LECTURE 15: AMERICAN OPTIONS

1. Introduction

All of the options that we have considered thus far have been of the *European* variety: exercise is permitted only at the termination of the contract. These are, by and large, relatively simple to price and hedge, at least under the hypotheses of the Black-Sholes model, as pricing entails only the evaluation of a single expectation. *American* options, which may be exercised at any time up to expiration, are considerably more complicated, because to price or hedge these options one must account for many (infinitely many!) different possible exercise policies. For certain options with convex payoff functions, such as *call* options on stocks that pay no dividends, the optimal policy is to exercise only at expiration. In most other cases, including *put* options, there is also an optimal exercise policy; however, this optimal policy is rarely simple or easily computable.

We shall consider the pricing and optimal exercise of American options in the simplest nontrivial setting, the Black-Sholes model, where the underlying asset Stock pays no dividends and has a price process S_t that behaves, under the risk-neutral measure Q, as a simple geometric Brownian motion:

$$S_t = S_0 \exp\left\{\sigma W_t + (r - \sigma^2/2)t\right\} \tag{1}$$

Here $r \geq 0$, the riskless rate of return, is constant, and W_t is a standard Wiener process under Q. For any American option on the underlying asset Stock, the admissible exercise policies must be *stopping times* with respect to the natural filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ of the Wiener process W_t. If $F(s)$ is the payoff of an American option exercised when the stock price is s, and if T is the expiration date of the option, then its value V_t at time $t \leq T$ is

$$V_t = \sup_{\tau : t \leq \tau \leq T} E(F(S_\tau)e^{-r(\tau - t)} \,|\, \mathcal{F}_t). \tag{2}$$

2. Call Options

Recall that a *call* option has payoff $(s-K)_+$ if exercised when the stock price is s. Here, as always, K is the strike price of the option. Note that, for each fixed K, the payoff function $(s-K)_+$ is convex in the argument s.

Proposition 1. *The optimal exercise policy for the owner of an American call option is to hold the option until expiration, that is,* $\tau = T$.

Proof. Let $\tau \leq T$ be any stopping time. If the American option were exercised at time τ, the payoff would be $(S_\tau - K)_+$, and so the value at time zero to a holder of the option planning to exercise at the stopping time τ would be

$$E(S_\tau - K)_+ e^{-r\tau} \leq E(S_\tau e^{-r\tau} - Ke^{-rT})_+,$$

using the fact that $\tau \leq T$. Now recall that the discounted price process $\left\{S_t e^{-rt}\right\}_{t \geq 0}$ is a martingale. Since the function $x \mapsto (x - C)_+$ is convex, the following lemma implies that

$$E(S_\tau e^{-r\tau} - Ke^{-rT})_+ \leq E(S_T e^{-rT} - Ke^{-rT})_+.$$

Thus, the value of the call at time zero to the holder planning to exercise it at time τ would be no larger than that of the *European* call with strike K and expiration T. It follows by (2) that the value of the American call is equal to that of the European call, and therefore that holding to expiration is optimal.[1] □

Lemma 1. *Let $\{M_t\}_{0\leq t\leq T}$ be a martingale relative to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, and let $\tau \leq T$ be a stopping time. Then for any convex function φ,*

$$E\varphi(M_\tau) \leq E\varphi(M_T). \tag{3}$$

Proof. This is a simple consequence of Jensen's inequality. Since M_t is a martingale relative to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, for any stopping time $\tau \leq T$ with associated stopping $\sigma-$algebra $\mathcal{F}_\tau$,

$$E(M_T \,|\, \mathcal{F}_\tau) = M_\tau \qquad \text{a.s.}$$

Hence, by Jensen's inequality for conditional expectation,

$$\begin{aligned} E(\varphi(M_T)) &= E(E(\varphi(M_T) \,|\, \mathcal{F}_\tau) \\ &\geq E\varphi(E(M_T \,|\, \mathcal{F}_\tau)) \\ &= E\varphi(M_\tau). \end{aligned}$$

□

3. Arbitrage Price of the American Put

A *put* option with strike K has payoff $(K-s)_+$ if exercised when the price of the underlying asset Stock is s. Denote by $V_A(t, S_t)$ and $V_E(t, S_t)$ the prices at time $t \leq T$ of the American and European put options on the asset Stock, both with expiration date T and strike price K.

3.1. Comparison of the American and European Put Options.

Proposition 2. *If the risk-free rate r is positive, then for every $t < T$,*

$$V_E(t, S_t) < V_A(t, S_t). \tag{4}$$

Consequently, the optimal exercise policy is *not* to hold until expiration no matter what. As we shall see, the optimal policy has the following form: For a certain differentiable, strictly increasing function $s_*(t)$ (for which, unfortunately, there is no simple closed form expression) one should exercise the put at the first time $\tau \leq T$ such that

$$S_\tau \leq s_*(\tau). \tag{5}$$

If (5) does not occur by the expiration date T then one must allow the option to expire worthless. Later we shall discuss the problem of computing the function $s_*(t)$ (called the *exercise boundary*) and derive some of its basic properties.

Proof of Proposition 2. The following argument was related to me by George Kordzakhia. To prove (4), it suffices to consider the case $t = 0$. Consider the following (not necessarily optimal) exercise policy for the American put: Exercise at $\min(\tau, T)$, where

$$\tau = \min\left\{t \geq 0 : S_t \leq K - Ke^{r(t-T)}\right\}. \tag{6}$$

[1] This argument may be improved to show that the expected discounted payoff for any policy such that $\tau < T$ with positive probability is *strictly* smaller than $E(S_T - K)_+e^{-rT}$. Therefore, hold until exercise is the *only* rational policy.

On the event that $\tau < T$, the put will be exercised at a time τ when $S_\tau \leq K(1 - e^{r(t-T)})$, and so the payoff is at least $Ke^{r(t-T)}$. If this payoff is invested in the riskless asset for the remaining time $\tau \leq t \leq T$, then its value at time T will be K. This is greater than any possible payoff from the European put with strike K, since $S_T > 0$ with probability one.

On the event that $\tau \geq T$, the option will be exercised at time T, and the payoff will be exactly the same as for the European put. Thus, in both cases ($\tau < T$ and $\tau \geq T$) the payoff (at expiration T) of the American put is at least the payoff from the European put, and on the event $\tau < T$ the payoff of the American put is strictly greater than tha of the European put. Since the event $\tau < T$ has positive probability, regardless of the initial price S_0 of Stock, it follows that the expected discounted payoff of the American put, exercised according to the policy described above, is greater than that of the European put.

3.2. **Bermuda Options.** A Bermuda option is one that may be exercised only at one of a finite, discrete set of times.[2] If the set of allowable exercise times is just $\{T\}$ (the expiration time) then the option reduces to the European variety. On the other hand, if the set of allowable exercise times is $\{k\Delta : k = 1, 2, \dots, [T/\Delta]\} \cup \{T\}$ where $\Delta > 0$ is small, then the option approximates the American option. Henceforth, we shall denote by $\mathcal{B}_\Delta$ the Bermuda put option with allowable exercise times $\{k\Delta : k = 1, 2, \dots, [T/\Delta]\} \cup \{T\}$, and by $V(t, S_t; T, \Delta)$ its value at time t.

Proposition 3. *As $\Delta \downarrow 0$, the time-zero value $V(0, S_0; T, \Delta)$ of the Bermuda put option $\mathcal{B}_\Delta$ converges to the value $V_A(0, S_0)$ of the American put option.*

Proof Sketch. Fix $\varepsilon > 0$ small. For all sufficiently small $\Delta > 0$, the probability that the stock price S_t varies by more than ε during any time interval of duration Δ is less than ε. In fact, if Δ is sufficiently small then

$$E \sup_{0 \leq t \leq T-\Delta} \max_{t \leq s \leq t+\Delta} |S_s - S_t| < 2\varepsilon. \tag{7}$$

(Exercise: Prove this!) Now consider any exercise policy (i.e., stopping time) τ for the American put. Define τ_Δ to be the first allowable exercise time of $\mathcal{B}_\Delta$ after τ. If the Bermuda option $\mathcal{B}_\Delta$ is exercised at time τ_*, then the difference in payoff with the American option exercised at τ is at most $|S_{\tau_*} - S_\tau|$. Moreover, since $\tau_* - \tau \leq \Delta$, the appreciation in value of the payoff S_τ by time τ_* is at most $e^{r\Delta}$. Consequently, the difference in the expected discounted values of the payoffs of the American and Bermuda options is at most

$$2\varepsilon e^{r\Delta},$$

by inequality (7).

Bermuda options are of interest in part because they are actually traded but also because their arbitrage prices may be computed numerically by an iterative scheme called *dynamic programming* or *backward induction*.[3] The idea, roughly, is that if one decides *not* to exercise the option at the first possible exercise time, then it is converted to another Bermuda option, but with one fewer possible exercise time. Thus, one may relate the price of the original Bermuda option to that of a Bermuda option with one less allowable exercise time. The price of this option may, similarly, be related to that of yet another Bermuda option with still one less allowable exercise time, and so

[2] Bermuda is a small island in the Atlantic midway between America and Europe. It is also an important banking center. This is not because it is the center of trading activity in Bermudan options, but rather because the U. S. and European governments do not bother to collect taxes on transactions conducted in Bermuda.

[3] This method was introduced in a related but somewhat different context by Bellman in the early 1950s.

on, until finally all prices are related to that of a put option with just one allowable exercise time. Since such an option is necessarily a *European* put, its price is given by the Black-Sholes formula.

Let $V_n(S_0;T)$ be the time-zero price of a Bermuda put option with strike K, expiration T, and allowable exercise times $T/n, 2T/n, \ldots, (n-1)T/n, T$.

Proposition 4. $V_n(s;T) = \max((K-s)_+, e^{-rT/n} EV_{n-1}(S_{T/n}; (n-1)T/n))$.

Proof. Exercise. □

The difficulty in using this for numerical purposes is that the computation of the inner expectation requires, at least in principle, the computation of V_{n-1} for *all* possible values of $S_{T/n}$. In practice, one would discretize the problem. The most efficient discretization involves replacing the Black-Sholes model (1) by a binomial model, where a binary tree can be used for the computation.

Finally, comparison of American, European, and Bermuda options can be used to give another proof of Proposition 2).

Another Proof of Proposition 2. To prove Proposition 2, we shall compare the values of the American and European puts with the value of an intermediate option called a *Bermuda* option. For the purpose of proving Proposition 2, we shall consider two Bermuda put options, denoted by $\mathcal{B}_0$ and $\mathcal{B}_1$, both with strike K and expiration T. The allowable exercise times for $\mathcal{B}_0$ are 0 and T, and the allowable exercise times for $\mathcal{B}_1$ are $0, T/2$, and T. Denote by $U_i(t, S_t)$ the value at time t of the option $\mathcal{B}_i$. Observe that the European, Bermuda, and American options provide their owner(s) successively larger sets of possibilities, and so their values must be ordered as follows:

$$V_E(t) \leq U_0(t) \leq V_A(t). \tag{8}$$

Consider the simpler Bermuda option $\mathcal{B}_0$. At time $t = 0$, the owner of this option has a choice: He/she must either exercise the option immediately, for a payoff $(K - S_0)_+$, or must forego the chance of immediate exercise, in which case the option becomes a standard European put option. Thus, in effect, the Bermuda option $\mathcal{B}_0$ is nothing more than a choice between a European put and a cash payment of $(K - S_0)_+$. Clearly, rational behavior is to choose the more valuable of the two. Which is more valuable? This depends on the current value S_0 of the underlying asset STOCK. If $S_0 \geq K$ then of course immediate exercise gives no payoff, and so it is better to use the option as a European put. On the other hand, if S_0 is sufficiently small, then the value $V_E(0, S_0)$ of the European put may be smaller than $(K - S_0)_+$. Here is why: By put-call parity,

$$V_E(0, S_0) + S_0 = C_K(0, S_0) + Ke^{-rT},$$

where $C_K(t, S_t)$ is the time-t value of a call option on STOCK with expiration $T$and strike K. Thus, on the event $S_0 \leq K$,

$$\begin{aligned} (K - S_0)_+ - V_E(0, S_0) &= K - S_0 - V_E(0, S_0) \\ &= K - Ke^{-rT} - C_K(0, S_0). \end{aligned}$$

But the call price $C_K(0, s)$ converges to 0 as the initial STOCK price $s \downarrow 0$. Thus, for sufficiently small $S_0 = s$, the difference between the payoff $K - s$ for immediate exercise and the value of the European put is *positive*. This argument may be modified to give the optimal exercise policy for $\mathcal{B}_0$ in terms of the initial STOCK price: Since the call price $C_K(0, s)$ is a continuous, decreasing function[4] of s, the Intermediate Value theorem of calculus implies that there must be a *unique* value $s_* = s_*(K;T)$ such that

$$C_K(0, s_*) = K - Ke^{-rT}.$$

[4]If you are unsure of this, look up the Black-Sholes formula and stare at it for a brief spell.

For this value s_*,

$$
\begin{aligned}
s > s_* &\Longrightarrow V_E(0,s) > (K-s)_+ \\
s < s_* &\Longrightarrow V_E(0,s) < (K-s)_+.
\end{aligned} \tag{9}
$$

Thus, the optimal exercise policy is to exercise immediate if $s < s_*$ and to convert the option to a European put if $s > s_*$.

The preceding argument shows that if $S_0 < s_*$ then the value of the Bermuda option $\mathcal{B}_0$ at time $t = 0$ is strictly greater than that of the European put, and hence, by inequality (8), for such initial values s the value of the American put is also strictly greater than that of the European put. In fact, the American put is more valuable than the European put for *all* $t < T$ and all S_t. It suffices to prove this for $t = 0$ and $S_0 = s \geq s_*$. Consider now the second of the Bermuda options, option $\mathcal{B}_1$. The holder of this option at time $t = 0$ has two choices: either immediate exercise, or the option reverts to a Bermuda option of type $\mathcal{B}_0$ with possible exercise times $T/2$ and T. For $s \geq s_*$, immediate exercise is suboptimal, because one would not choose immediate exercise for such s even if the only alternative were to hold until expiration T. Thus, rational behavior for the owner of a $\mathcal{B}_1$ if $S_0 \geq s_*$ is to hold the option at least until the next possible exercise time $T/2$. Now because the price process S_t of the underlying evolves as a geometric Brownian motion, there is positive probability, given any initial value $S_0 = s \geq s_*$, that

$$
S_{T/2} < s_*(K; T/2). \tag{10}
$$

But on this event, the value of the option at time $T/2$ will be strictly greater than the value of the European option, by the result of the preceding paragraph. It follows that the discounted expected value at time $t = 0$ of $\mathcal{B}_1$ is strictly greater than that of the European put. Since the value of the American option is never less than that of $\mathcal{B}_1$, this proves that the American put is always more valuable than the European put at time $t = 0$, regardless of the initial value $S_0 = s$ of the underlying asset STOCK.

4. The Perpetual American Put Option

The exercise boundary for the American put with expiration T does not have a closed form, and it is difficult even to establish simple qualititative properties of the boundary, such as smoothness. To illustrate some of the mathematical techniques that are used in studying optimization problems such as the determination of the optimal exercise policy for the American put, we shall analyze a simpler American option, the *perpetual put option.*[5] The perpetual put works the same way as the American put option, *except* that there is no expiration date. Thus, one may hold it (or pass it along to one's offspring) until the universe collapses into a final black hole, and even beyond.[6] If at some time τ the owner chooses to exercise the option, the payoff is $(K - S_\tau)_+$, where S_τ is the share price of the underlying asset STOCK at the instant of exercise.

Theorem 1. *The optimal policy for the use of the perpetual put with strike K is to exercise at time*

$$
\tau = \inf\{t : S_t \leq s_*\} \tag{11}
$$

where

$$
s_* = \frac{2Kr}{2r + \sigma^2}. \tag{12}
$$

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[5]This was first studied by MERTON (I think).

[6]Thinking in these terms helps one to appreciate why economists believe in discounting future gains.

The value V_t of the perpetual put at time t (on the event that it has not yet been exercised) depends only on the current share price S_t of the underlying asset, and is given by

$$V_t = u(S_t) = K\left\{\frac{K}{S_t}\left(1 - \frac{2r}{2r+\sigma^2}\right)\right\}^{2r/\sigma^2}. \tag{13}$$

The remainder of this section is devoted to the proof of this theorem.

4.1. **Existence of an optimal policy.** One's natural inclination is to presume that there is at least one optimal policy for the exercise of the perpetual put, and in fact there is. However, lest we begin to be too cavalier about such matters, let's look at a perpetual option of a different type for which there is *not* an optimal exercise policy. The option, which we shall call a ZENO option, gives the owner the right to exercise at any time $t > 0$, for a payoff of $(1 - 1/t)$. Assume that the riskless rate of return is $r = 0$. Then the value of the (unexercised) option at any time t is 1, because (a) one can never obtain a payoff greater than 1, and (b) by waiting, one can obtain a payoff arbitrarily close to 1. It is clear that there is no exercise policy that will yield a payoff of 1, and so there is no optimal policy!

4.2. **Time-homogeneity of the value function.** Let's return our attention to the perpetual put option with strike K. The key to the analysis of the option is the time-homogeneity of the problem, which, as we shall argue, implies that the value function depends only on the current stock price. Time-homogeneity follows from the nature of the option itself and the model assumptions: (i) that the riskless rate of return is constant, and (ii) that the stock price evolves as a geometric Brownian motion. In particular, the stock price S_t is given by

$$S_t = S_0 Z_t = S_0 \exp\left\{\sigma W_t + (r - \sigma^2/2)t\right\}, \tag{14}$$

where W_t is, under the risk-neutral probability measure P, a standard Brownian motion. Observe that, because Brownian motion has stationary, independent increments,

$$S_{t+t'} = S_t \tilde{Z}_{t'} \tag{15}$$

where the process $\tilde{Z}_{t'}$ is independent of the $\sigma-$algebra of all events observable by time t, and has the same law as the process $Z_{t'}$. Thus, the future evolution of the stock price, given its value $S_t = s$ at time t, follows the same law as it would have done starting at time 0 if $S_0 = s$.

In discussing the problem of determining the optimal exercise policy, we will find it necessary to compare the value function for different initial values s of the stock price. The geometric Brownian motion model facilitates such comparisons, because by (14) all possible initial values s of the stock price may be used with the *same* fluctuation process Z_t. In particular, observe that exercise policies are just stopping times for the process Z_t, or equivalently W_t (that is, they are stopping times relative to the natural filtration of the driving Brownian motion W_t).

Proposition 5. *Let V_t be the value of the perpetual put option at time t on the event that it has not yet been exercised. Then V_t is a function only of S_t :*

$$V_t = u(S_t). \tag{16}$$

Why should this be true? Suppose that you held the option at time $t = 0$ when the stock price was $S_0 = 17$, and that now, at time t, you still hold it, unexercised, and the stock price is again $S_t = 17$. Because the future evolution $\{S_{t+s}\}_{s\geq 0}$ of the stock price follows the same law, conditional on the event $S_t = 17$, as the process $\{S_s\}_{s\geq 0}$, conditional on $S_0 = 0$, and because the discount rate r is constant in time, there can be no advantage to behaving any differently now (at time t, with $S_t = 17$) than at time 0, when $S_0 = 17$. Therefore, the value of the option must be the same!

A complete, fully rigorous proof of Proposition 5 involves some subtleties that would be more distracting than illuminating at this point. Thus, we shall leave the proof until later.

4.3. **Elementary properties of the value function** $u(s)$**.**

Proposition 6. *The value function* $u(s)$ *is*

(a) strictly positive;
(b) nonincreasing in s*;*
(c) Lipshitz continuous in s*; and*
(d) bounded below by $(K-s)_+$*.*

Proof. All of these are fairly obvious except perhaps (c), which may be explained as follows. Consider two possible values s_1, s_2 for the initial stock price S_0 such that $|s_2 - s_1| < \varepsilon$, where $\varepsilon > 0$ is small. The time evolution of the stock price for the two different initial values is

$$S_t = s_i \exp\left\{\sigma W_t + (r - \sigma^2/2)t\right\} := s_i Z_t \qquad \text{for } S_0 = s_i.$$

Suppose that, given $S_0 = s_2$, you chose to exercise the perpetual put option at exactly the same time τ you would have if the initial value of the stock price had been s_1. Then the magnitude of the difference in payoffs would be

$$|(K - s_1 Z_\tau)_+ - (K - s_2 Z_\tau)_+| \le |s_1 - s_2| Z_\tau \mathbf{1}\left\{Z_\tau \le \max(K/s_1, K/s_2)\right\}.$$

It follows by taking discounted expectation that

$$|u(s_1) - u(s_2)| \le \varepsilon \max(K/s_1, K/s_2).$$

□

In fact, the function $u(s)$ is not only Lipshitz continuous, but is everywhere differentiable. This is more difficult to show, but is of crucial in the analytic approach to determining the optimal exercise policy. More on this later.

4.4. **Monotonicity of the optimal exercise policy.** Why are we so concerned with the value function $u(s)$? Among other reasons is this: it tells you when to exercise! Recall that $u(s) \ge (K-s)_+$ (part (d) of Proposition 6), since you can always cash in your option for a payoff of $(K-s)_+$. If $u(s) > (K-s)_+$, then immediate exercise is suboptimal, because you would be exchanging your option, valued at $u(s)$, for a payment $(K-s)_+$ of smaller value. Thus, if there is an optimal exercise policy, then it will only permit exercise when $u(s) = (K-s)_+$. On the other hand, if $u(s) = (K-s)_+$, then your option is worth exactly what you would obtain by exercising it immediately; since it may (and it fact does, but this requires further argument) lose value if it is not exercised, it is optimal to cash it. Thus, *assuming* that an optimal policy exists, we have found one:

Proposition 7. *An optimal exercise policy for the perpetual put option is to exercise at time* τ*, where*

$$\tau = \inf\{t : u(S_t) = (K - S_t)_+\}. \tag{17}$$

So now the problem is to determine when $u(s) = (K-s)_+$. The next result shows that the set of possible values of the stock price where this equality holds is an *interval* $[0, s_*]$ of real numbers.

Proposition 8. *There exists* $s_* = s_*(K) > 0$ *such that* $s_* \le K$ *and*

$$u(s) = (K-s)_+ \qquad \text{if and only if} \qquad s \le s_*. \tag{18}$$

Proof. We have already noted that $u(s) > 0$ for all $s > 0$ (there is always a cchance of getting a payoff with the option!), so $u(s) = (K - s)_+$ is only possible if $s < K$. Suppose that $s < s' \leq K$ and that $u(s) > (K - s)$; we will show that $u(s') > K - s'$. If $u(s) > (K - s)$ then there must be an exercise policy for which the expected discounted payoff is strictly greater than the payoff for immediate exercise; thus, there is a stopping time τ such that

$$Ee^{-r\tau}(K - sZ_\tau)_+ \mathbf{1}_F > (K - s),$$

where $F$is the event $\{\tau < \infty\}$. Without loss of generality, we may assume that this stopping time is such that $sZ_\tau < K$, because otherwise the exercise policy could be improved by merely waiting until $sZ_t < K/2$. Then we may rewrite the last inequality as

$$\begin{aligned} Ee^{-r\tau}(K - sZ_\tau)\mathbf{1}_F > (K - s) & \qquad \Longrightarrow \\ s(1 - Ee^{-r\tau}\mathbf{1}_F) > K - KEe^{-r\tau}\mathbf{1}_F & \qquad \Longrightarrow \\ s'(1 - Ee^{-r\tau}\mathbf{1}_F) > K - KEe^{-r\tau}\mathbf{1}_F & \qquad \Longrightarrow \\ Ee^{-r\tau}(K - s'Z_\tau)\mathbf{1}_F > (K - s'). & \end{aligned}$$

But the final inequality shows that if the initial stock price is $S_0 = s'$, the policy τ also gives an expected discounted payoff strictly greater than the payoff for immediate exercise, and so $u(s') > (K - s')_+$. □

Corollary 1. *An optimal exercise policy is to exercise at time*

(19) $$\tau = \inf\{t : S_t = s_*\}.$$

4.5. **Determination of s_*.** We now have a nearly complete description of the optimal exercise policy: by Corollary 1, it is optimal to exercise at the first passage time of the stock price process to the level s_*. Only the value of s_* remains to be calculated. But this may be approached as an elementary maximization problem: We can calculate the expected discounted payoff for all possible levels x_*, then maximize over s_*.

Denote by P^s and E^s the risk-neutral probability and expectation operators when the initial stock price is $S_0 = s$.

Lemma 2. *For all $s \geq s_*$,*

(20) $$E^s \exp\{-r\tau\}\,\mathbf{1}\{\tau < \infty\} = \left(\frac{s_*}{s}\right)^{2r/\sigma^2}$$

Proof. By equation (1), the discounted stock price process is given by

(21) $$e^{-rt}S_t = s\exp\left\{\sigma W_t - \sigma^2 t/2\right\},$$

which the reader will recognize as a Cameron-Martin likelihood ratio. By the Cameron-Martin theorem, the probability measure P_σ gotten by "tilting" the risk-neutral measure P^s by the likelihood ratio $\exp\left\{\sigma W_t - \sigma^2 t/2\right\}$ makes the process W_t a Wiener process with drift σ. Since

$$\tau = \inf\{t : S_t = s_*\} = \inf\left\{t : \sigma W_t - \sigma^2 t/2 = \log(s_*/s)\right\},$$

we have

$$\begin{aligned} s_* E^s e^{-r\tau}\mathbf{1}\{\tau < \infty\} &= sE^s \exp\left\{\sigma W_t - \sigma^2 t/2\right\}\mathbf{1}\{\tau < \infty\} \\ &= sP_\sigma\left\{\sigma W_t - \sigma^2 t/2 \text{ ever hits } \log(s_*/s)\right\} \\ &= s\left(\frac{s_*}{s}\right)^{1+2r/\sigma^2}, \end{aligned}$$

by Lemma 3 below. □

Lemma 3. *Let W_t be a standard Wiener process under Q. Then for any alpha < 0 and $\beta > 0$*

$$Q\{W_t + \beta t = \alpha \textit{ for some } t\} = e^{2\beta\alpha}. \tag{22}$$

Proof. Read the Cameron-Martin lecture again. □

Now consider the expected discounted payoff of the perpetual put option when exercised according to the policy (19). There is no reason to consider values of $s_* \geq K$, because if the policy (19) were used with $s_* \geq K$ then the payoff would be zero. So consider values of $s_* < K$, and consider the expected discounted payoff of the perpetual put when the initial value of the stock price is $S_0 = s \geq K$. By Lemma 2, this expected discounted payoff is

$$\begin{aligned} E^s(K - S_\tau)_+ e^{-r\tau}\mathbf{1}\{\tau < \infty\} &= (K - s_*)E^s e^{-r\tau}\mathbf{1}\{\tau < \infty\} \\ &= (K - s_*)\left(\frac{s_*}{s}\right)^{2r/\sigma^2}. \end{aligned} \tag{23}$$

It is an elementary exercise in (ordinary) calculus to find the value of s_* that maximizes this expression: it is

$$\boxed{s_* = \frac{2Kr}{2r + \sigma^2}} \tag{24}$$

The value function may now be obtained by substituting this value in the expression (23). The result is the formula (13) given in the statement of Theorem 1.