LECTURE 7: BLACK–SCHOLES THEORY

1. Introduction: The Black–Scholes Model

In 1973 Fisher Black and Myron Scholes ushered in the modern era of derivative securities with a seminal paper[1] on the pricing and hedging of (European) call and put options. In this paper the famous Black-Scholes formula made its debut, and the Itô calculus was unleashed upon the world of finance.[2] In this lecture we shall explain the Black-Scholes argument in its original setting, the pricing and hedging of European contingent claims. In subsequent lectures, we will see how to use the Black–Scholes model in conjunction with the Itô calculus to price and hedge all manner of exotic derivative securities.

In its simplest form, the Black–Scholes(–Merton) model involves only two underlying assets, a riskless asset Cash Bond and a risky asset Stock.[3] The asset Cash Bond appreciates at the *short rate*, or *riskless rate of return* r_t, which (at least for now) is assumed to be *nonrandom*, although possibly time–varying. Thus, the price B_t of the Cash Bond at time t is assumed to satisfy the differential equation

$$\frac{dB_t}{dt} = r_t B_t, \tag{1}$$

whose unique solution for the value $B_0 = 1$ is (as the reader will now check)

$$B_t = \exp\left(\int_0^t r_s\, ds\right). \tag{2}$$

The share price S_t of the risky asset Stock at time t is assumed to follow a stochastic differential equation (SDE) of the form

$$dS_t = \mu_t S_t\, dt + \sigma S_t\, dW_t, \tag{3}$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, μ_t is a nonrandom (but not necessarily constant) function of t, and $\sigma > 0$ is a constant called the *volatility* of the Stock.

Proposition 1. *If the drift coefficient function μ_t is bounded, then the SDE (3) has a unique solution with initial condition S_0, and it is given by*

$$S_t = S_0 \exp\left(\sigma W_t - \sigma^2 (t/2) + \int_0^t \mu_s\, ds\right) \tag{4}$$

Moreover, under the risk–neutral measure, it must be the case that

$$r_t = \mu_t. \tag{5}$$

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[1] "The pricing of options and corporate liabilities" in *Journal of Political Economy*, volume 81, pages 637–654

[2] In fact, the use of the Wiener process in financial models dates back to the early years of the 20th century, in Bachelier's thesis. However, the success of the Black-Scholes model assured that the Itô calculus would have a permanent place in the world of mathematical finance.

[3] Keep in mind that the designation of a riskless asset is somewhat arbitrary. Recall that *any* tradeable asset whose value at the termination time T is strictly positive may be used as *numeraire*, in which case it becomes the riskless asset, with rate of return $r = 0$. This is the principle of *numeraire invariance*. Note, however, that the equilibrium measure depends on the choice of riskless asset. More on this when we talk about the **Cameron–Martin–Girsanov theorem** later in the course.

Proof. As in many arguments to follow, the magical incantation is "Itô's formula". Consider first the formula (4) for the share price of STOCK; to see that this defines a solution to the SDE (3), apply Itô's formula to the function

$$u(x,t) = \exp\left(\sigma x - \sigma^2(t/2) + \int_0^t \mu_s \, ds\right).$$

To see that the solution is unique, check your favorite reference on the theory of SDEs. Finally, to see that the drift coefficient μ_t must coincide with the riskless rate of return r_t in the risk–neutral world, recall that under the risk–neutral measure the *discounted* share price of STOCK must be a martingale. The appropriate discount factor is B_t, so the discounted share price of STOCK is

$$S_t^* = S_t/B_t = S_0 \exp\left(\sigma W_t - (\sigma^2 t/2) + \int_0^t (\mu_s - r_s) \, ds\right).$$

Applying Itô's formula once more, one finds that

$$dS_t^* = \sigma S_t^* \, dW_t + S_t^*(\mu_t - r_t)dt.$$

In order that S_t^* be a martingale, it is necessary that the dt term be 0; this implies that $\mu_t = r_t$. □

Corollary 1. *Under the risk–neutral measure, the log of the discounted stock price at time t is normally distributed with mean* $\log S_0 - \sigma^2 t/2$ *and variance* $\sigma^2 T$.

2. The Black–Scholes Formula for the Price of a European Call Option

Recall that a European CALL on the asset STOCK with strike K and expiration date T is a contract that allows the owner to purchase one share of STOCK at price K at time T. Thus, the value of the CALL at time T is $(S_T - K)_+$. According to the Fundamental Theorem of Arbitrage Pricing,[4] the price of the asset CALL at time $t = 0$ must be the discounted expectation, under the risk–neutral measure, of the value at time $t = T$, which, by Proposition 1, is

$$C(S_0, 0) = C(S_0, 0; K, T) = E(S_T^* - K/B_T)_+ \tag{6}$$

where S_T^* has the distribution specified in Corollary 1. A routine calculation, using integration by parts, shows that $C(x, T; K)$ may be rewritten as

$$\boxed{C(x, 0; K, T) = x\Phi(z) - \frac{K}{B_T}\Phi(z - \sigma\sqrt{T}),} \tag{7}$$

where

$$z = \frac{\log(xB_t/K) + \sigma^2 t/2}{\sigma\sqrt{T}}$$

and Φ is the cumulative normal distribution function.

And that's all there is to the Black–Scholes formula! There are other derivations, some with pedagogical merit, others just long and painful, most based somehow on the Itô formula. For the cultural enlightenment of the reader, we shall present another argument, based on PDE theory, in the next section. Before we leave this derivation, though, we should observe that it works essentially unchanged for *all* European–style derivative securities. Let $F : \mathbb{R} \to \mathbb{R}$ be a function of polynomial

[4]We only proved this in the case of discrete–time, finite–scenario markets. However, as we mentioned at the time, the theorem is true in much greater generality, and the essential idea of the proof remains the same.

growth, and consider the derivative security whose value at $t = T$ is $F(S_T)$. Then the value at time $t = 0$ of this derivative security is

$$V_0 = EB_T^{-1}F(B_T S_T^*), \tag{8}$$

where S_T^* has the lognormal distribution specified in Corollary 1 above.

3. The Black–Scholes PDE

Next, another derivation of the Black–Scholes formula. This one proceeds by finding a PDE for the price function $C(x,T) = C(x,T;K)$ and then verifying that the function defined by (7) solves the PDE. It has the disadvantage that the issues of *uniqueness* and *smoothness* of solutions to the PDE must be tackled separately (and we won't do this here). For the sake of simplicity, we shall only consider the case where the short rate is constant, that is, $r_t \equiv r$.

Let $C(S_t, t)$ be the price of the Call at time t when the share price of Stock is S_t. By the Fundamental Theorem of Arbitrage Pricing, the discounted Call price must be a martingale under the risk–neutral measure. Now the Itô formula[5] implies that the Call price must satisfy the SDE

$$\begin{aligned} B_t d(B_t^{-1}C(S_t,t)) \\ &= C_x(S_t,t)\,dS_t + \frac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2\,dt + C_t(S_t,t)\,dt - r_t C(S_t,t)\,dt \\ &= C_x(S_t,t)\sigma S_t\,dW_t + \{C_x(S_t,t)r_t S_t + \frac{\sigma^2 S_t^2}{2}C_{xx}(S_t,t) + C_t(S_t,t) - r_t C(S_t,t)\}\,dt. \end{aligned} \tag{9}$$

Here C_x, C_{xx}, C_t, etc. represent partial derivatives of the call price function $C(x,t)$ with respect to the indicated variables. Multiplying by the discount factor B_t^{-1}, one sees that the discounted Call price can be a martingale only if the dt term on the right side vanishes. Thus, the call price function $C(x,t)$ must satisfy the **Black–Scholes PDE**:

$$\boxed{-r_t C(x,t) + C_t(x,t) + r_t x C_x(x,t) + \frac{\sigma^2 x^2}{2}C_{xx}(x,t) = 0} \tag{10}$$

with the terminal condition

$$C(x,T) = (x-K)_+. \tag{11}$$

It may now be verified by differentiation that the function defined by the Black–Scholes formula (7) solves the Black–Scholes PDE (10), and converges to the terminal value as $t \to T-$. This isn't an especially enlightening way to spend one's time. What might make a nice Exercise, though, is to check that (7) solves the Black–Scholes PDE (10) by using a computer package that does symbolic differentiation.

Observe that nowhere in this argument did we use the specific form of the terminal payoff, except in determining the terminal condition (11). Thus, the argument applies, unchanged, to the price function $C(x,t)$ for *any* derivative security whose terminal payoff is a function of the terminal Stock price S_T; and so the Black–Scholes PDE (10) must hold for the price function of any such derivative security.

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[5]Notice that, to use the Itô formula, we must know *a priori* that the function $C(x,t)$ is twice differentiable in x and once differentiable in t. But we don't know this! So the argument of this section does not give a complete proof of the Black–Scholes formula. However, since we already know that the Black–Scholes formula is true, by the argument of the preceding section, we know that $C(x,t)$ is infinitely differentiable; hence, the argument of this section does at least give a proof of the Black–Scholes PDE (10).

4. Hedging in Continuous Time

Can one *hedge* a call option in the traded assets Cash Bond and Stock? If so, how does one do it? The answer comes by examination of the SDE (9) satisfied by the call price function $C(S_t, t)$. To obtain this SDE, we once again invoke the Itô formula to get

$$
\begin{aligned}
dC(S_t,t) &= C_x(S_t,t)\,dS_t + \left(C_t(S_t,t) + \frac{\sigma^2 S_t^2}{2} C_{xx}(S_t,t)\right) dt \\
&= C_x(S_t,t)\,dS_t + \left(-r_t S_t C_x(S_t,t) + r_t C(S_t,t)\right) dt \\
&= C_x(S_t,t)\,dS_t + \left(-\frac{S_t C_x(S_t,t)}{B_t} + \frac{C(S_t,t)}{B_t}\right) dB_t
\end{aligned} \tag{12}
$$

Observe that the second equality follows from the first because the call price function $C(x,t)$ must satisfy the Black–Scholes PDE (10); and the third equality follows from the second because the Cash Bond price B_t satisfies the ODE (1).

Equation (12) shows that the instantaneous fluctuation in the price of the Call at any time t is a linear combination of the instantaneous fluctuations in the share prices of Stock and Cash Bond. The coefficients in this linear combination are expressions involving the function $C(x,t)$ and its first partial derivative $C_x(x,t)$; since the call price function $C(x,t)$ is explicitly given by the Black–Scholes formula (7), these coefficients may be computed to any desired degree of accuracy, at any time t. Thus, the formula (12) tells us how to *replicate* a European Call by holding a time–dependent portfolio in Cash Bond and Stock:

Hedging Strategy: *At time $t \le T$, hold*

$$
\begin{array}{ll}
C_x(S_t,t) & \textit{shares of } \text{Stock } \textit{and} \\
(-S_t C_x(S_t,t) + C(S_t,t))/B_t & \textit{shares of } \text{Cash Bond.}
\end{array} \tag{13}
$$

There is only one problem[6]: how do we know that, if we start with $C(S_0, 0)$ dollars at time $t = 0$ and invest it according to the Hedging Strategy, we will have enough assets at time $t > 0$ to buy the number of shares of Stock and Cash Bond required? This is, after all, what we would like a hedging strategy to do: if someone pays us $C(S_0, 0)$ dollars[7] at time $t = 0$ for a Call option, we would like to arrange things so that there is *absolutely no risk* to us of having to pump in any of our own money later to cover the Call at expiration T.

Definition 1. *A* **portfolio** *in the assets* Cash Bond *and* Stock *consists of a pair of adapted processes $\{\alpha_t\}_{0\le t\le T}$ and $\{\beta_t\}_{0\le t\le T}$, representing the number of shares of* Cash Bond *and* Stock *that are owned (or shorted) at times $0 \le t \le T$. The portfolio is said to be* **self–financing** *if, with probability 1, for every $t \in [0, T]$,*

$$
\alpha_t B_t + \beta_t S_t = \alpha_0 B_0 + \beta_0 S_0 + \int_0^t \alpha_s\,dB_s + \int_0^t \beta_s\,dS_s \tag{14}
$$

A portfolio $\{(\alpha_t, \beta_t)\}_{0\le t\le T}$ **replicates** *a derivative security whose value at $t = T$ is V_T if, with probability 1,*

$$
V_T = \alpha_T B_T + \beta_T S_T. \tag{15}
$$

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[6]Actually, there is another problem: the Hedging Strategy entails buying and selling infinitely many shares of Stock between times $t = 0$ and $t = T$, because of the wild fluctuations in the Stock price. If there were transaction costs for buying and selling shares of Stock, the Hedging Strategy would be quite expensive to follow! More on this later.

[7]plus a transaction fee — that's how we make a living

The equation (14) states that the value $\alpha_t B_t + \beta_t S_t$ of the portfolio at any time t should be the initial value $\alpha_0 B_0 + \beta_0 S_0$ plus the accumulated changes in value due to fluctuations dB_s and dS_s in the values of the assets held at times up to t. A *self–financing* portfolio that replicates a derivative security is called a **hedge** or a **hedging strategy**.

Proposition 2. *The portfolio defined by equations (13) is a hedge for the* CALL.

Proof. The strategy (13) specifies the portfolio

$$\beta_t = C_x(S_t, t) \qquad \text{and} \tag{16}$$

$$\alpha_t = (-S_t C_x(S_t, t) + C(S_t, t))/B_t \tag{17}$$

The value of this portfolio at any time $t \leq T$ is

$$\alpha_t B_t + \beta_t S_t = (-S_t C_x(S_t, t) + C(S_t, t)) + C_x(x, t) S_t = C(S_t, t). \tag{18}$$

In particular, setting $t = T$, one sees that the portfolio replicates the CALL. To see that the portfolio is self–financing, integrate the stochastic differential equation (12) for the value of the call to obtain

$$C(S_t, t) = C(S_0, 0) + \int_0^t \alpha_s \, dB_s + \int_0^t \beta_s \, dS_s. \tag{19}$$

Substituting $C(S_t, t) = \alpha_t B_t + \beta_t S_t$ and $C(S_0, 0) = \alpha_0 B_0 + \beta_0 S_0$, by (18), yields (14). □

5. The Arbitrage Argument

It is possible to avoid the use of the Fundamental Theorem and risk–neutral measures in the derivation of the Black–Scholes formula altogether by resorting to an arbitrage argument. Although this argument is somewhat circuitous, it is instructive.

Assume that the price process of the CASH BOND obeys (1), and that the share price process of STOCK obeys the SDE (3).[8] Since we have not assumed that the underlying probability is risk–neutral, it is no longer necessary that $\mu_t = r_t$.

Consider the problem of pricing a derivative security whose payoff at expiration $t = T$ is $F(S_T)$, for some (measurable) function F of polynomial growth. Let V_0 be the $t = 0$ price of this derivative security.

Theorem 1. *Let $C(x, t)$, for $0 \leq t \leq T$ and $x \in \mathbb{R}_+$, be the unique*[9] *solution of the Black–Scholes PDE (10) that satisfies the terminal condition $C(x, T) = F(x)$. In the absence of arbitrage,*

$$V_0 = C(S_0, 0). \tag{20}$$

Note: The Black–Scholes PDE does not involve the drift coefficient μ_t that appears in the SDE 3), and so the function $C(x, t)$ does not depend in any way on the function μ_t. The volatility parameter σ and the short rate r_t do influence $C(x, t)$, but unlike μ_t these parameters are "observable" (in particular, the volatility σ is determined by the quadratic variation of the STOCK price process). Thus, two different investors could have two different drift functions in their models for the STOCK price process, but, according to the theorem, would price the derivative security *the same way.*

[8]Since we do *not* assume the existence of a risk–neutral measure, the underlying probability measure implied by the SDE (3) must be given a new interpretation. It seems that most authors implicitly take this probability measure to be "subjective", that is, an expression of the trader's prior opinions about the future behavior of the STOCK price.

[9]Uniqueness of solutions is proved by the *maximum principle.* Consult a book on the theory of PDEs for this.

Proof. (Sketch.) Define a portfolio $(\alpha_t, \beta_t)_{0 \le t \le T}$ by equations (16)–(17). By the same argument as in the proof of Proposition 2, the portfolio (α_t, β_t) replicates the derivative security, since the terminal condition assures that $C(x,T) = F(x)$. Moreover, the Itô formula applies, as earlier, to give the SDE (12)—here we use the assumption that the function $C(x,t)$ satisfies the Black–Scholes PDE. Now (12) may be integrated as in the proof of Proposition 2 to give (19), which implies, again by the same argument as in the proof of Proposition 2, to show that the portfolio (α_t, β_t) is self–financing. Thus, the portfolio (α_t, β_t) hedges the derivative security.

Now the arbitrage. Suppose that $V_0 < C(S_0, 0)$; in this case, one would at time $t = 0$

- buy 1 derivative security;
- short the portfolio (α_0, β_0); and
- invest the proceeds $C(S_0, 0) - V_0$ in Cash Bond.

In the trading period $0 < t < T$ one would dynamically update the short position $-(\alpha_0, \beta_0)$ to $-(\alpha_t, \beta_t)$, so that at the expiration time $t = T$ the short position would cancel the payoff from the 1 derivative security bought at $t = 0$. Note that no further infusion of assets would be needed for this dynamic updating, because the portfolio (α_0, β_0) is self–financing. At the expiration time $t = T$, the net proceeds would be

$$(C(S_0, 0) - V_0)B_T > 0,$$

with probability one. Since the position at time $t = 0$ was flat, this is an arbitrage. Consequently, V_0 cannot be less than $C(S_0, 0)$. The same argument, "reversed", shows that V_0 cannot be more than $C(S_0, 0)$. □

6. Exercises

In problems 1–2, assume that there are tradable assets Cash Bond and Stock whose price processes obey the differential equations

$$dB_t = r_t B_t \, dt \qquad \text{and} \tag{21}$$

$$dS_t = S_t(r_t \, dt + \sigma \, dW_t), \tag{22}$$

where W_t is a standard Wiener process (Brownian motion) and r_t is the "short rate". Assume that the "short rate" r_t is a continuous, non-random function of t.

Problem 1: Consider a contingent claim that pays S_T^n at time T, where n is a positive integer.

(A) Show that the value of this contingent claim at time $t \le T$ is

$$h(t,T)S_t^n$$

for some function h of (t,T). Hint: Use the fact that the process S_t is given by (4). You should not need the Itô formula if you start from (4).

(B) Derive an ordinary differential equation for $h(t,T)$ in the variable t, and solve it. Hint: Your ordinary differential equation should be first-order, and it should involve only the short rate r_t.

Problem 2: Let $C(S_t, t; K, T)$ be the price at time t of a European call option on the tradable asset (S_t) with strike price K and exercise time T.

(A) Show that the price function C satisfies the following symmetry properties:for any positive constant a,

$$C(S, t; K, T) = C(S, 0; K, T - t) \tag{23}$$

$$C(aS, t; aK, T) = aC(S, t; K, T) \qquad \forall\, a > 0. \tag{24}$$

(B) Use the result of part (A) to derive an identity relating the partial derivatives C_S and C_K.

(C) Find a PDE in the variables K, T for the function $C(x, 0; K, T)$. (The equation should involve first and second partial derivatives.)