

ANISOTROPIC CONTACT PROCESSES ON HOMOGENEOUS TREES

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ABSTRACT. Sufficient conditions for the existence of a weak survival phase are given for an anisotropic contact process on a homogeneous tree. These require that the contact process be *homogeneous*, that is, for any two vertices x, y of the tree there is an automorphism mapping x to y leaving the infection rates invariant; and that the contact process be *weakly symmetric*, that is, for each vertex there should be at least two incident edges with the same infection rate.

1. INTRODUCTION

The isotropic contact process on a homogeneous tree $\mathcal{T} = \mathcal{T}^d$ of degree $d \geq 3$ is known to have three distinct phases: an extinction phase, a weak survival phase, and a strong survival phase. In the weak survival phase, the set of infected sites remains nonempty forever with positive probability, but with probability one every fixed site eventually becomes and remains healthy (provided the set of initially infected sites is finite). By contrast, in the strong survival phase, the event that a fixed site is infected at arbitrarily large times has positive probability. The existence of the weak survival phase was proved by Pemantle [9] for $d \geq 4$, and later by Liggett [6] for $d = 3$; subsequently, Stacey [10] discovered a relatively simple proof for all $d \geq 3$. Their proofs all rely on the isotropy of the processes. (The methods of Pemantle and Stacey extend to contact processes on certain inhomogeneous trees; however, it is necessary that the vertices of these trees be arranged in “levels” of descent from a root vertex in such a way that the infection rates remain invariant under automorphisms of the tree that permute the vertices within levels.)

The purpose of this paper is to show that certain *anisotropic* contact processes on \mathcal{T} also have weak survival phases. An *anisotropic contact process* on \mathcal{T} has infection rates that are direction-dependent in the following sense: Each edge e of the tree is assigned a color $\kappa(e)$ in such a way that the coloring of the tree is homogeneous, that is, for each color k , the number of edges of color k incident to vertex x is the same for all vertices x . The infection rate $\lambda_e = \lambda_{\kappa(e)}$ across edge e is a function only of the *color* of e .

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The contact process with infection rates $\lambda_e = \lambda_{\kappa(e)}$ and recovery rate δ is a continuous-time Markov process whose state ξ_t at any time $t \geq 0$ is a finite set of vertices of \mathcal{T} and whose evolution is governed by the following laws: (1) Infected sites (vertices included in ξ_t) recover at rate $\delta > 0$ and upon recovery are removed from ξ_t . (2) A healthy site x (a vertex not included in ξ_t) becomes infected at rate $\sum_{e \in I_t(x)} \lambda_e$, where $I_t(x)$ is the set of all edges incident to x and a vertex $y \in \xi_t$. The default initial state is $\xi_0 = \{r\}$, where r is a distinguished vertex of \mathcal{T} called the *root*. In general, the initial state A will be denoted by a superscript ξ_t^A , so that $\xi_0^A = A$. Versions of anisotropic contact processes with all possible infection and recovery rates and initial states A may be constructed simultaneously on any probability space equipped with sequences of independent Poisson processes and random variables uniformly distributed on the unit interval using Harris' *graphical representation* – see [5] for details.

For each color k denote by q_k the number of edges of color k incident to a vertex x . (By the homogeneity of the coloring, this is the same for all vertices x). Say that the contact process is *weakly symmetric* if there is a color k such that $q_k \geq 2$. Note that if the contact process is weakly symmetric then for any vertex x there is a nontrivial automorphism (in fact, infinitely many) of \mathcal{T} that fixes x and leaves the infection rates invariant. The main result of this paper is the following theorem:

Theorem 1. *Every weakly symmetric, homogeneous, anisotropic contact process on the tree $\mathcal{T} = \mathcal{T}^d$ of degree $d \geq 3$ has a weak survival phase, that is, for each choice of infection rates satisfying the homogeneity and weak symmetry conditions specified above, there exist critical values $0 < \delta_u < \delta_c < \infty$ such that*

- (a) $\delta \geq \delta_c \implies$ *extinction*;
- (b) $\delta < \delta_u \implies$ *strong survival*; and
- (c) $\delta_u \leq \delta < \delta_c \implies$ *weak survival*.

We conjecture that the hypothesis of weak symmetry is extraneous, but have been unable to prove this. Note that the existence of a weak survival phase for anisotropic contact processes whose infection rates λ_i are nearly equal follows by a trivial comparison argument from the corresponding fact for the isotropic contact process: By the results of [9], [6], and [10], there exist critical values $0 < \lambda_c < \lambda_u < \infty$ such that, for any $\lambda \in (\lambda_c, \lambda_u]$, the isotropic contact process with infection rate λ and recovery rate 1 survives weakly. Consequently, any anisotropic contact process with infection rates $\lambda_e \in (\lambda_c, \lambda_u]$ and recovery rate 1 survives weakly. No such simple argument proves the existence of a weak survival phase when the infection rates are substantially different, however.

2. PRELIMINARIES

2.1. Edge-Labelled Trees. Let $\mathcal{T} = \mathcal{T}^d$ be an infinite, homogeneous tree of degree d with a distinguished vertex r , the *root*. An infinite, self-avoiding

path in \mathcal{T} beginning at a vertex x is called a *geodesic ray*; a finite segment of a geodesic ray is a *geodesic segment*. For any two vertices x, y of \mathcal{T} there is a unique geodesic segment connecting x to y , whose length $d(x, y)$ is defined to be the *distance* from x to y . For brevity, the distance from the root vertex r to x will be abbreviated as $|x|$.

Assume that the edges e of \mathcal{T} are assigned labels $\ell(e) \in [d] := \{1, 2, \dots, d\}$ in such a way that each vertex is incident to one edge of each label. Then there is a one-to-one correspondence between geodesic segments starting at r and finite reduced words with letters from the alphabet $[d]$ (A *reduced word* is a finite string of letters in which no two consecutive letters are identical.) Denote by $[d]_n$ the set of all reduced words of length n , and by $[d]^* = \cup_{n \geq 0} [d]_n$ the set of all finite reduced words. Since the geodesic segments starting at r are in one-to-one correspondence with the vertices of \mathcal{T} , there is a bijection between $[d]^*$ and the vertex set of \mathcal{T} . Henceforth, we shall not distinguish between a vertex x and the reduced word $i_1 i_2 \dots i_{|x|}$ that specifies the geodesic segment from r to x . For any vertex $x \neq r$, define the *type* of x to be the label of the last edge (the edge incident to x) of the geodesic segment from r to x .

Removal of any vertex x disconnects the tree, leaving d non-overlapping subtrees $\mathcal{T}_{x;i}$ rooted at the vertices neighboring x . These subtrees are indexed by the edge labels $i \in [d]$ in the natural way: the subtree $\mathcal{T}_{x;i}$ is connected to the vertex x in \mathcal{T} by an edge with label i . For any vertex $x \neq r$, define \mathcal{T}_x to be the subtree consisting of all vertices y such that the geodesic segment from r to y passes through x ; thus, if x is of type i , then

$$(2.1) \quad \mathcal{T}_x = \mathcal{T} \setminus \mathcal{T}_{x;i}.$$

2.2. Symmetries of \mathcal{T} . The identification of words and vertices exhibits \mathcal{T} as the Cayley graph of the free product \mathcal{F}^d of d copies of the two element group \mathbb{Z}_2 . Multiplication in this group is accomplished as follows: For any two words x, y , concatenate x and y and then successively eliminate all spurs (pairs of adjacent letters that are not distinct) to obtain xy . The multiplicative inverse of a word x is obtained by reversing its letters. The elements of the group \mathcal{F}^d act as automorphisms of \mathcal{T} under which the labelling of edges remains invariant. Since the group \mathcal{F}^d acts transitively on the vertices of \mathcal{T} , it follows that for any two vertices x, y of \mathcal{T} there is a unique label-preserving automorphism $\varphi_{x,y}$ of \mathcal{T} that maps x to y . Note that for each edge label i the automorphism $\varphi_{x,y}$ maps $\mathcal{T}_{x;i}$ bijectively onto $\mathcal{T}_{y;i}$; consequently, if $x, y \neq r$ are vertices of the same type, then $\varphi_{x,y}$ also maps \mathcal{T}_x bijectively onto \mathcal{T}_y . The importance of label-preserving automorphisms resides in the fact that these are the automorphisms of \mathcal{T} under which anisotropic contact processes are invariant: If ξ_t^A is an anisotropic contact process with infection rates $\lambda_{\ell(e)}$, recovery rate δ , and initial state $\xi_0^A = A$, and if $\varphi \in \mathcal{F}^d$ is a label-preserving automorphism of \mathcal{T} , then $\varphi(\xi_t^A)$ is a version of the anisotropic contact process with the same infection and recovery rates, and initial state $\varphi(A)$.

For any *weakly symmetric* anisotropic contact process, there is a non-injective mapping that assigns a *color* to every label; the infection rate λ_e across any edge is determined solely by its color (that is, the color assigned to its label). If all labels are mapped to the same color then the contact process is *isotropic*. The law of a weakly symmetric, anisotropic contact process is invariant by any color-preserving automorphism of \mathcal{T} . Note that any *label-preserving* automorphism is a *color-preserving* automorphism, but that there are many more color-preserving than label-preserving automorphisms.

2.3. Nonamenability of \mathcal{T} . For any nonempty finite set F of vertices, define the *outer boundary* ∂F to be the set of all vertices y at distance 1 from F such that F intersects only one of the subtrees $\mathcal{T}_{y;i}$.

Proposition 2.1.

$$(2.2) \quad |\partial F| \geq |F|.$$

Proof. The proof is by induction on $|F|$. If F is a singleton, then its outer boundary consists of $d - 1$ vertices. Now suppose that the inequality (2.2) is true for sets of cardinality less than n , for some integer $n \geq 2$, and let $|F| = n$. Then F has at least one vertex y such that $\mathcal{T}_y \cap F = \{y\}$. The outer boundary of F includes the $d - 1$ vertices of \mathcal{T}_y at distance 1 from y ; these are *not* included in the outer boundary of $F - \{y\}$. The outer boundary of $F - \{y\}$ includes at most one vertex that is not in ∂F . Thus,

$$\begin{aligned} |\partial F| &\geq |\partial(F - \{y\})| + d - 2 \\ &\geq |F| + d - 3 \\ &\geq |F|. \end{aligned}$$

□

2.4. The Harris Coupling. Stochastic comparisons and continuity properties of various expectations in the recovery rate δ are most easily approached using the *Harris coupling* (also known as the *graphical representation*). The main features of this coupling are set forth in the following proposition:

Proposition 2.2. *Versions $\xi_t(\lambda, \delta)$ of the contact process with common initial state $\xi_0 = A$ but different (bounded) values λ_e of the infection rates and δ of the recovery rate may be constructed on a common probability space in such a way that*

$$(2.3) \quad \xi_t(\lambda, \delta) \subset \xi_t(\lambda, \delta') \quad \text{if } \delta > \delta', \text{ and}$$

$$(2.4) \quad \xi_t(\lambda, \delta) = \lim_{\delta' \rightarrow \delta} \xi_t(\lambda, \delta').$$

See [5] for the construction.

Corollary 2.3. *Let $\xi_t = \xi_t^A$ be an anisotropic contact process on \mathcal{T} with infection rates $\lambda_e = \lambda_{\ell(e)}$ and recovery rate δ , started from the initial state $\xi_0 = A$. Then for any bounded real-valued function f on the vertex set of \mathcal{T} and any $t < \infty$, the expectation*

$$U_f(t) := E \sum_{x \in \xi_t} f(x)$$

varies continuously with the recovery rate δ .

Proof. In view of the Harris coupling, continuity of $U_f(t)$ in δ follows from the dominated convergence theorem, provided it is shown that $E|\xi_t| < \infty$ for every value of the recovery rate δ . But for any value of δ , the random variable $|\xi_t|$ is stochastically dominated by Z_t , where $\{Z_t\}_{t \geq 0}$ is a binary fission process with fission rate $\sum_{i=1}^d \lambda_i$. Consequently,

$$(2.5) \quad E|\xi_t| \leq \exp\left\{t \sum_i \lambda_i\right\} < \infty.$$

□

Expectations of functionals that depend on the entire history of the contact process, and not just its state at a fixed time, need not vary continuously. Of particular importance is the *hitting probability* functional, defined as follows:

$$(2.6) \quad u(x) := P\{x \in \cup_{t \geq 0} \xi_t\}.$$

Corollary 2.4. *For each vertex x , the hitting probability functional $u(x)$ is right-continuous in the recovery rate parameter δ .*

Proof. Fix a vertex x , and let $A(x; \delta)$ be the event that $x \in \cup_{t \geq 0} \xi(t; \delta)$. Then by relations (2.3) and (2.4), $A(x; \delta)$ is the increasing limit of the events $A(x; \delta')$ as $\delta' \downarrow \delta$. Therefore, by the monotone convergence theorem, $u(x)$ is right-continuous in δ . □

2.5. Critical Values of Contact Processes. The existence of extinction and strong survival phases for anisotropic contact processes on the tree \mathcal{T} may be established by simple comparison arguments. To prove that the anisotropic contact process with infection rates $\lambda_e = \lambda_{\ell(e)}$ survives strongly with positive probability when the recovery rate δ is small, recall that this is known to be true for the *isotropic* contact process with infection rate $\lambda_e = \lambda_*$, where λ_* is the minimum of the rates λ_i for $i \in [d]$. By the Harris coupling, the anisotropic contact process can be constructed on the same probability space in such a way that it dominates the isotropic contact process pathwise, and hence will survive strongly whenever the isotropic contact process survives strongly.

Similarly, the anisotropic contact process can be constructed together with a dominating isotropic contact process with infection rates $\lambda_e = \lambda^*$,

where λ^* is the *maximum* of the rates λ_i for $i \in [d]$. Because the isotropic contact process becomes trapped at \emptyset with probability one when δ is large, the same must be true of the anisotropic contact process. Alternatively, one may dominate the anisotropic contact process by a continuous-time Galton-Watson process with death rate δ and fission rate $\sum_i \lambda_i$, where the sum is over all edge labels. This coupling shows that if δ_c is defined to be the supremal value of the recovery rate at which extinction occurs with probability one, then

$$(2.7) \quad \delta_c \leq \sum_{i=0}^d \lambda_i < \infty.$$

Henceforth we shall refer to a contact process with recovery rate $\delta = \delta_c$ as *critical*.

3. GROWTH OF THE CRITICAL CONTACT PROCESS

Proposition 3.1. *Let ξ_t be a critical homogeneous contact process on \mathcal{T} with initial state $\xi_0 = \{r\}$. Then for some constant $C < \infty$ (depending on the infection rates $\lambda_{\ell(e)}$ but not on t)*

$$(3.1) \quad 1 \leq E|\xi_t| \leq C \quad \text{for all } t \geq 0.$$

The corresponding result for the critical *isotropic* contact process was proved by MORROW, SCHINAZI, and ZHANG [8], using an earlier result of MADRAS and SCHINAZI [7]. The proof is easily adapted to the anisotropic case; a sketch is provided below. Before proceeding to this, we make note of the following consequence of Proposition 3.1.

Corollary 3.2. *Ultimate extinction is certain for the critical contact process.*

Proof. A standard argument using Lévy's extension of the Borel-Cantelli lemma shows that, on the event of survival, $\lim_{t \rightarrow \infty} |\xi_t| = \infty$ almost surely. If the event of survival had positive probability, then it would follow, by the monotone convergence theorem, that $\lim_{t \rightarrow \infty} E|\xi_t| = \infty$, contradicting Proposition 3.1. \square

Following is an extension to anisotropic contact processes of a result of MADRAS and SCHINAZI, which implies that for any contact process — critical, subcritical, or supercritical — the expected number of infected sites has sharp exponential growth in t :

Proposition 3.3. *There exist constants γ and C , depending on the infection and recovery rates $\lambda_{\ell(e)}$ and δ , such that*

$$(3.2) \quad \exp\{\gamma t\} \leq E|\xi_t| \leq C \exp\{\gamma t\}$$

for all $t \geq 0$. Furthermore, the exponential rate γ varies continuously with the infection and recovery parameters $\lambda_{\ell(e)}$ and δ .

Proof. As in the isotropic case, the expectation $E|\xi_t|$ is sub-multiplicative in t , and so by the subadditivity lemma, the limit $\gamma := \lim_{t \rightarrow \infty} t^{-1} \log E|\xi_t|$ exists and

$$E|\xi_t| \geq e^{\gamma t} \quad \text{for all } t \geq 0.$$

This proves the first inequality in (3.2). To prove the second, it suffices to show that there exists a constant $\epsilon > 0$ such that

$$(3.3) \quad E|\xi_{t+s+1}| \geq \epsilon E|\xi_t| E|\xi_s| \quad \text{for all } t, s > 0,$$

because this implies that $E|\xi_t|^n \leq \epsilon^{-n} E|\xi_{nt+n}|$ for every $n = 2, 3, \dots$, and letting $n \rightarrow \infty$ then yields $E|\xi_t| \leq \epsilon^{-1} \exp\{\gamma(t+1)\}$. For the proof of inequality (3.3), the nonamenability of the tree, as encapsulated in Proposition 2.1, is essential. Proposition 2.1 implies that the cardinality of the outer boundary $\partial\xi_t$ is at least the cardinality of ξ_t . For each vertex $x \in \partial\xi_t$, at least one of the subtrees $\mathcal{T}_{x;i}$ does not intersect ξ_t . Moreover, for distinct vertices $x, x' \in \partial\xi_t$ these subtrees do not overlap; consequently, the cardinality of ξ_{t+s+1} is no smaller than the sum of the cardinalities of its intersections with the subtrees $\mathcal{T}_{x;i}$ that do not overlap ξ_t , where $x \in \partial\xi_t$. The expected cardinalities of these intersections may be estimated from below using inequality (3.5) of Lemma 3.4 following: for any vertex $x \in \partial\xi_t$, if $x' = xj \in \xi_t$ is the neighboring vertex in ξ_t to x , then for each edge label i ,

$$E(|\xi_{s+t+1} \cap \mathcal{T}_{x;i}| | \xi_t) \geq E|\xi_{s+1} \cap \mathcal{T}_{j;i}| \geq \epsilon E|\xi_s|.$$

This, together with inequality (3.3) above, proves the second inequality in (3.2).

Continuity of the Malthusian parameter γ in the infection and recovery rates is a consequence of the continuity of $E|\xi_t|$ (Corollary 2.3 above) and the inequalities (3.2). The first inequality guarantees that γ is lower semicontinuous; the second, together with the submultiplicativity of $E|\xi_t|$ and the fact that $e^\gamma = \lim E|\xi_t|^{1/t}$, implies that γ is upper semicontinuous. \square

Lemma 3.4. *For any homogeneous contact process on \mathcal{T} with initial state $\xi_t = \{r\}$, there exists $\epsilon > 0$ such that for all $t \geq 0$ and all edge labels $i \in [d]$,*

$$(3.4) \quad E|\xi_{t+1} \cap \mathcal{T}_i| \geq \epsilon E|\xi_t|.$$

Similarly, there exists $\epsilon' > 0$ such that for all $t \geq 0$ and all pairs $i, j \in [d]$,

$$(3.5) \quad E|\xi_{t+1} \cap \mathcal{T}_{j;i}| \geq \epsilon' E|\xi_t|.$$

The constants $\epsilon, \epsilon' > 0$ may be chosen so as to vary continuously with the infection and recovery rates.

Proof. Let A be the event that at time 1 all vertices at distance 2 from the root are infected. The probability of A is positive, and in fact is bounded below by a positive constant ϵ that varies continuously in the infection and recovery rates. The conditional distribution of $|\xi_{t+1} \cap \mathcal{T}_i|$, given the event A , stochastically dominates the unconditional distribution of $|\xi_t|$, because each intersection $\xi_t \cap \mathcal{T}_j$ may be matched with an intersection $\xi_{t+1} \cap \mathcal{T}_{xj}$ for some

vertex $x \in \mathcal{T}_i$ at distance 2 from r . The inequality (3.4) now follows. The inequality (3.5) follows by a similar argument. \square

Proof of Proposition 3.1. In view of Proposition 3.3, what must be proved is that for any critical contact process, $\gamma = 0$. To show that $\gamma \geq 0$, it suffices to show that $E|\xi_t| \geq 1$ for every $t \geq 0$. As was observed by MORROW, SCHINAZI and ZHANG, this is true for a critical homogeneous contact process on *any* homogeneous graph:

Proof that $E|\xi_t| \geq 1$ at Criticality. For any contact process — not necessarily critical — and any $t > 0$, the sequence $\{|\xi_{nt}|\}_{n \geq 1}$ is dominated by a Galton-Watson process Z_n whose offspring distribution is the distribution of $|\xi_t|$. If $E|\xi_t| < 1$, then the Galton-Watson process Z_n is subcritical, and so, with probability one, $\xi_{nt} = \emptyset$ eventually. Thus, if $E|\xi_t| < 1$ for some t , then the contact process cannot survive, either weakly or strongly, with positive probability.

Now suppose that $E|\xi_t| < 1$ for some value of t when the recovery rate is $\delta = \delta_c$. Then by the continuity of $E|\xi_t|$ in the infection and recovery rates (Proposition 2.3), there is some smaller value of the recovery rate $\delta < \delta_c$ at which $E|\xi_t| < 1$, and so for this value of the recovery rate the contact process is in the extinction phase. But this contradicts the fact that δ_c is the smallest value of the recovery rate at which the contact process becomes extinct almost surely. \square

Proof that $\gamma \leq 0$ at Criticality. To complete the proof, we must show that for a critical contact process, $\gamma \leq 0$. Suppose that $\gamma > 0$ at $\delta = \delta_c$; then since γ is continuous in the recovery rate δ , for all $\delta > \delta_c$ sufficiently near δ_c it will also be the case that $\gamma > 0$. We shall argue that if $\gamma > 0$ then the contact process survives with positive probability. Since the probability of survival is zero for all $\delta > \delta_c$, this will imply that it is impossible for $\gamma > 0$ at $\delta = \delta_c$.

For any edge e of the tree \mathcal{T} and any vertex x , define the e -severed contact process $\chi_t^{e,x}$ started at x to be the contact process with initial state $\{x\}$ whose infection and recovery rates are the same as those of the homogeneous contact process ξ_t^x except that the infection rate across edge e is set to 0.

Lemma 3.5. *If the Malthusian parameter γ for the homogeneous contact process ξ_t is positive, then for each edge label $i \in [d]$, the corresponding contact process $\chi_t^{i,r}$ severed across the edge incident to r with label i satisfies*

$$(3.6) \quad \limsup_{t \rightarrow \infty} E|\chi_t^{i,r}| = \infty.$$

Proof. The proof is virtually the same as that of the corresponding result for the isotropic contact process in [8]. Fix a vertex $x \in \mathcal{T} \setminus \mathcal{T}_{r,i}$ and a time $t > 0$. On the event that $x \in \xi_t$, it must be the event that there is an “infection trail” leading from the root r to x at time t ; this trail must eventually stay in $\mathcal{T} \setminus \mathcal{T}_{r,i}$, after crossing the label- i edge incident to r a last time. As the

process of infection arrows across this edge is Poisson of rate λ_i , it follows that

$$P\{x \in \xi_t\} \leq \lambda_i \int_{s=0}^t P\{x \in \chi_s^{i,r}\} ds.$$

Summing over all vertices $x \in \mathcal{T} \setminus \mathcal{T}_{r;i}$ and using inequality (3.4) of Lemma 3.4 gives

$$\epsilon E|\xi_t| \leq \lambda_i \sup_{s \leq t} E|\chi_s^{i,r}| \quad \forall t > 0.$$

If the Malthusian parameter γ for the contact process ξ_t is positive, then the left side of this inequality grows exponentially in t , and so (3.6) follows. \square

The proof of Proposition 3.1 may now be completed. Suppose that $\gamma > 0$ at $\delta = \delta_c$, and therefore also at some $\delta > \delta_c$. Then by Lemma 3.5, for each edge type $i \in [d]$ the contact process $\chi^{i,r}$ started at r and severed across the edge incident to r with label i satisfies (3.6). Thus, by Proposition 2.1, the expected number of vertices in the outer boundary of $\chi_t^{i,r}$ can be made arbitrarily large, and in particular, at least $1/\epsilon$, where $\epsilon > 0$ is a lower bound for the probability that ξ_1 includes all vertices at distance 2 from r . Consequently, for at least one of the edge types i , there exists $t > 0$ such that

$$(3.7) \quad E|\partial^i \chi_t^{i,r}| > 1,$$

where $\partial^i \chi_t^{i,r}$ denotes the set of all vertices x in $\chi_t^{i,r}$ of type i such that $\chi_t^{r,i} \subseteq \mathcal{T}_{x;i}$. But this implies that the sequence $|\xi_{nt}|$ dominates a supercritical Galton-Watson process. This is built inductively as follows: First, sever across the edge incident to r with label i , then run until time t , then discard all vertices not in $\partial^i \chi_t^{i,r}$. Then for each of the non-discarded vertices x , sever across the edge incident to x of type i and proceed as in the initial step. Continue in this manner to obtain generations $n = 2, 3, \dots$. Inequality (3.7) implies that the resulting Galton-Watson process is supercritical, and thus survives with positive probability. But then the contact process ξ_t must survive with positive probability, contradicting the hypothesis that $\delta > \delta_c$. \square

4. SPARSE SUBTREES

A subtree \mathcal{S} of \mathcal{T} will be called k -sparse if for every vertex x of \mathcal{S} the distance from x to the complement of \mathcal{S} is no greater than k . Observe that any geodesic ray is 1-sparse, as is any geodesic segment.

Proposition 4.1. *Let ξ_t be a critical contact process on \mathcal{T} . There exist constants $C_k < \infty$, depending only on the infection rates of ξ_t , such that for any k -sparse subtree \mathcal{S} of \mathcal{T} containing the root vertex r ,*

$$(4.1) \quad EN(\infty, \mathcal{S}) \leq C_k,$$

where $N(\infty, \mathcal{S})$ is defined to be the number of distinct vertices of \mathcal{S} that are ever infected.

Proof. By Lemma 3.4 and the first inequality of (3.1), there exists a constant $\epsilon > 0$ such that for every vertex $x \neq r$, every edge label i , and each $t > 0$,

$$E|\xi_{t+1}^x \cap \mathcal{T}_{x;i}| \geq \epsilon E|\xi_t| \geq \epsilon$$

Thus, by the strong Markov property, if τ_x is defined to be the smallest time t at which $x \in \xi_t$, then for any vertex $x \neq r$,

$$E(|\xi_{t+1} \cap \mathcal{T}_x|) \geq \epsilon P\{\tau_x \leq t\}.$$

For any subtree \mathcal{S} of \mathcal{T} , the outer boundary $\partial\mathcal{S}$ is the set of vertices not included in \mathcal{S} but each at distance 1 from \mathcal{S} . If the subtree \mathcal{S} contains the root, then for each vertex $x \in \partial\mathcal{S}$ the subtree \mathcal{T}_x is disjoint from \mathcal{S} ; moreover, if x and y are distinct boundary vertices of \mathcal{S} , then the trees \mathcal{T}_x and \mathcal{T}_y do not intersect. Hence, for any $t \geq 0$,

$$E|\xi_{t+2}| \geq \epsilon \sum_{x \in \partial\mathcal{S}} P\{\tau_x \leq t\}.$$

Suppose now that \mathcal{S} is a k -sparse subtree of \mathcal{T} , so that each vertex x of \mathcal{S} is within distance k of some vertex $y \in \partial\mathcal{S}$. Let (x, y) be any such pair of vertices; then, conditional on the event that $\tau_x \leq t$, the event $\tau_y \leq t + 1$ occurs with (conditional) probability at least

$$\alpha_k := \frac{\lambda_*^k}{k!} \exp\{-\lambda_* - \delta\} > 0,$$

where λ_* is the minimum of the infection rates $\lambda_e = \lambda_{\ell(e)}$. Since each vertex $y \in \partial\mathcal{S}$ is within distance k of at most d^k other vertices, it follows that

$$\sum_{y \in \partial\mathcal{S}} P\{\tau_y \leq t + 1\} \geq \alpha_k d^{-k} \sum_{x \in \mathcal{S}} P\{\tau_x \leq t\} := \alpha_k d^{-k} EN(t, \mathcal{S}),$$

where $N(t, \mathcal{S})$ is the number of distinct vertices of \mathcal{S} that have been infected by time t . Thus, for any $t \geq 0$,

$$C \geq E|\xi_{t+2}| \geq \epsilon \alpha_k d^{-k} EN(t, \mathcal{S}),$$

where $C < \infty$ is the constant provided by Proposition 3.1. Since the random variables $N(t, \mathcal{S})$ converge monotonically to $N(\infty, \mathcal{S})$, the monotone convergence theorem implies that

$$EN(\infty, \mathcal{S}) \leq Cd^k / \alpha_k \epsilon.$$

□

Corollary 4.2. *Let ξ_t be a critical anisotropic contact process on \mathcal{T} . Then for any k -sparse subtree \mathcal{S} of \mathcal{T} containing the root vertex,*

$$(4.2) \quad \lim_{t \rightarrow \infty} E|\xi_t \cap \mathcal{S}| = 0.$$

Proof. Let B_m be the set of all vertices at distance no more than m from the root. By Corollary 3.2, the contact process ξ_t eventually dies out, with probability one, and so by the dominated convergence theorem, for each $m \geq 0$,

$$\lim_{t \rightarrow \infty} E|\xi_t \cap B_m| = 0.$$

On the other hand, by Proposition 4.1, for any $\epsilon > 0$ there exists m sufficiently large that for all $t > 0$,

$$E|\xi_t \cap \mathcal{S} \cap B_m^c| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the corollary follows. \square

5. WEAK SURVIVAL

Assume now that ξ_t is a *weakly symmetric* anisotropic contact process on \mathcal{T} . Recall that an anisotropic contact process is weakly symmetric if the edge labels $i \in [d]$ may be assigned colors $\pi(i)$ in such a way that (a) the infection rates depend only on edge colors; and (b) there is at least one color k (hereafter designated RED) such that at least two different edge labels are assigned color k .

5.1. The Zigzag Norm. Recall that geodesic segments in \mathcal{T} with initial vertex r are uniquely specified by reduced words $x = i_1 i_2 \dots i_n$ in the edge labels. For any such word x , define $\pi(x) = (\pi(i_1), \pi(i_2), \dots, \pi(i_n))$ to be the *chromatic signature* of x . Observe that chromatic signatures need not be reduced words in the different colors, because the assignment π of colors to labels is not injective. If two vertices x and y of \mathcal{T} correspond to reduced words in $[d]^*$ of the same chromatic signature, then there is a color-preserving automorphism of \mathcal{T} that fixes r and maps x to y . Consequently, for any two such vertices,

$$(5.1) \quad u(x) = u(y) \quad \text{and}$$

$$(5.2) \quad u(t, x) = u(t, y) \quad \text{for all } t \geq 0,$$

where

$$u(x) := P\{x \in \cup_{t \geq 0} \xi_t\} \quad \text{and} \quad u(t, x) := P\{x \in \xi_t\}.$$

Let x be a vertex of \mathcal{T} with chromatic signature $\pi(x)$. Define $\|x\|$ to be the number of RED blocks in the word $\pi(x)$, equivalently, the number of RED entries not preceded by RED entries. (Thus, for example, if $\pi(x) = \text{BYRRYRRGR}$ then $\|x\| = 3$, and if $\pi(y) = \text{RRGRBRRR}$ then $\|y\| = 3$.) Henceforth we shall refer to $\|\cdot\|$ as the *zigzag norm* (despite the fact that it is not a norm). For distinct vertices x, y of \mathcal{T} , define the *zigzag distance* $d_{zz}(x, y) = \|x^{-1}y\|$. Observe that the zigzag distance between distinct vertices may be zero.

Lemma 5.1. *For any vertex x of \mathcal{T} , the number of vertices y of the same chromatic signature is at least $2^{\|x\|}$.*

Proof. Let $\kappa = k_1 k_2 \dots k_m$ be the chromatic signature of x . The number of vertices y with chromatic signature κ is the number of distinct reduced words $\iota = i_1 i_2 \dots i_m$ in the edge labels that are mapped to κ by π . For any admissible assignment i_j of edge labels to the indices j corresponding to the non-RED entries of κ , there are at least $2^{\|x\|}$ admissible ways to assign labels to the RED blocks in κ , since there are at least two distinct labels assigned to the color RED. \square

Remark: This is the only place where the hypothesis of weak symmetry enters the argument. However, the result, though simple, will play a central role in the proof that $\delta_u < \delta_c$ – see the proof of Proposition 5.5 below.

Lemma 5.2. *For any vertex x , define $\nu(x)$ to be the number of vertices y of \mathcal{T} with the same chromatic signature as x . Let x, y, z be any three vertices of \mathcal{T} such that z lies on the geodesic segment from x to y . Then*

$$(5.3) \quad \nu(x^{-1}z)\nu(z^{-1}y) \leq d\nu(x^{-1}y).$$

Proof. The chromatic signature a of $x^{-1}y$ is the concatenation of the chromatic signatures b and c of $x^{-1}z$ and $z^{-1}y$. The number $\nu(x^{-1}y)$ (respectively, $\nu(x^{-1}z)$ and $\nu(z^{-1}y)$) of reduced words in $[d]^*$ that are mapped by π to a (respectively, b and c) is the number of ways of assigning edge types to monochromatic blocks of a (respectively, b and c) in such a way that each block of a given color k is assigned an edge type i of color $\pi(i) = k$. Thus, $\nu(x^{-1}y)$ would be the product of $\nu(x^{-1}z)$ with $\nu(z^{-1}y)$ but for the fact that if the monochromatic blocks at the end of b and the beginning of c have the same color, then in the assignment of edge types to monochromatic blocks of a the edge type for the first block of c is forced to agree with that of the last block of b . This reduces the number of choices by at most a factor of d , since there are only d edge types. \square

Lemma 5.3. *For any integer $k \geq 0$, define \mathcal{S}_k to be the set of all vertices x such that $\|x\| \leq k$. If there are at least two different edge colors (that is, if the contact process is not isotropic) then \mathcal{S}_{k-1} is a $2k$ -sparse subtree of \mathcal{T} .*

Proof. Let x be any vertex of \mathcal{S}_{k-1} , and let $i_1 i_2 \dots i_n$ be the reduced word in $[d]^*$ that specifies the geodesic segment from r to x . Since there are at least two distinct colors, RED and BLUE, it is possible to adjoin letters $i_{n+1}, i_{n+2}, \dots, i_{n+2k}$ with alternating colors RED and BLUE to the end of $i_1 i_2 \dots i_n$ in such a way that $i_1 i_2 \dots i_{n+2k}$ is a reduced word. If z is the vertex corresponding to the reduced word $i_1 i_2 \dots i_{n+2k}$, then $\|z\| \geq k$, and the distance from x to z is $2k$. \square

5.2. Analysis of a Lyapunov Function. We assume now that the contact process ξ_t is *weakly symmetric* and that there are at least two distinct edge colors, one of them (RED) as in the preceding section. Let $\lambda_{\pi(\ell(e))}$ and δ be the infection and recovery rates, respectively. For any $0 < a \leq 1$, any $t \geq 0$, and any finite initial set $F = \xi_0$ of vertices, define

$$(5.4) \quad U_a(t) = U_{a,\delta}^F(t) = E \sum_{x \in \xi_t} a^{\|x\|}.$$

Here $\|\cdot\|$ is the zigzag norm. The functional $\xi \mapsto \sum_{x \in \xi} a^{\|x\|}$ plays the role of a *Lyapunov function*. The main result of this section will be an estimate that implies that its expectation decays to zero exponentially as $t \rightarrow \infty$.

Lemma 5.4. *Assume that the contact process ξ_t is critical, that is, that $\delta = \delta_c$. If $a < 1$ then for every finite initial set $F = \xi_0$,*

$$(5.5) \quad \lim_{t \rightarrow \infty} U_a(t) = 0.$$

Proof. By Lemma 5.3, each of the trees \mathcal{S}_k is sparse. Hence, by Corollary 4.2, $\lim_{t \rightarrow \infty} E|\xi_t \cap \mathcal{S}_k| = 0$ for every $k = 1, 2, \dots$. But by Proposition 3.1,

$$\begin{aligned} U_a(t) &\leq E|\xi_t \cap \mathcal{S}_k| + a^k E|\xi_t \cap \mathcal{S}_k^c| \\ &\leq E|\xi_t \cap \mathcal{S}_k| + a^k E|\xi_t| \\ &\leq E|\xi_t \cap \mathcal{S}_k| + C|F|a^k. \end{aligned}$$

The relation (5.5) now follows easily. \square

Proposition 5.5. *For $1/\sqrt{2} < a < 1$, all $s, t \geq 1$, and all δ ,*

$$(5.6) \quad U_{a,\delta}(s+t) \leq U_{a,\delta}(s)U_{a,\delta}(t)d/a(1-1/2a^2).$$

Proof. Each vertex y included in ξ_{t+s} is an ‘‘offspring’’ of at least one vertex x in ξ_t , that is, there is an infection trail from (x, t) to $(y, t+s)$. For each $x \in \xi_t$, denote by $\xi_{t+s}(x)$ the set of all offspring of x in ξ_{t+s} . Note that the sets $\xi_{t+s}(x)$ and $\xi_{t+s}(x')$ may overlap for different vertices x, x' . Since ξ_{t+s} is contained in the union of the sets $\xi_{t+s}(x)$, where $x \in \xi_t$, it follows that

$$\begin{aligned} (5.7) \quad U_a(s+t) &\leq E \sum_{x \in \xi_t} \sum_{y \in \xi_{t+s}(x)} a^{\|x\|} a^{\|y\| - \|x\|} \\ &\leq E \sum_{x \in \xi_t} \sum_{y \in \xi_{t+s}(x)} a^{\|x\|} a^{\|x^{-1}y\|} a^{\|y\| - \|x\| - \|x^{-1}y\|} \\ &= \sum_x u(t, x) a^{\|x\|} \sum_y u(s, x^{-1}y) a^{\|x^{-1}y\|} a^{\|y\| - \|x\| - \|x^{-1}y\|}. \end{aligned}$$

For any two vertices x, y , let $x \wedge y$ be the last common vertex on the geodesic segments from r to x and from r to y ; then

$$\|x^{-1}y\| - \|y\| + \|x\| \leq 2d_{zz}(x, x \wedge y) + 1.$$

For each pair of vertices x, y , the number of vertices with the same chromatic signature as $x^{-1}y$ is $\nu(x^{-1}y)$; of these vertices y' , the number such that $x \wedge y' = z$ is bounded by

$$\nu(z^{-1}y) \leq d\nu(x^{-1}y)/\nu(x^{-1}z) \leq d\nu(x^{-1}y)/2^{d_{zz}(x,zy)},$$

by Lemma 5.2. Thus, grouping the terms y of the inner sum \sum_y in (5.7) according to the chromatic signature of $x^{-1}y$ and the value $k = d_{zz}(x, x \wedge y)$, one obtains the bound

$$(5.8) \quad \sum_y u(s, x^{-1}y) a^{\|x^{-1}y\|} a^{\|y\| - \|x\| - \|x^{-1}y\|} \\ \leq \left(\sum_y u(s, x^{-1}y) a^{\|x^{-1}y\|} \right) \left(\sum_{k=0}^{\infty} da^{-2k-1} 2^{-k} \right) \\ = U_a(s) d/a (1 - a^{-2}/2).$$

The desired inequality is now obtained by substituting this bound on the inner sum \sum_y in inequality (5.7). \square

5.3. The Weak Survival Phase.

Proposition 5.6. *For a symmetric, anisotropic contact process on \mathcal{T} , the weak survival phase is an interval of positive length, that is,*

$$(5.9) \quad \delta_u < \delta_c.$$

Proof. Fix $a \in (1/\sqrt{2}, 1)$. By Lemma 5.4, at $\delta = \delta_c$ the expectation $U_{a,\delta}(t)$ converges to zero as $t \rightarrow \infty$, and by Proposition 2.3, $U_a(t)$ varies continuously with δ for each $t < \infty$. Hence, for some $t < \infty$ and all $\delta < \delta_c$ sufficiently close to δ_c ,

$$U_{a,\delta}(t) \leq \epsilon \quad \text{where} \quad \epsilon d/a(1 - 1/2a^2) < 1.$$

It therefore follows from Proposition 5.5 that $U_{a,\delta}(nt)$ converges to zero exponentially fast as $n \rightarrow \infty$, and from this it is easily deduced that $U_{a,\delta}(s)$ converges to zero exponentially fast as $s \rightarrow \infty$. Since $U_{a,\delta}(s)$ is no smaller than $P\{r \in \xi_s\}$, this implies that eventually $r \notin \xi_s$, with probability one. \square

6. WEAK SURVIVAL AT THE UPPER CRITICAL POINT

6.1. Hitting Probabilities. Recall that, for any vertex x , the hitting probability $u(x)$ is defined to be the probability that x is ever infected by the contact process with initial state $\xi_0 = \{r\}$, that is,

$$u(x) := P\{x \in \cup_{t \geq 0} \xi_t\}.$$

For an isotropic contact process, $u(x)$ depends only on the distance $|x|$ from the root, but for anisotropic contact processes this is not generally true: the function u is invariant under automorphisms of \mathcal{T} that fix the root vertex r and leave the infection rate structure invariant, but u is not necessarily

invariant under other automorphisms. However, as in the isotropic case u is supermultiplicative:

Lemma 6.1. *For any two vertices x, y of $\mathcal{T} = \mathcal{F}^d$,*

$$(6.1) \quad u(xy) \geq u(x)u(y).$$

Proof. Let $\tau(x)$ denote the first hitting time of vertex x , that is, the smallest t such that $x \in \xi_t$, or ∞ if there is no such t . By the Strong Markov Property, the additivity of the contact process, and invariance by the automorphism $\varphi_{e,x}$,

$$\begin{aligned} u(xy) &\geq P\{xy \in \cup_{t \geq \tau(x)} \xi_t\} \\ &= EP(xy \in \cup_{t \geq \tau(x)} \xi_t \mid \xi_{\tau(x)}) \mathbf{1}\{\tau(x) < \infty\} \\ &\geq P\{xy \in \cup_{t \geq \tau(x)} \xi_t^x\} \\ &= u(y)u(x). \end{aligned}$$

□

Recall that the vertices of \mathcal{T} are identified with finite reduced words from the alphabet $[d]$, and that the identification is a bijection. Denote by Σ the set of *infinite* reduced words $x = x_1x_2\dots$, that is, the set of all infinite sequences with entries in $[d]$ in which no letter appears twice in succession. Note that if $x_1x_2\dots$ is an element of Σ then for every integer $n \geq 1$ the word $x_1x_2\dots x_n$ is reduced, and consequently is the address of a vertex in \mathcal{T} .

Corollary 6.2. *Let μ be an ergodic, shift-invariant probability measure on Λ , and let X_1, X_2, \dots be the coordinate random variables on Σ . Then there exists a constant $0 < \beta(\mu) \leq 1$ such that*

$$(6.2) \quad \lim_{n \rightarrow \infty} u(X_1X_2\dots X_n)^{1/n} = \beta(\mu) \quad \text{almost surely } \mu.$$

Furthermore, for every integer $n \geq 1$,

$$(6.3) \quad E_\mu \log u(X_1X_2\dots X_n) \leq n \log \beta(\mu).$$

Proof. In view of the supermultiplicativity of the function u , this follows directly from Kingman's subadditive ergodic theorem. □

The hitting probability $u(x)$ is never smaller than the probability that ξ_t survives strongly, since on the event of strong survival *every* vertex is eventually infected. Consequently, in the strong survival phase,

$$(6.4) \quad \beta(\mu) = 1$$

for every ergodic, shift-invariant probability measure on Σ . The main result of this section is that, in the weak survival phase, $\beta(\mu) < 1$ for every *reversible* ergodic probability measure μ with positive entropy. (A probability measure μ on Σ is said to be *reversible* if for every $n \geq 1$ the joint distribution of the coordinate random variables X_n, X_{n-1}, \dots, X_1 is the same as that of X_1, X_2, \dots, X_n .)

Proposition 6.3. *Let μ be a reversible, ergodic, shift-invariant probability measure on Σ with entropy $h(\mu)$. If the probability of strong survival is zero, then*

$$(6.5) \quad 2 \log \beta(\mu) + h(\mu) \leq 0.$$

For isotropic contact processes this result follows from the main result of [3], since in the isotropic case $h(\mu) \leq \log(d-1)$ and $\beta(\mu) = \beta$ is the same for all μ . The proof is very similar to the proof of Theorem 1 in [3], and so we will only provide a brief sketch in paragraph 6.3 below.

6.2. Weak Survival at the Upper Critical Point. Before discussing the proof of Proposition 6.3 we shall show how the result implies that the contact process survives only weakly at the upper critical point $\delta = \delta_u$. Let μ be the Shannon (maximal entropy) measure on Σ , that is, the unique invariant probability measure on Σ under which the coordinate variables X_1, X_2, \dots comprise a Markov chain with transition probabilities $p(i, j) = 1/(d-1)$ for $i \neq j$ and $p(i, i) = 0$. The entropy of this measure is

$$h := h(\mu) = \log(d-1) > 0,$$

and so by Proposition 6.3,

$$\beta := \beta(\mu) \leq -h/2 < 0.$$

Consequently, by inequality (6.3), for any $\delta > \delta_u$, where strong survival occurs with probability zero,

$$(6.6) \quad E_\mu \log u(X_1 X_2 \dots X_n) \leq -nh/2.$$

By Corollary 2.4, the hitting probability function $u(x)$ is right-continuous in δ , and by the Harris coupling it is monotone. Since $u(X_1 X_2 \dots X_n) \geq \epsilon^n$, where $\epsilon = \lambda_*/(\lambda_* + \delta)$, the bounded convergence theorem implies that (6.6) holds at $\delta = \delta_u$. This implies that the event of strong survival has probability zero at $\delta = \delta_u$, because for any vertex x ,

$$u(x) \geq P\{\text{strong survival}\}.$$

6.3. The Entropy Inequality. We shall discuss the proof of Proposition 6.3 only in the special case where μ is the Shannon measure, since only this special case is needed for the proof of weak survival at the upper critical point. The proof in the general case is similar, but requires an additional argument based on the Shannon-MacMillan-Breiman theorem – see [4] for a similar argument involving the use of this theorem.

The strategy of the proof is the same as used in [3] to prove the analogous theorem for the isotropic contact process. It entails showing that, if the inequality (6.5) is violated, then with positive probability there exist infection trails extending from the root r to vertices at arbitrarily large distances from r and then back to r . On this event, the contact process must survive strongly, because the event implies that infinitely many vertices are infected before the last time that r is infected.

Suppose, then, that the inequality (6.5) is false. Denote by S_n the set of all vertices at distance n from r , and let N_n be the number of vertices $x \in S_n$ such that there is an infection trail that begins at r at time 0, passes through x , and returns to r . We shall argue that

$$(6.7) \quad \inf_{n \geq 1} P\{N_n \geq 1\} > 0;$$

this implies that $P(\cap_{n \geq 1}\{N_n \geq 1\}) > 0$, and therefore that the contact process survives strongly with positive probability.

Exponential Growth of EN_n . That (6.7) should be expected when inequality (6.5) is false may be seen by a crude estimation of the expectation of N_n . Fix $\epsilon > 0$, and consider a vertex $x \in S_n$ such that

$$(6.8) \quad \min(u(x), u(x^{-1})) \geq (\beta - \epsilon)^n.$$

For any such vertex x , the probability that there is an infection trail leading from r to x and then back to r is at least $u(x)u(x^{-1}) \geq (\beta - \epsilon)^{2n}$. Since under the Shannon measure the random vector (X_1, X_2, \dots, X_n) is uniformly distributed on the set of all $d(d-1)^{n-1}$ reduced words of length n , the ergodic theorem (6.2) for u implies that for any $\epsilon > 0$ and all sufficiently large n , the number of vertices $x \in S_n$ satisfying (6.8) is at least $e^{nh-n\epsilon}$, and consequently

$$(6.9) \quad EN_n \geq \exp\{n(h - \epsilon)\}(\beta - \epsilon)^{2n}.$$

If inequality (6.5) is false, then (6.9) implies that for sufficiently small $\epsilon > 0$ the exponential rate of growth of EN_n is positive. Therefore, one should expect that for large n there will be large numbers of vertices in S_n included in ‘‘infection loops’’ beginning and ending at the root.

Directed Infection Trails. Let x, y be two distinct vertices of \mathcal{T} at distance $n \geq 1$. A *directed* infection trail from a vertex x to a vertex y is an infection trail that (a) lies entirely in the subtree $\mathcal{T}_{x;i}$ containing y ; (b) first exits the ball $B_{n-1}(x)$ of radius $n-1$ centered at x at vertex y ; and (c) terminates upon reaching y . For any vertex $x \neq r$, define

$$(6.10) \quad u_*(x) = P\{\exists \text{ directed infection trail from } r \text{ to } x\}.$$

(More precisely, $u_*(x)$ is the probability that there is a directed infection trail beginning at the root at time $t = 0$ that ends at x .)

Lemma 6.4. *Fix $\epsilon > 0$ and an edge label $i \in [d]$, and define $G_n = G_n(\epsilon; i)$ to be the set of all vertices x of type i at distance n from the root such that*

$$(6.11) \quad \min(u_*(x), u_*(x^{-1})) > (\beta - \epsilon)^n.$$

For all sufficiently large n ,

$$(6.12) \quad |G_n| \geq e^{nh-n\epsilon}.$$

Proof. The argument is similar to that used in section 3 of [3]. For any integer $K \geq 0$ and any vertex x define $u_K(x)$ to be the probability that there is an infection trail from r to x that remains within distance K of the geodesic segment from r to x . By the monotone convergence theorem, $\lim_{K \rightarrow \infty} u_K(x) = u(x)$ for each vertex x . Since for large n the set of vertices $x \in S_n$ that satisfy (6.8) has cardinality in excess of $e^{nh-n\epsilon}$, there exist integers $K \geq m \geq 1$ and edge labels $i \neq i'$ such that

$$(6.13) \quad |G_m^K(\epsilon; i, i')| \geq e^{mh-m\epsilon},$$

where $G_m^K = G_m^K(\epsilon; i, i')$ is defined to be the set of vertices $x \in S_m$ such that

- (a) $\min(u_K(x), u_K(x^{-1})) > (\beta - \epsilon)^m$;
- (b) x is of type i ; and
- (c) x^{-1} is of type i' .

Consider any sequence w_1, w_2, \dots, w_n of elements of G_m^K . Since each element of G_m^K begins with the letter i' and ends with the letter $i \neq i'$, the product $y = w_1 w_2 \dots w_n$ is a word of length nm . Because successive infection trails can be concatenated, the probability $u_K(y)$ cannot be less than $\prod_{j=1}^n u_K(w_j)$, nor can $u_K(y^{-1})$ be less than $\prod_{j=1}^n u_K(w_j^{-1})$; hence,

$$\min(u_K(y), u_K(y^{-1})) \geq (\beta - \epsilon)^{nm}.$$

Now let z and z' be words of length $K + 1$ such that z and z' are of type i and z^{-1} and $(z')^{-1}$ are of type i' . Then for any product $y = w_1 w_2 \dots w_n$ of elements in G_m^K , the concatenation zyz' will be a word of length $nm + 2K + 2$; furthermore, there will be a directed infection trail from r to zyz' if there is a directed infection trail from r to z , followed by an infection trail from z to zy that stays within distance K of the geodesic segment from z to zy , followed by a directed infection trail from zy to zyz' . Consequently,

$$\begin{aligned} u_*(zyz') &\geq \epsilon'(\beta - \epsilon)^{nm} && \text{and} \\ u_*((zyz')^{-1}) &\geq \epsilon'(\beta - \epsilon)^{nm}, \end{aligned}$$

where $\epsilon' > 0$ is a lower bound for the probability that there are directed infection trails from r to the vertices z, z' . Thus, for all sufficiently large n , the set $G_{nm+2K+2}(2\epsilon; i)$ has cardinality at least that of $(G_m^K)^n$, and therefore at least $e^{nmh-nm\epsilon}$. The lemma now follows routinely. \square

Embedded Branching Processes. Fix an edge label $i \in [d]$. For each integer $n \geq 1$, define $A_n = A_n(\epsilon; i)$ to be set of all vertices $x \in G_n(\epsilon; i) \cap \mathcal{T}_i^c$ of type i such that there exists a directed infection trail leading from the root at time zero to x . Here $G_n(\epsilon; i)$ is, as in Lemma 6.4, the set of vertices $x \in S_n$ such that both $u(x)$ and $u(x^{-1})$ exceed $(\beta - \epsilon)^n$.

Lemma 6.5. *If $h + \log \beta > 0$ then the probability that the contact process ξ_t survives is positive, and for any $\epsilon > 0$,*

$$(6.14) \quad \limsup_{n \rightarrow \infty} P\{|A_n(\epsilon; i)| \geq e^{nh-2n\epsilon}(\beta - \epsilon)^n\} > 0.$$

Proof. Lemma 6.4 implies that, for all sufficiently large n , the number of vertices x of type i at distance n from the root that satisfy inequality (6.11) is at least $e^{nh-n\epsilon}$. Hence, the expected number of such vertices that are reached by directed infection trails from the root is at least $e^{nh-n\epsilon}(\beta-\epsilon)^n$. Under the hypothesis that $h + \log \beta > 0$, this expectation is strictly greater than 1, provided $\epsilon > 0$ is sufficiently small.

We now construct a supercritical Galton-Watson process inductively as follows: (i) Run a restricted contact process in $\mathcal{T}_{r;i} \cap B_m(r)$, and mark all vertices $x \in G_m^\epsilon$ of type i that are infected: these vertices constitute the first generation of the Galton-Watson process. (ii) From each vertex in the first generation, run a restricted contact process in $\mathcal{T}_x \cap B_m(x)$, beginning at the time τ_x when x is first infected, and mark all infected vertices y of type i at distance m from x such that $x^{-1}y \in G_m^\epsilon$; these vertices form the second generation. (iii) Continue this process indefinitely. That the process so defined is in fact a Galton-Watson process follows because the restricted contact processes in successive generations are run in nonoverlapping regions of the tree, and therefore are independent. By the preceding paragraph, the mean offspring number is at least $e^{mh-m\epsilon}(\beta-\epsilon)^m$, provided m is sufficiently large, and so the process is supercritical. Moreover, on the event of survival, the exponential growth rate of the process is at least $e^{mh-m\epsilon}(\beta-\epsilon)^m$, by the Kesten-Stigum theorem. Since the size of the n th generation is dominated by $|A_{nm}|$, the result follows. \square

Proof of (6.7). The proof uses the same strategy as was used in section 5 of [3] for the isotropic case. Assume that the inequality (6.5) is false. Fix $\epsilon > 0$, and set $h_* = h - 2\epsilon$ and $\beta_* = \beta - \epsilon$. If $\epsilon > 0$ is small, then $e^{h_*}\beta_*^2 > 1$. Consequently, by Lemma 6.5, there exist $0 < \alpha < 1$, an integer $m \geq 1$, and an edge label $i \in [d]$ such that

$$\begin{aligned} P\{|A_m^\epsilon(i)| \geq e^{mh_*}\beta_*^m\} &> 2\alpha \quad \text{and} \\ (1 - \alpha\beta_*^m)^{\exp\{mh_*\}\beta_*^m} &< 1/2. \end{aligned}$$

We shall prove that the infimum in (6.7) is at least $\alpha\beta_*^m$.

For any vertex $x \neq r$, define τ_x to be the first time that x is infected by a directed infection trail beginning at the root at time 0 (or ∞ if there is no such directed infection trail). Clearly, τ_x is a stopping time for the contact process. For any integer $k \geq 1$, let $H_k(x)$ be the event that $\tau_x < \infty$ and that there is an infection trail that begins at x at time τ_x , remains entirely in \mathcal{T}_x , reaches a vertex at distance k from x , and then returns to x , where it terminates. Similarly, define $H_k(r; i)$ to be the event that there is an infection trail that begins at r at time 0, does not enter the subtree \mathcal{T}_i , reaches a vertex in S_k and then returns to r , where it terminates. For any vertex x of type i ,

$$P(H_k(x) | \tau_x < \infty) = P(H_k(r; i)) := p(k; i)$$

We shall prove by induction on n that

$$(6.15) \quad p(nm; i) \geq \alpha \beta_*^m$$

for every n . Since $P\{N_n \geq 1\} \geq P(H_n(r; i))$, and since these probabilities are nonincreasing in n , this will imply (6.7).

Consider first the case $n = 1$. On the event that $|A_m(\epsilon; i)| \geq 1$, there is at least one vertex $x \in G_m(\epsilon; i)$ that is reached by an infection trail beginning at the root at time 0. Recall that $u_*(x^{-1}) > \beta_*^m$ for each $x \in G_m(\epsilon; i)$; thus, the conditional probability that the root is reinfected by the first vertex $x \in A_m(\epsilon; i)$ (that is, the one for which τ_x is smallest) is at least β_*^m . Thus,

$$p(m; i) \geq 2\alpha \beta_*^m.$$

Suppose now that inequality (6.15) is valid for all positive integers $\leq n$. Consider the event $H_{nm+m}(r; i)$. This will occur if all three of the following events occur:

- (A) $|A_m(\epsilon; i)| \geq \exp\{mh_*\beta_*^m\}$.
- (B) For at least one $x \in A_m(\epsilon; i)$ the event $H_{nm}(x)$ occurs.
- (C) For the first $x \in A_m(\epsilon; i)$ such that $H_{nm}(x)$ occurs, there is an infection trail leading from x back to the root.

Given that (A) occurs, the probability that (B) occurs is at least $p(nm; i)$, because event (B) involves a part of the tree disjoint from that which determines (A). Similarly, given that both (A) and (B) occur, the probability that (C) occurs is at least β_*^m , since $u_*(x^{-1}) \geq \beta_*^m$ for all vertices $x \in A_m(\epsilon; i)$. Therefore, by the induction hypothesis,

$$\begin{aligned} P(H_{nm+m}(r; i)) &\geq 2\alpha(1 - p(nm; i))^{\exp\{mh_*\}\beta_*^m} \beta_*^m \\ &\geq 2\alpha(1 - \alpha\beta_*^m)^{\exp\{mh_*\}\beta_*^m} \beta_*^m \\ &\geq \alpha\beta_*^m. \end{aligned}$$

□

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