SCHUR MODULES AND SEGRE VARIETIES

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1. Introduction

This paper is an elementary introduction to the methods of Landsberg and Manivel, [3] for finding the ideals of secant varieties to Segre varieties. We cover only the most basic topics from [3], but hope that since this is a topic which is rarely made explicit, these notes will be of some use. We assume the reader is familiar with the basic operations of multilinear algebra: tensor, symmetric, and wedge products. For background on these topics, see [1, Appendix B].

For problems in algebraic statistics [2] and linear algebra [4] it is important to determine the ideals of secant varieties of Segre varieties.

A secant variety is defined as follows. Let X be a projective variety. Then the $r-1$-st secant variety to X, denoted $\sigma_r(X)$, is the algebraic closure of the set of secant $\mathbb{P}^{r-1}$'s to X. For example, if $r = 2$, this is the set of all lines passing through 2 points on X.

We are interested in the case of the Segre variety, $X = \mathrm{Seg}(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_k^*)$. (For simplicity of notation, since we are primarily concerned with the ideal of X, we will write X as the product of the dual spaces.) Notice that $GL(A_1) \times \cdots \times GL(A_k)$ acts on X as well as on the space of degree d polynomials, $S^d(A_1 \otimes \cdots \otimes A_K)$. Therefore we can decompose this space into irreducible representations of $GL(A_1) \times \cdots \times GL(A_k)$.

In order to do this, we shall first describe the irreps of $GL(V)$. Then we use these irreps to give a decomposition of the homogeneous parts of the ideals of secant varieties of Segre varieties. Finally, we show how to explicitly construct this decomposition for the case $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1)$.

2. Irreducible representations of $GL(V)$

First, a little background on partitions. We say that $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$ is a partition of d with length k if $\lambda_1 + \cdots + \lambda_k = d$. The Young diagram of λ is the collection of unit boxes, where the first row has λ_1 boxes, up through the k-th row with λ_k boxes. The dual partition to λ is constructed by flipping the young diagram of λ so that columns and rows are switched. Alternatively, it is the partition of d with $\lambda_1 - \lambda_2$ parts of size 1, $\lambda_2 - \lambda_3$ parts of size 2, etc.

Irreducible representations of $GL(V)$ are indexed by partitions λ. The irreps are called *Schur modules* and are denoted by $S_\lambda V$. A good reference for this material is [1, Chapters 6 and 15]. Suppose λ is a partition of d. Then a basic, (but not terribly useful), definition of $S_\lambda V$ is as the space of S_d equivariant maps

$$S_\lambda V = \mathrm{Hom}_{S_d}([\lambda], V^{\otimes d}),$$

where $[\lambda]$ is the irrep of S_d associated to λ.

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FIGURE 1. Calculating the dimension of $S_{311}V$

Two examples are well known. First, $S_d V = S^d V$. Second, if $\lambda = (1, \ldots, 1)$ consists of d ones, then $S_\lambda V = \wedge^d V$. As a generalization of the second example, if λ has more parts than $\dim V$, then $S_\lambda V = 0$.

Let λ be a partition of d, and suppose that the dual partition μ has a_1 parts of size 1, up through a_k parts of size k. Then the irrep $S_\lambda V$ naturally lives in

$$S_\lambda V \subset S^{a_1}(V) \otimes S^{a_2}(\wedge^2 V) \otimes \cdots \otimes S^{a_k}(\wedge^k V) \subset V^{\otimes d}. \tag{1}$$

We will use this fact in Section 4 to explicitly construct the irreps $S_\lambda V$.

Now we describe how to compute the dimension of $S_\lambda V$. Let $g \in GL(V)$ be a linear operator with eigenvalues $x_1, \ldots, x_n$. The character of the action of g on $S_\lambda V$ is given by the Schur polynomial ([1, Theorem 6.3]),

$$\chi_{S_\lambda V}(g) = \mathbb{S}_\lambda(x_1, \ldots, x_n).$$

As a side remark, this implies that multiplication of Schur modules follows the Littlewood-Richardson rule. The only fact from this that we will use is that in the tensor $S_\lambda V \otimes S_\mu V$, the only modules that appear are $S_\nu V$ where ν has at least as many parts as λ and μ.

The dimension of $S_\lambda V$ is obtained by evaluating the Schur polynomial at $(1, \ldots, 1)$. The following classical formula [1, Exercise 6.4] gives an easy way to calculate the dimension. Suppose $\dim V = k$, then

$$\dim S_\lambda V = \prod \frac{k - i + j}{h_{ij}},$$

where the product is over all boxes in the Young diagram of λ, and h_{ij} is the *hook length* of λ at i, j

Example 1. *Let* $\lambda = (3, 1, 1)$ *(as shown in Figure 1), and let* $\dim V = k$. *Then expanding the above product down the columns of the diagram gives*

$$\begin{aligned} \dim S_{311} V &= \frac{k-1+1}{5} \cdot \frac{k-2+1}{2} \cdot \frac{k-3+1}{1} \cdot \frac{k-1+2}{2} \cdot \frac{k-1+3}{1} \\ &= \frac{(k+2)(k+1)k(k-1)(k-2)}{20}. \end{aligned}$$

In particular if $k = 4$, *we have* $\dim S_{311}\mathbb{C}^4 = 36$.

Example 2. *A similar calculation for* $S_{2111}V$ *gives*

$$\dim S_{2111} V = \frac{(k+1)k(k-1)(k-2)(k-3)}{30}.$$

Thus $\dim S_{2111}\mathbb{C}^4 = 4$.

Example 3. *Let $A = B = C = \mathbb{C}^4$. In* [3, Proposition 6.5]*, it is stated that*

$$\begin{aligned} I_5(\sigma_4(\mathrm{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))) = & S_{311}A \otimes S_{2111}B \otimes S_{2111}C \\ & \oplus S_{2111}A \otimes S_{311}B \otimes S_{2111}C \\ & \oplus S_{2111}A \otimes S_{2111}B \otimes S_{311}C, \end{aligned}$$

is of dimension 1728. We can now check this calculation. By Examples 1 and 2, $\dim S_{311}\mathbb{C}^4 = 36$ *and* $\dim S_{2111}\mathbb{C}^4 = 4$*. Therefore the dimension is* $36 \cdot 4 \cdot 4 + 4 \cdot 36 \cdot 4 + 4 \cdot 4 \cdot 36 = 1728$.

Furthermore, Landsberg and Manivel show that $I_9(\sigma_4(\mathrm{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)))$ *contains* $S_{333}A \otimes S_{333}B \otimes S_{333}C$*. By our remark earlier about the Littlewood-Richardson rule, this module can't appear in the ideal generated by* I_5 *since* I_5 *contains partitions with 4 parts and this has only 3 parts. Therefore* $I(\sigma_4(\mathrm{Seg}(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)))$ *is not generated in degree 5.*

3. Decomposition into irreps

Now that we know what the irreps of $GL(V)$ look like, we can decompose $S^d(A_1 \otimes \cdots \otimes A_K)$ (the space of degree d polynomials) into $GL(A_1) \times \ldots GL(A_k)$ irreps. The following is a straightforward usage of Schur duality ([3, Proposition 4.1]).

Theorem 1.

$$S^d(A_1 \otimes \cdots \otimes A_k) = \bigoplus_{|\lambda_1| = \cdots = |\lambda_k| = d} ([\lambda_1] \otimes \cdots \otimes [\lambda_k])^{S_d} S_{\lambda_1}A_1 \otimes \cdots \otimes S_{\lambda_k}A_k,$$

where $([\lambda_1] \otimes \cdots \otimes [\lambda_k])^{S_d}$ *denotes the space of instances of the trivial representation in the tensor product.*

Now we have all the tools to describe the basic algorithm of [3]. To find the degree d part of the ideal of $\mathrm{Seg}(\mathbb{P}A_1^* \times \cdots \times \mathbb{P}A_k^*)$,

1. First compute the decomposition of $S^d(V)$, where $V = A_1 \otimes \cdots \otimes A_k$.
2. Next, for each element $S_{\lambda_1} \otimes \cdots \otimes S_{\lambda_k}$ in this decomposition, write down a (non-zero) polynomial.
3. Test if this polynomial vanishes on the variety, either symbolicly (using the parameterization) or by picking several random points and evaluating numerically.

We will give explicit examples of this algorithm for the case $\mathbb{P}^1 \times \mathbb{P}^1$.

As a side remark, it is a well known fact that the number of partitions of d, denoted $p(d)$ grows as

$$p(d) \approx \frac{1}{4d\sqrt{3}} e^{\pi\sqrt{2d/3}}.$$

Therefore, this method will not be effective for large d.

4. The ideal of the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^1$

Our running example through the rest of the paper will be $A = B = \mathbb{C}^2$, $X = \mathrm{Seg}(\mathbb{P}A^* \times \mathbb{P}B^*)$. In this case the first decompositions from Theorem 1 are

$$S^2(A \otimes B) = S^2A \otimes S^2B \oplus \wedge^2 A \otimes \wedge^2 B \tag{2}$$

and

$$S^3(A \otimes B) = S^3A \otimes S^3B \oplus S_{21}A \otimes S_{21}B \oplus \wedge^3 A \otimes \wedge^3 B. \tag{3}$$

Of course, we know that $I(X)$ is the principal ideal generated by the determinant $x_{11}x_{22} - x_{12}x_{21}$ and thus $I_2(X)$ is one-dimensional. Looking at the decomposition (2), we see that

$$I(X) = \wedge^2 A \otimes \wedge^2 B.$$

However, let's pretend that we don't know this fact and try to use our algorithm.

First, we have to construct polynomials from the two modules in (2). We construct these modules by first embedding them in $A^{\otimes 2} \otimes B^{\otimes 2}$ and then using the natural bijection with $(A \otimes B)^{\otimes 2}$. We use the notation $x_{ij} = a_i \otimes b_j$.

For example, the element $(a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$ maps to $(a_1 \otimes a_2 - a_2 \otimes a_1) \otimes (b_1 \otimes b_2 - b_2 \otimes b_1)$. Multiplying out, we get $x_{11} \otimes x_{22} - x_{21} \otimes x_{12} - x_{12} \otimes x_{21} + x_{22} \otimes x_{11}$. In $S^2(A \otimes B)$, this is just the element $x_{11}x_{22} - x_{12}x_{21}$, the determinant.

Now we can check that this polynomial vanishes, for example symbolically using the parameterization of $X = \mathbb{P}^1 \otimes \mathbb{P}^1$, $x_{ij} = a_i b_j$. Therefore $I_2(X) \supset S_{11}A \otimes S_{11}B$.

A similar process for the second module in (2), takes (for example) the element $a_1^2 \otimes b_1 b_2 \in S^2 A \otimes S^2 B$ to the polynomial $2(x_{11}x_{12})$ which doesn't vanish on X. Therefore we know $I_2(X)$ contains only the determinant.

We continue with (3) for the degree 3 polynomials. We can simplify (3) by noticing that the wedge products are zero, so we have

$$S^3(A \otimes B) = S^3 A \otimes S^3 B \oplus S_{21}A \otimes S_{21}B \tag{4}$$

The dimension of $S^3 A \otimes S^3 B$ is 16 and the dimension of $S_{21}A \otimes S_{21}B$ is 4. Notice that $S^3 A \otimes S^3 B$ can not be contained in the ideal generated by $I_2(X)$ for two reasons. First, the degree 3 part could have at most dimension 4, not 16. Second, the partitions associated to $I_2(X)$ have length 2, while the partitions associated with $S^3 A \otimes S^3 B$ have length 1.

Since we know that the determinant is the only generator, we can rule out $S^3 A \otimes S^3 B$, however, lets check a typical polynomial from it. Take $a_1^2 a_2 \otimes b_1 b_2^2 \in S^3 A \otimes S^3 B$. The first term becomes $2a_1 \otimes a_1 \otimes a_2 + 2a_1 \otimes a_2 \otimes a_1 + 2a_2 \otimes a_1 \otimes a_1$, and after multiplying and collecting terms we are left with $4x_{11}x_{12}x_{22} + 2x_{12}^2 x_{21}$. This doesn't vanish on X.

Finally, we have to construct a non-trivial Schur module, namely $S_{21}A \otimes S_{21}B$. By (1), we can construct $S_{21}A$ by letting

$$F_A = a_1 \otimes (a_1 \wedge a_2).$$

Then as $a_1, a_2 \in A$ vary, the subspace of $A^{\otimes 3}$ generated by F_A is a copy of $S_{21}A$.

Since $S_{21}A$ is 2-dimensional, we see that the elements $a_1 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_1$ and $a_2 \otimes a_2 \otimes a_1 - a_2 \otimes a_1 \otimes a_2$, where a_1, a_2 is a basis of A, generate $S_{21}A$.

Now we multiply them by the corresponding elements of $S_{21}B$. For example, $(a_1 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_1) \otimes (b_1 \otimes b_1 \otimes b_2 - b_1 \otimes b_2 \otimes b_1)$ is

$$x_{11} \otimes x_{11} \otimes x_{22} - x_{11} \otimes x_{12} \otimes x_{21} - x_{11} \otimes x_{21} \otimes x_{12} + x_{11} \otimes x_{22} \otimes x_{11} \in (A \otimes B)^{\otimes 3}$$

However, we have to make sure at the end that we land inside $S^3(A \otimes B)$, instead of just in the tensor product, so we have to average over S_3. In the end, we just get the polynomial $x_{11}(x_{11}x_{22} - x_{12}x_{21})$, which is just x_{11} times the determinant. The other combinations give similar results.

Combining the results for the two terms on the right hand side of (4), we see that there are no new generators needed of degree 3 in the ideal $I(x)$. As we see, a notable drawback of this algorithm is that it just goes degree by degree. However, for many ideals where other techniques fail, this may be the best way to partially describe the ideal.

References

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