RUELLE'S PERRON-FROBENIUS THEOREM AND THE CENTRAL LIMIT THEOREM

FOR ADDITIVE FUNCTIONALS OF ONE-DIMENSIONAL GIBBS STATES

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A central limit theorem for a class of stationary sequences is given. The proof is based on the spectral analysis of an associated Perron-Frobenius type operator.

1. Introduction.

The central limit problem for sums of stationary sequences has such a long history (dating back to Markov and Bernstein, at least) that one might have hoped the final chapter had by now been written. Alas, this is not the case.

It is now known that the central limit theorem will generally hold for a stationary sequence when there is a certain degree of "mixing" present. A variety of general results specifying sufficient conditions for the validity of the central limit theorem are presented in the treatise by Ibragimov and Linnik (1971), and further references to the literature are made there. So far as I have been able to discern, there are four methods for obtaining such results: (i) Markov's method of moments; (ii) Bernstein's method of approximation by i.i.d. sequences; (iii) Doeblin's method of finding "regeneration times"; and (iv) a martingale method which is apparently due to Gordin (1969). Although these methods are very powerful, enabling one to work with very general

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processes, all suffer form the disadvantage of being ill-suited to making more precise asymptotic estimates than those in the central limit theorem.

The purpose of this paper is to present another approach to the central limit theorem which gives refined estimates of a local character (including estimates of large deviation probabilities). This method is based on spectral analysis, the main tools being an infinite-dimensional Perron-Frobenius type theorem of Ruelle and an extension due to Pollicott. Unfortunately it is much more limited in scope than the Markov and Bernstein methods; but it does include under its aegis functionals on many of the concrete dynamical systems of interest in ergodic theory. Examples are sequences of the form $\{f(T^n x)\}_{n>0}$, where $f: M \to R$ is a "smooth" function on a compact space M and $T: M \to M$ is a transformation which admits a "Markov partition", such as an ergodic endomorphism of a compact abelian group, an expanding map of the unit interval, and certain Axiom A diffeomorphisms. The reduction of such systems to the study of one-dimensional Gibbs states is carried out in Bowen (1974). (The simplest such system is the sequence $\{f(2^n x)\}_{n>0}$, where f is a smooth periodic function on R of period 1. This system alone has received a lot of attention: cf. Kac (1949).

The basic idea in the approach is very simple (although some of the technical details of the Fourier analysis are rather-technical). It is Kac's idea that additive functionals of a process may be studied by means of an associated "Feynman-Kac" semigroup. It is quite likely that this approach may yield similar results in other contexts, such as stationary random fields in dimensions higher than one.

The main results of the paper are in section 3. Sections 1 and 2 are devoted to the necessary operator theory and some basic properties of one-dimensional Gibbs states. I have attempted to follow the notational conventions of Pollicott (1983) and Bowen (1974). Many of the proofs are omitted, but may be found in Lalley (1985).

Important note: After writing this paper I learned that a similar idea was used by Rousseau-Egele (1983) in his study of expansive transformations of the unit interval.

2. Ruelle's Perron-Frobenius Theorem and Gibbs Measures.

Let A be an irreducible aperiodic kxk matrix of zeros and ones, and let

$$\Sigma_A^+ = \{x \in \prod_0^\infty \{1,2,\ldots,k\}: A(x_n, x_{n+1}) = 1 \text{ for every } n \geq 0\}.$$

The space Σ_A^+ is compact and metrizable in the product topology (in the special case k = 2 and A(i,j) = 1 for i,j = 1,2, Σ_A^+ is homeomorphic to the Cantor set). Let $\sigma: \Sigma_A^+ \to \Sigma_A^+$ be the forward shift operator:

$$(\sigma x)_n = x_{n+1}, \quad n \geq 0.$$

The fact that A is irreducible and aperiodic implies that σ is topologically mixing.

For $f,g \in C(\Sigma_A^+)$ (here $C(\Sigma_A^+)$ denotes the Banach space of continuous complex-valued functions on Σ_A^+) defined $L_f g \in C(\Sigma_A^+)$ by

(1.1) $$L_f g(x) = \sum_{y:\ \sigma y = x} e^{f(y)} f(y);$$

L_f will be called an RPF operator (for Ruelle, Perron, and Frobenius). Clearly $L_f: C(\Sigma_A^+) \to C(\Sigma_A^+)$ is a bounded linear operator which is positive when f is real. Notice that

(1.2) $$L_f g = L_0(e^f g)$$

and

(1.3) $$L_f^n g(x) = L_0^n(e^{S_n f} g)(x) = \sum_{y:\ \sigma^n y = x} e^{S_n f(y)} g(y),$$

where

(1.4) $$S_0 f \equiv 0,$$
$$S_n f = f + fo_\sigma + \ldots + fo_\sigma^{n-1}, \quad f \in C(\Sigma_A^+).$$

For $f \in C(\Sigma_A^+)$ define $\mathrm{var}_n\ f = \sup\{|f(x)-f(y)|: x_i=y_i;$ for every i, $0 \leq i < n\}$ and for $0 < \Theta < 1$ define the Hölder norm $|\cdot|_\Theta$ by $|f|_\Theta = \sup_{n>0} \mathrm{var}_n f/\Theta^n$. Let $\mathbf{A}_\Theta = \{f \in C(\Sigma_A^+): |f|_\Theta < \infty\}$; then $\mathbf{A}_\Theta$ is a Banach algebra when endowed with the norm $||f||_\Theta = |f|_\Theta + ||f||_\infty$. For any $0 < \Theta < 1$ $\mathbf{A}_\Theta$ is dense in $C(\Sigma_A^+)$. Also, for $f \in \mathbf{A}_\Theta$, $\mathbf{L}_f(\mathbf{A}_\Theta) \subset \mathbf{A}_\Theta$.

THEOREM A (Ruelle): For each real $f \in \mathbf{A}_\Theta$ there exists a real $\lambda_f \in (0,\infty)$ which is a simple eigenvalue of $\mathbf{L}_f: \mathbf{A}_\Theta \to \mathbf{A}_\Theta$, with strictly positive eigenfunction h_f. Moreover, spectrum $(\mathbf{L}_f)$ $\{\lambda_f\}$ is contained in a disc of radius strictly less than λ_f.

Proofs may be found in Bowen (1974), Ruelle (1978) and Pollicott (1984). Bowen and Ruelle also prove

THEOREM B: For each real $f \in A_\Theta$ there is a positive Borel measure $\nu_f \in C(\Sigma_A^+)^*$ such that $(\mathbf{L}_f^*)\nu_f = \lambda_f \nu_f$ and such that

$$\lim_{n\to\infty} ||\mathbf{L}_f^n\ g/\lambda_f^m - (\int g d\nu_f) h_f||_\infty = 0$$

for all $g \in C(\Sigma_A^+)$. Furthermore, support $\nu_f = \Sigma_A^+$.

(Bowen does not prove explicitly that support $\nu_f = \Sigma_A^+$ but this follows easily from the fact that σ is topologically mixing.)

The spectrum of $\mathbf{L}_f$ for complex $f \in \mathbf{A}_\Theta$ was described by Pollicott (1984).

THEOREM C (Pollicott): Suppose $f = u + iv \in \mathbf{A}_\Theta$.

(i) If for some $a \in [-\pi,\pi]$, $(v-a)/2\pi$ is homologous to an integer-valued function then $e^{ia}\lambda_u$ is a simple eigenvalue of L_f and the rest of the spectrum is contained in a disc of radius strictly less than λ_u.

(ii) Otherwise the spectral radius of L_f is strictly less than λ_u.

(Two functions $f,g \in C(\Sigma_A^+)$ are said to be homologous if there exists $\psi \in C(\Sigma_A^+)$ such that $f-g = \psi - \psi\circ\sigma$).

The mapping $f \to \log \lambda_f$ is usually called the pressure; it plays the same role in the study of Gibbs states as does the cumulant generating function in the study of sums of i.i.d. random variables. This analogy will be exploited in section 3.

It will be convenient to normalize the eigenfunction and eigenmeasure in a different manner than is customary. Thus, assume that for $f \in A_\Theta$ real

$$\int_{\Sigma_A^+} 1 \, d\nu_0 = 1,$$

$$\int_{\Sigma_A^+} h_f d\nu_0 = 1,$$

and

$$\int_{\Sigma_A^+} h_f d\nu_f = 1.$$

This normalization will be used (implicitly) in the perturbation arguments of the next section.

Bowen and Ruelle used Theorems 1 and 2 to prove the existence of "equilibrium states" (Gibbs measures) for σ. The Gibbs measure μ_f is defined for real $f \in \mathbf{A}_\Theta$ by

$$\frac{d\mu_f}{d\nu_f} = h_f. \tag{1.5}$$

It is easily verified that μ_f is a σ-invariant probability measure on Σ_A^+ (cf. Bowen, Lemma 1.13) and in fact that $(\Sigma_A^+,\sigma,\mu_f)$ is strongly mixing (it is even Bernoulli: cf. Bowen, Theorem 1.25 for the easy argument). It is easily verified that if f is constant then under μ_f the coordinate variables x_n are i.i.d. Bernoulli (1/2); if $f(x) = f(x_0)$ then under μ_f the coordinate variables are i.i.d. Bernoulli (p) for suitable p; and if $f(x) = f(x_0,x_1,\ldots,x_{k-1})$ then under μ_f the coordinate variables are k-step Markov dependent.

The following result is helpful in understanding the structure of μ_f. (However, it will not be used in obtaining the results of sections 2-3.)

<u>Proposition 1</u>. For any sequence $x = (x_0,x_1,\ldots) \in \Sigma_A^+$, any real $f \in \mathbf{A}_\theta$, and any $m > 1$

$$(1.6)\qquad \mu_f(x_0,x_1,\ldots,x_{m-1}|x_m,x_{m-1},\ldots) = \frac{e^{S_m f(x)}\, h_f(x)}{\lambda_f^m\, h_f(\sigma^m x)}$$

Here the LHS is the conditional probability of $x_0,x_1,\ldots,x_{m-1}$ appearing in the first m slots given that $x_m,x_{m+1},\ldots$ appear in the slots $m,m+1,\ldots$. (Technically

$$\begin{aligned}\mu_f(x_0,x_1,\ldots,x_{m-1}|x_m x_{m+1},\ldots) &= \lim_{n\to\infty} \mu_f(x_0,x_1,\ldots,x_{m-1}|x_m,x_{m+1},\ldots,x_{m+n}) \\ &= \lim_{n\to\infty} \frac{\mu_f\{y:\ y_j=x_j,\ 0<j<m+n\}}{\mu_f\{y:\ y_j=x_j,\ m<j<m+n\}};\end{aligned}$$

the limit exists with μ_f-measure one, by the martingale convergence theorem.) The equation (1.6) explains why μ_f is called a "Gibbs state": note that f(x) plays the role of an energy function. The proof is given in Lalley (1985).

<u>Proposition 2</u>: For each real $f \in \mathbf{A}_\theta$, all $g,\psi \in C(\Sigma_A^+)$, and every integer $m > 0$

$$(1.7)\qquad E_{\mu_f}(e^{S_m\psi} g) = \lambda_f^{-m} \int_{\Sigma_A^+} L_{f+\psi}^m (gh_f)(x)\,d\nu_f(x).$$

(Henceforth E_{μ_f} will be used to denote the expectation operator $\int \cdot \, d\mu_f$).

Proof. Using (1.3) and (1.5), we have

$$E_{\mu_f}(e^{S_m\psi}g) = \int e^{S_m\psi(x)} g(x)h_f(x)\nu_f(dx)$$

$$= \lambda_f^{-m} \int e^{S_m\psi(x)} g(x)\ h_f(x)\ ((\mathbf{L}_f^*)^m\ \nu_f)(dx)$$

$$= \lambda_f^{-m} \int \mathbf{L}_f^m(e^{S_m\psi} gh_f)(x)\nu_f(dx)$$

$$= \lambda_f^{-m} \int L_{f+\psi}^m(gh_f)(x)\nu_f(dx).$$

2. Perturbation Theory for RPF Operators.

The results of this section stem from the observation that the assignment $f \to \mathbf{L}_f$ is analytic, i.e., for each $f_1,\ldots,f_k \in \mathbf{A}_\Theta$, the function

$$(z_1,z_2,\ldots,z_k) \to \mathbf{L}_{\sum_1^k z_i f_i}$$

is an (entire) analytic function of the complex variables $z_1,\ldots,z_k$. In fact it is easily verified from (1.1) that

$$(2.1)\qquad (\partial/\partial z_j)\mathbf{L}_{\sum_1^k z_i f_i}(g) = \mathbf{L}_{\sum_1^k z_i f_i}(f_j g).$$

The following results follow from Theorem A and standard results in regular perturbation theory (cf. Kato (1980), Ch. 7, #1, Ch. 4, #3, and Ch. 3, #5). More detail can be found in Lalley (1985).

Proposition 3. Suppose $f,\psi \in \mathbf{A}_\Theta$ are real-valued. Then $\lambda_{f+z\psi}$ and $h_{f+z\psi}$ have analytic extensions to a neighborhood $\eta(f,\psi)$ of the real line $\{z\colon \operatorname{Im} z = 0\}$, such that

$$\mathbf{L}_{f+z\psi}h_{f+z\psi} = \lambda_{f+z\psi}h_{f+z\psi} \tag{2.2}$$

and

$$\int_{\Sigma_A^+} h_{f+z\psi}d\nu_0 = 1 \tag{2.3}$$

hold throughout this neighborhood. Furthermore the mapping $z \to \nu_{f+z\psi}$ may be extended to an $M(\Sigma_A^+)$-valued function in $\eta(f,\psi)$ which is weak-* analytic, and such that

$$\mathbf{L}^*_{f+z\psi}\nu_{f+z\psi} = \lambda\nu_{f+z\psi} \tag{2.4}$$

and

$$\int h_{f+z\psi}\, d\nu_{f+z\psi} = 1 \tag{2.5}$$

hold throughout the neighborhood. Finally, for each real z_0 there exists an $\varepsilon > 0$ such that if $|z-z_0| < \varepsilon$, then

$$|\lambda_{f+z\psi} - \lambda_{f+z_o\psi}| < \delta \tag{2.6}$$

and

$$(\text{spectrum } \mathbf{L}_{f+z\psi})\ \{\lambda_{f+z\psi}\} \subset \{\xi\colon |\xi| < \lambda_{f+z_o\psi}-2\delta\} \tag{2.7}$$

for some $\delta > 0$.

<u>Note</u>: Weak-* analytic means that for each $h \in \mathbf{A}_\Theta$ the function $z \to \int h\, d\nu_{f+z\psi}$ is analytic. It is generally impossible for $z \to \nu_{f+z\psi}$ to be analytic in (total variation) norm: in fact, for real $f_1, f_2 \in \mathbf{A}_\Theta$ such that f_1 is

not homologous to f_2, ν_{f_1} and ν_{f_2} are mutually singular.

Corollary 1: For all $g \in \mathbf{A}_\Theta$ and real $f,\psi \in \mathbf{A}_\Theta$ there exists a neighborhood $\Omega = \Omega_{f,y}$ of the real line $\{z: \text{Im } z = 0\}$ such that

$$(2.8) \qquad \frac{\mathbf{L}^m_{f+z\psi}\, g}{\lambda^m_{f+z\psi}} \to \left(\int_{\Sigma^+_A} g\, d\nu_{f+z\psi}\right) h_{f+z\psi}$$

uniformly for z in any compact subset of Ω. In fact $\Omega_{f,\psi}$ may be chosen so that for each compact $K \subset \Omega$ there exists an $\varepsilon > 0$ such that

$$(2.9) \qquad (1+\varepsilon)^m \left\| \frac{\mathbf{L}^m_{f+z\psi}\, g}{\lambda^m_{f+z\psi}} - \left(\int_{\Sigma^+_A} g\, d\nu_{f+z\psi}\right) h_{f+z\psi} \right\|_\Theta \to 0$$

uniformly for $z \in K$ and $\{g \in \mathbf{A}_\Theta: \|g\|_\Theta < 1\}$.

3. Central Limit Theorem and Saddlepoint Approximation.

Throughout this section $f,\psi \in \mathbf{A}_\Theta$ are fixed real-valued functions; let

$$(3.1) \qquad \beta(z) = \log(\lambda_{f+z\psi}/\lambda_f)$$

for all $z \in C$ where $\lambda_{f+z\psi}$ is defined.

Let Φ_γ denote the normal distribution with mean zero and variance γ (when $\gamma = 0$, Φ_γ is just the point mass at 0).

THEOREM 1 (Central Limit Theorem): For every $c \in R$

$$(3.2) \qquad \mu_f\{x: n^{-1/2}(S_n\psi(x) - n\beta'(0)) < c\} \to \int_{-\infty}^{c} d\Phi_{\beta''(0)}(t)$$

as $n \to \infty$. Moreover, for every continuous function $u: R \to R$ with at most polynomial growth

$$(3.3) \qquad E_{\mu_f} u(n^{-1/2}(S_n\psi - n\beta'(0))) \to \int_{-\infty}^{\infty} u(t)\, d\Phi_{\beta''(0)}(t)$$

as $n \to \infty$.

This theorem is apparently due to Ratner (1973), who deduced it from another central limit theorem for mixing sequences due to Ibragimov. A direct proof based on the method of moments is outlined in Ruelle (1978). Neither Ratner's nor Ruelle's approach is suitable for obtaining the more refined estimates given below.

It is obviously of interest to know when $\beta''(0) = 0$, since in this case the limit distribution is degenerate.

Proposition 4. The function $\beta''(z)$ is strictly positive for $z \in R$ unless ψ is homologous to a constant, in which case $\beta''(z) \equiv 0$.

Ruelle (1978) showed that $\beta(z)$ is strictly convex but by a roundabout method using the variational principle. Ratner (1973) states that $\beta''(0) > 0$ unless there is an $\eta \in L^2(\mu_f)$ such that $\psi = (\text{const}) + \eta - \eta\circ\sigma$, but this is weaker than Proposition 4. A direct proof of Proposition 4 is given in the Appendix.

It is worth noting that (3.3) implies

$$E_{\mu_f} \psi = \beta'(0).$$

This could also be proved directly.

Proof of Theorem 1. To prove (3.2) it suffices to show that the Laplace transform of $n^{-1/2}(S_n\psi - n\beta'(0))$ converges to the Laplace transform of the normal distribution $\Phi_{\beta''(0)}$ (this is the "continuity theorem", cf. Feller (1971), XIII. 1, Th. 2). By (1.7) and (2.8), for any $z \in R$

$$
(3.4)\quad
\begin{aligned}
&E_{\mu_f} \exp\{zn^{-1/2}(S_n\psi - n\beta'(0))\}\\
&= e^{-z\beta'(0)n^{-1/2}} \lambda_f^{-n} \int (L^n_{f+zn^{-1/2}\psi} h_f) d\nu_f\\
&= \exp\{n(-z\beta'(0)n^{-1/2} + \beta(zn^{-1/2})\} \int \Big(\frac{L^n_{f+zn^{-1/2}\psi} h_f}{\lambda^n_{f+zn^{-1/2}\psi}}\Big) d\nu_f\\
&\sim \exp\{z^2\beta''(0)/2\} \int (\int h_f \, d\nu_{f+zn^{-1/2}\psi}) h_{f+zn^{-1/2}\psi} \, d\nu_f\\
&\sim \exp\{z^2\beta''(0)/2\}
\end{aligned}
$$

since $\int h_f d\nu_f = 1$ and since by Proposition 3 $\nu_{f+z\psi}$ and $h_{f+z\psi}$ are continuous in z. This proves (3.2).

The relation (3.4) incidentally proves that if $u\colon R \to R$ has polynomial growth then the random variables $\{u(n^{-1/2}(S_n\psi - n\beta'(0)))\}_{n \geq 1}$ are uniformly integrable with respect to the probability measure μ_f. This and (3.2) imply (3.3).

The essential idea in the foregoing proof of the Central Limit Theorem is that Proposition 3 and Corollary 1 give an explicit asymptotic representation for the Laplace transform of $S_n\psi$, to wit,

$$
(3.5)\qquad E_{\mu_f} e^{zS_n\psi} \sim e^{n\beta(z)} (\int h_f d\nu_{f+z\psi})(\int h_{f+z\psi} d\nu_f)
$$

uniformly for z in any compact subset of R. This formula suggests that sharper results than (3.2) might be obtained by Fourier inversion.

Assume that ψ is not homologous to a constant. Then by Proposition 4 $\beta''(\theta) > 0$ for $\theta \in R$, hence $\beta'(\theta)$ is strictly increasing. Let

$$
(3.6)\qquad \Gamma_\psi = \{\beta'(\theta)\colon \theta \in R\};
$$

then for each $a \in \Gamma_\psi$ there is a unique $\rho(a) \in R$ such that

$$\beta'(\rho(a)) = a. \tag{3.7}$$

The function $\rho(a)$ is clearly a strictly increasing, real analytic function of $a \in \Gamma_\psi$ onto R.

THEOREM 2. Assume that there do not exist scalars $a_1 > 0$, $a_2 \in R$ such that $a_1(\psi - a_2)$ is homologous to an integer-valued function. Then for every $b \in (0,\infty)$, $a \in \Gamma_\psi$,

$$\mu_f\{x\colon 0 < S_n\psi(x) - na < b\} \sim \int_0^b e^{-\rho(z)t}dt \int_{\Sigma_A^+} h_{f+\rho(a)\psi}d\nu_f \int_{\Sigma_A^+} h_f d\nu_{f+\rho(a)\psi} \cdot (2\pi n\beta''(\rho(a)))^{-1/2}\exp\{n(\beta(\rho(a)) - a\rho(a))\}, \tag{3.8}$$

as $n \to \infty$, and the convergence holds uniformly for a in any compact subset of Γ_ψ. Moreover,

$$\mu_f\{x\colon 0 < S_n\psi(x) - n\beta'(0) - n^{-1/2}c < b\} \sim b(2\pi n\beta''(0))^{-1/2}e^{-c^2/2\beta''(0)} \tag{3.9}$$

uniformly for any compact subset of R. Finally, if $\rho(a) > 0$, then (3.8) also holds for $b = +\infty$, uniformly for a in any compact subset of $\{a \in \Gamma_\psi\colon \rho(a) > 0\}$.

Notice that (3.9) implies the Central Limit Theorem.

THEOREM 3. Assume that $\eta \in \mathbf{A}_\theta$ is real-valued, that $\Psi \in \mathbf{A}_\theta$ is integer-valued, and that

$$\psi = \Psi + \eta - \eta\circ\sigma. \tag{3.10}$$

Assume further that there do not exist scalars $a_1, a_2 \in R$ with $0 < a_1 < \frac{1}{2}$ such that $a_1(\psi - a_2)$ is homologous to an integer-valued function. Then for every $a \in \Gamma_\psi$ and all $0 \le b \le c \le 1$,

$$(3.11)\quad \mu_f\{x:\ b < S_n\psi(x) - na < c\} \sim (2\pi n\beta''(\rho(a)))^{-1/2}\exp\{n(\beta(\rho(a)) - a\rho(a))\}$$

$$\times \iint_{x,x'\in\Sigma_A^+} \exp\{-\rho(a) < \eta(x) - \eta(x') + an > \}\ \ 1\{b < < \eta(x)-\eta(x') + an > < c\}$$

$$\cdot h_{f+\rho(a)\psi}(x)\ h_f(x')d\nu_f(x)d\nu_{f+\rho(a)\psi}(x').$$

This relation holds uniformly for a in any compact subset of Γ_ψ. Moreover,

$$(3.12)\quad \mu_f\{x:\ b < S_n\psi(x) - n\beta'(0) - n^{1/2}\delta < c\} \sim (2\pi n\beta''(0))^{-1/2}e^{-\delta^2/2\beta''(0)}$$

$$\cdot \iint_{x,x'\in\Sigma_A^+} 1\{b < < \eta(x) - \eta(x') + n\beta'(0) + n^{1/2}\delta > < c\}d\mu_f(x)d\mu(x')$$

uniformly for δ in any bounded subset of R. Finally, if $\rho(a) > 0$ then

$$(3.13)\quad \mu_f\{x:\ a < S_n\psi(x)/n\} \sim (2\pi n\beta''(0))^{-1/2}\exp\{n(\beta(\rho(a)) - a\rho(a))\}\ .\ (1-e^{-\rho(a)})^{-1}$$

$$\cdot \iint_{x,x'\in\Sigma_A^+} \exp\{-\rho(a)<\eta(x) - \eta(x') + an>\}$$

$$\cdot\ h_{f+\rho(a)\psi}(x)\ h_f(x')d\nu_f(x')d\nu_f(x)d\nu_{f+\rho(a)\psi}(x')$$

uniformly for a in any compact subset of $\{a \in \Gamma_\psi:\ \rho(a) > 0\}$.

<u>Note</u>: $<t>$ denotes the fractional part of $t \in R$.

If $\eta \equiv 0$ in (3.10) then the expressions in (3.11)-(3.13) simplify somewhat, yielding relations which more closely resemble the corresponding results for sums of i.i.d. integer-valued random variables. Observe that (3.13) and (3.8) ($b = \infty$) imply that when $\rho(a) > 0$,

$$(3.14)\qquad n^{-1}\log \mu_f\{x:\ a < S_n\psi(x)/n\} \to \beta(\rho(a)) - a\rho(a),$$

which is the analogue of Chernoff's (1952) theorem for sums of i.i.d. random variables. The relations (3.13) and (3.9) (with $b=\infty$) are themselves the analogues of results of Bahadur and Ranga Rao (1960) concerning sums of i.i.d. random variables. The relations (3.8) and (3.11) are analogous to results of Daniels (1954) concerning i.i.d. random variables.

Theorems 2 and 3 are proved by Fourier inversion. The details are similar to those in the proofs of the corresponding theorems for sums of i.i.d. random variables, but are slightly more difficult because there is no suitable analogue of the Riemann-Lebesgue Lemma here. The gruesome details may be found in Lalley (1985).

It is also possible to use standard Fourier techniques to obtain analogues of the classical renewal theorems. However, better results may be obtained by coupling methods (cf. Lalley (1984)).

Appendix: Strict Convexity of $\beta(z)$.

The proof of Proposition 4 relies on the following fact, which may be of some interest in its own right.

Proposition 5. Suppose $f,\psi \in \mathbf{A}_\Theta$ are real-valued. Then ψ is homologous to 0 if the sequence $\{S_n\psi\}_{n>0}$ is bounded in $L^2(\mu_f)$.

Proof. If $\psi = \Psi - \Psi o\sigma$ for some $\Psi \in C(\Sigma_A^+)$ then $S_n\psi = \Psi - \Psi o\sigma^n$, so $\{S_n\psi\}_{n>0}$ is clearly bounded in $L^2(\mu_f)$.

Suppose $\{S_n\psi\}_{n>0}$ is bounded in $L^2(\mu_f)$. Define

$$W_m(r) = (1-r)\sum_{k=0}^{\infty} r^k(S_{m+k}\psi - S_m\psi) = W_o(r)o\sigma^m;$$

then the set $\{W_m(r): m=0,1,\dots;\ 0 < r < 1\}$ is bounded in $L^2(\mu_f)$. Notice that

$$\lim_{r\uparrow 1}(W_0(r)-W_1(r)) = \lim_{r\uparrow 1} \psi + \sum_{k=1}^{\infty}(S_k\psi-\psi)(1-r)(r^k-r^{k-1}) = \psi.$$

Now the unit ball of $L^2(\mu_f)$ is weakly compact so there exist $r_n\uparrow 1$ such that

$$W_0(r_n) \overset{w}{\to} W \in L^2(\mu_f)$$

and

$$W_1(r_n) \overset{w}{\to} W o\sigma \in L^2(\mu_f).$$

Consequently

$$\psi = W - W o\sigma \quad \text{a.s. } (\mu_f). \tag{A.1}$$

It remains to be shown that there is a <u>continuous</u> W for which this relation holds.

Define $f^* \in \mathbf{A}_\Theta$ by $f^* = f + \log(h_f/\lambda_f h_f o\sigma)$; then $\mathbf{L}_{f^*}g = (\lambda_f h_f)^{-1}\mathbf{L}_f(h_f g)$, so the leading eigenvalue and eigenfunction of $\mathbf{L}_{f^*}: \mathbf{A}_\Theta \to \mathbf{A}_\Theta$ are 1 and 1, respectively. Also, μ_f is an eigenmeasure of the adjoint operator $\mathbf{L}^*_{f^*}$ as is easily verified, since $\mathbf{L}^*_f \nu_f = \lambda_f \nu_f$ and $d\mu/d\nu = h_f$. It follows that the operator $\mathbf{L}_{f^*}: L^1(\mu_f) \to : L^1(\mu_f)$ has norm 1, because for any $g \in L^1(\mu_f)$

$$\int |L_{f^*}g|d\mu_f \le \int L_{f^*}(|g|)d\mu_f = \int |g|d\mu_f. \tag{A.2}$$

Now suppose (A.1) holds but that ψ is <u>not</u> homologous to 0. Then there is a constant $a > 0$ such that $(2\pi a)^{-1}\psi$ is <u>not</u> homologous to an integer-valued function (otherwise $(2\pi a)^{-1}S_n\psi(x) \in Z$ for every x such that $\sigma^n x = x$ and all $a > 0$, which is clearly impossible unless $\psi \equiv 0$). Consequently by Theorem 3 the spectral radius of the operator $\mathbf{L}_{f^*+ia\psi}: \mathbf{A}_\Theta \to \mathbf{A}_\Theta$ is strictly less than 1, and therefore by the spectral radius formula (cf. Yosida (1965), section VII. 2, Theorem 3 and 4)

$$||\mathbf{L}^n_{f^*+ia\psi}\, g||_\Theta \to 0$$

for all $g \in \mathbf{A}_\Theta$. But $||\cdot||_\infty \le ||\cdot||_\Theta$ so for $g \in \mathbf{A}_\Theta$

$$\text{(A.3)} \qquad \int |\mathbf{L}^n_{f*+ia\psi}\, g| d\mu_f \to 0.$$

Consider now the function $e^{-iaW} \in L^1(\mu_f)$, where W is as in (A.1). (Note: W may clearly be chosen to be real-valued, because $\psi = W + W\circ\sigma$ implies $\psi = (\mathrm{Re}W) + (\mathrm{Re}W)\circ\sigma$. Hence $|e^{-iaW}| = 1$ a.s. (μ_f).) Because

$$\begin{aligned} L_{f*+ia\psi} e^{-iaW} &= L_{f*}\, e^{ia\psi - iaW} \\ &= L_{f*} e^{-ia(W\circ\sigma)} \\ &= e^{-iaW} L_{f*}\, 1 \\ &= e^{-iaW}, \end{aligned}$$

it follows that

$$\text{(A.4)} \qquad \int |\mathbf{L}^n_{f*+ia\psi} e^{-iaW}| d\mu_f = 1$$

for all $n \ge 1$. But the relations (A.3) and (A.4) are contradictory because $\mathbf{A}_\Theta$ is dense in $L^1(\mu_f)$ and because (A.2) implies that $\mathbf{L}_{f*+ia\psi}: L^1(\mu_f) \to L^1(\mu_f)$ has norm ≤ 1. Thus the assumption that ψ is not homologous to 0 is incompatible with (A.1).

Proof of Proposition 4. Without loss of generality we may assume that $\beta'(0) = 0$, since this may always be accomplished by subtracting $\beta'(0)$ from ψ. By (3.3) it follows that

$$E_{\mu_f} \psi = \int \psi h_f d\nu_f = 0$$

Now by (2.9) there is an $\varepsilon > 0$ such that $(1+\varepsilon)^n \lambda_f^{-n} ||\mathbf{L}^n_f(\psi h_f)||_\Theta \to 0$. Hence by (1.7)

$$(1+\varepsilon)^n|E_{\mu_f}\psi(\psi\circ\sigma^n)| = |(1+\varepsilon)^n\lambda_f^{-n}\int L_f^n(\psi(\psi\circ\sigma^n)h_f)d\nu_f|$$

$$= |(1+\varepsilon)^n\lambda_f^{-n}\int(L_f^n(\psi h_f))\psi d\nu_f| \to 0 \text{ as } n \to +\infty.$$

Therefore if $F(z) = \sum_{-\infty}^{\infty} z^n E_{\mu_f}\psi(\psi\circ\sigma^n)$, then $F(z)$ has an analytic extension to an annulus containing the unit circle $\{e^{i\alpha}, \alpha \in [-\pi,\pi)\}$.

It follows from elementary Fourier analysis that

(A.5) $$E_{\mu_f}(S_n\psi)^2 = n\int_{-\pi}^{\pi} G_n(\theta)\, F(e^{i\theta})d\theta/2\pi,$$

where G_n is Fejer's kernel

$$G_n(\theta) = \sum_{m:\ |m|<n} (1 - \frac{|m|}{n})\, e^{im\theta} = n^{-1}\{\frac{\sin(\frac{n+1}{2})\theta}{\sin\frac{1}{2}\theta}\}.$$

Since $G_n(\theta)$ is an approximate identity (summability kernel) (A.5) implies

(A.6) $$\lim_{n\to\infty} E_{\mu_f}(S_n\psi)^2/n = F(1).$$

Now (3.3) implies that the limit in (A.6) is $\beta''(0)$. Suppose $\beta''(0) = 0$; then $F(1) = 0$. It is evident from the definition of F that $F(e^{i\theta})$ is an even function of θ, so that if $F(1) = 0$, then $F(e^{i\theta})$ is (essentially) quadratic in θ near $\theta = 0$, i.e.,

$$F(e^{i\theta}) = 0(\theta^2) \text{ as } \theta \to 0.$$

Thus

$$E_{\mu_f}(S_n\psi)^2 = \int_{-\pi}^{\pi}\{\frac{\sin(\frac{n+1}{2})\theta}{\sin\frac{1}{2}\theta}\}\, F(e^{i\theta})d\theta/2\pi$$

is bounded as $n\to\infty$. It follows from Proposition 2 that ψ is homologous to zero.

This shows that $\beta''(0) > 0$ if ψ is not homologous to a constant. To show that $\beta''(z) > 0$ for z real, just replace f by $f+z\psi$.

REFERENCES

Bahadur, R., and Ranga Rao, R. (1960). On deviations of the sample mean. Ann. Math. Statist. 31 1015-1027.

Bowen, R. (1974). Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Math. 470. Springer, New York.

Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. Ann. Math. Statist. 23

Daniels, H. (1954). Saddlepoint approximations in statistics. Ann. Math. Statist. 25 631-650.

Feller, W. (1971). An Introduction to Probability Theory and its Applications II, second edition. Wiley, New York.

Gordin, I. V. (1969). On the central limit theorem for stationary processes. Doklady AN SSSR 188, 4.

Ibragimov, I. A. and Linnik, Y. (1971). Independent and stationary of random variables. Wolters-Noordhoff, Groningen.

Kac, M. (1949). Probabilistic methods in some problems of analysis and number theory. Bull. Amer. Math. Soc. 55 641-665.

Kato, T. (1980). Perturbation Theory for Linear Operators. Springer, New York.

Lalley, S. (1984). A renewal theorem for a class of stationary sequences, submitted for publication.

Lalley, S. (1985). Ruelle's Perron-Frobenius theorem and saddle-point approximations for additive functionals of one-dimensional Gibbs states. Columbia University technical report.

Pollicott, M. (1984). A complex Ruelle-Perron-Frobenius theorem and two counterexamples. Ergod. Th. and Dynam. Sys. 4 135-146.

Ratner, M. (1973). The central limit theorem for geodesic flows on n-dimensional manifolds of negative curvature. Israel J. Math. 16 181-197.

Rousseau-Egele, J. (1983). Un theoreme de la limite locale pour une classe de transformations dilatentes et monotones par morceaux. Annals Prob. **11** 772-788.

Yosida, K. (1965). Functional Analysis. Springer, New York.