta) - E\Psi(S_n + v)) \quad \text{and}$$
$$-2v \geq \limsup_{\Delta \to 0+} \frac{1}{\Delta}(E\Psi(S_n + v - \Delta) - E\Psi(S_n + v)).$$

Because Ψ is continuously differentiable with compactly supported derivative ψ, the limsups can be taken under the expectations, where they become limits, and so

$$2v = E\psi(v + S_n).$$

A similar argument applies to $u < 0$. $\square$

C Hoeffding's Inequality

Hoeffding's inequality [7] is a substantial sharpening of the Chebyshev bound for sums of *bounded* independent random variables. Following are two variants of the inequality adapted to the needs of this paper; the second shows that for sums of bounded, i.i.d. random variables the probabilities of "large deviations" are exponentially decaying in the number of summands.

Hoeffding's Inequality . *Let $X_1, X_2, \cdots, X_n$ be independent random variables satisfying $a \leq X_i \leq b$, where $a < b$ are two real constants. Then for any real $t > 0$,*

$$P\{|S_n - ES_n| \geq t\} \leq 2e^{-2t^2/n(b-a)^2} \quad \Longrightarrow \tag{85}$$
$$P\{|S_n - ES_n| \geq nt\} \leq 2e^{-2nt^2/(b-a)^2} \tag{86}$$

D The Berry-Esseen Theorem

The Berry-Esseen theorem provides sharp bounds on the error in the central limit theorem. For the proof and further discussion, see [3], section XVI. 5.

Berry-Esseen Theorem . *Let $Y_1, Y_2, \cdots$ be independent, identically distributed random variables such that*

$$EY_1 = 0, \quad EY_1^2 = \sigma^2 > 0, \quad \text{and } E|Y_1|^3 = \gamma < \infty. \tag{87}$$

Then for all $x \in \mathbb{R}$ and all inegers $n \geq 1$,

$$\left| P\left\{ \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} Y_i \leq x \right\} - \Phi(x) \right| \leq \frac{3\gamma}{\sigma^3\sqrt{n}}, \tag{88}$$

where Φ denotes the standard normal cumulative distribution function.

Proof of Proposition 13. First, we claim that it suffices to prove the inequality (33) for intervals of length α. To see this, observe that any interval J of length larger than α is contained in an interval J' whose length $|J'| = n\alpha$ is an integer multiple of α satisfying $n\alpha \leq 2|J|$. Clearly, the probability that $S_n^Y := \sum_{i=1}^n Y_i$ falls in J is no larger than the probability that it falls in J', and by the Bonferroni inequality this probability is no larger than n times the maximal probability over all intervals of length α; thus, if the inequality holds for intervals of length α then it will hold (with ϵ replaced by 2ϵ) for all intervals of length $\geq \alpha$.

Second, we can assume, without loss of generality, that $EY_1 = 0$ and $\alpha = 1$, because this can always be accomplished by translation and rescaling. Thus, we must show that for any $\epsilon > 0$ and $C < \infty$ there exist $C' = C'(\epsilon, C)$ and $n' = n'(\epsilon, C)$ such that if $n \geq n'$ and if the summands Y_i are i.i.d. and satisfy the moment constraints (32), then for every real number a,

$$P\left\{S_n^Y \in [a, a+1]\right\} \leq \epsilon.$$

Let $\sigma^2 = EY_i^2$ be the variance and $\gamma = E|Y_i|^3$ the third absolute moment of the summands. By hypothesis, $\gamma \leq C\sigma^3$; consequently, by the Berry-Esseen theorem,

$$P\left\{S_n^Y \in [a, a+1]\right\} \leq \Phi(\frac{a+1}{\sigma\sqrt{n}}) - \Phi(\frac{a}{\sigma\sqrt{n}}) + 6C/\sqrt{n}.$$

If the variance satisfies $\sigma^2 \geq C'n$ then the interval $[a/\sigma\sqrt{n}, (a+1)/\sigma\sqrt{n}]$ has length $(\sigma\sqrt{n})^{-1}$ bounded above by $1/\sqrt{C'}$; thus, if C' is chosen large enough that $1/\sqrt{2\pi C'} < \epsilon/2$ then for every $a \in \mathbb{R}$,

$$\Phi(\frac{a+1}{\sigma\sqrt{n}}) - \Phi(\frac{a}{\sigma\sqrt{n}}) = \frac{1}{\sqrt{2\pi}} \int_{\frac{a}{\sigma\sqrt{n}}}^{\frac{a+1}{\sigma\sqrt{n}}} e^{-t^2/2}\, dt \leq \epsilon/2.$$

Finally, if n' is chosen so large that $6C/\sqrt{n'} < \epsilon/2$, then for all $n \geq n'$ we will have

$$P\left\{S_n^Y \in [a, a+1]\right\} \leq \Phi(\frac{a+1}{\sigma\sqrt{n}}) - \Phi(\frac{a}{\sigma\sqrt{n}}) + 6C/\sqrt{n} \leq \epsilon.$$

□

E Proof of Proposition 2

We shall assume throughout that $\delta < 1/\sqrt{2}$, and that the function Ψ and its derivatives satisfy the model assumptions (M1)–(M4) of section 1.1. Thus, $\psi/2$ is an even,

C^∞ probability density with support $[-\delta, \delta]$; it has positive derivative ψ' on $(-\delta, 0)$ (and hence negative derivative on $(0, \delta)$); and there is a single point ι of inflection in the interval $(-\delta, 0)$ such that ψ' is strictly increasing in $[-\delta, \iota]$ and strictly decreasing in $[\iota, 0]$

Define

$$H(\alpha, w) = (1 - \Psi(w))|\underline{u}| - (\alpha - w)^2. \tag{89}$$

Proposition 2 asserts that, under the assumption $\delta < 1/\sqrt{2}$, there is a unique value $\xi > \delta$ such that (i) the maximum value of the function $w \mapsto H(\xi, w)$ for $w \in [-\delta, \delta]$ is 0, and (ii) this maximum is attained at at a unique point $w \in (-\delta, \delta)$. The next lemma establishes the uniqueness of the value ξ; Lemma 31 below will show the uniqueness of the maximizer w.

Lemma 30. *There is a unique $\xi > \delta$ such that $\max_{w \in [-\delta, \delta]} H(\xi, w) = 0$. Moreover,*

(U1) if $\alpha > \xi$ then $\max_{w \in [-\delta, \delta]} H(\alpha, w) < 0$; and
(U2) if $\alpha < \xi$ then $\max_{w \in [-\delta, \delta]} H(\alpha, w) > 0$.

Proof. For each fixed $w \in [-\delta, \delta]$ the function $\alpha \mapsto H(\alpha, w)$ is strictly decreasing in the interval $\alpha \in [\delta, \infty)$, because its derivative $-2(\alpha - w)$ is negative throughout this interval. Hence, since H is jointly continuous in its arguments and since the interval $[-\delta, \delta]$ is compact, the function

$$h(\alpha) := \max_{w \in [-\delta, \delta]} H(\alpha, w)$$

is continuous and strictly decreasing for $\alpha \in [\delta, \infty)$. Thus, to complete the proof it suffices to show that there exists $\xi \in [\delta, \infty)$ such that $h(\xi) = 0$.

Clearly, $\lim_{\alpha \to \infty} h(\alpha) = -\infty$, because to first order the maximum value of $H(\alpha, w)$ over $w \in [-\delta, \delta]$ is controlled by the quadratic term in (89), so by the Intermediate Value Theorem of calculus it is enough to show that $h(\delta) > 0$. This is where the hypothesis that $\delta < 1/\sqrt{2}$ comes in, as it implies that $(\alpha - (-\delta))^2 = 4\delta^2 < 2$ when $\alpha = \delta$. By the standing model assumptions (cf. sec. 1.1), $|\underline{u}| \geq 1$, so

$$(1 - \Psi(-\delta))|\underline{u}| = 2|\underline{u}| \geq 2 \quad \Longrightarrow H(\delta, -\delta) > 0.$$

□

Lemma 31. *Under the standing hypotheses (M1)–(M4) of section 1.1, there is a unique point $w_* \in (-\delta, \delta)$ where the function $w \mapsto H(\xi, w)$ attains the value 0.*

Proof. Local minima and maxima of the function $w \mapsto H(\xi, w)$ must be critical points, that is, points w where the first partial derivative $\partial H / \partial w$ vanishes. We will argue that there are either two or three critical points in the interval $(-\delta, \delta)$, and that one of these is the unique point in $[-\delta, \delta]$ where $w \mapsto H(\xi, w)$ achieves its maximum value.

The first and second partial derivatives of $H(\xi, w)$ with respect to w are

$$\frac{\partial H}{\partial w} = -\psi(w)|\underline{u}| + 2(\xi - w) \quad \text{and} \quad \frac{\partial^2 H}{\partial w^2} = -\psi'(w)|\underline{u}| - 2.$$

By hypothesis, the function Ψ is odd, and hence so is its second derivative ψ'. Moreover, ψ' is positive in $(-\delta, 0)$ and consequently negative in $(0, \delta)$. By assumption (M4), there is a unique $\iota \in (-\delta, 0)$ such that ψ' is strictly increasing on $[-\delta, \iota]$ and strictly decreasing on $[\iota, 0]$, and so ψ' is strictly decreasing on $[0, -\iota]$ and strictly increasing on $[\iota, \delta]$. Therefore, the function $\partial^2 H/\partial w^2$ has at most two zeros in $[-\delta, \delta]$, both in the interval $(0, \delta)$. It now follows by a sign change argument that the first partial $\partial H/\partial w$ has at most three zeros in $[-\delta, \delta]$; these must be separated by zeros of $\partial^2 H/\partial w^2$.

According to our standing assumptions, the reward function Ψ is identically 1 in the interval $[\delta, \infty)$. Hence, since $\xi > \delta$, the function $w \mapsto H(\xi, w)$ is *negative* in the interval $[\delta, \xi)$. Define $w_* \in (-\delta, \delta)$ by

$$w_* := \max \{w \in [-\delta, \delta] \, : \, H(\xi, w) = 0\} \, ;$$

Lemma 30 ensures that w_* is well-defined, and that w_* is a critical point. To complete the proof, we must show that there are no other points $w \neq w_*$ where $H(\xi, w) = 0$.

Since $H(\xi, w) = 0$ at the endpoints $w = w_*$ and $w = \xi$ and $H(\xi, w) < 0$ for $w_* < w < \xi$, it follows that the function $w \mapsto H(\xi, w)$ attains a minimum value at some point $w_+ \in (w_*, \xi)$; this must also be a critical point. This accounts for two of the (at most) three critical points. Now suppose that there were a second point $w_{**} < w_*$ in the interval $(-\delta, \delta)$ where $H(\xi, w_{**}) = 0$. Since this point w_{**} would be a local maximum of $w \mapsto H(\xi, w)$, it would necessarily be a third critical point. But because $H(\xi, w) < 0$ for all $w \in (w_{**}, w_*)$, there would be at least one local minimum of $w \mapsto H(\xi, w)$ in the interval (w_{**}, w_*); this would be a fourth critical point, contradicting the fact that there are at most three.

□

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