CLIFFORD ALGEBRAS AND SPIN GROUPS

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1. Introduction

Among the classical groups, the special orthogonal group is unique because it is the only one that is not simply connected. This fact leads to difficulty when one studies the representation theory of SO_n and its associated Lie algebra, $\mathfrak{so}_n$. To find all representations of $\mathfrak{so}_n$, one must study the universal cover of SO_n called the spin group of order n, Spin_n. However, this is complicated by the fact that Spin_n is usually difficult to construct directly. Thus we are led on a more circuitous route: we first construct an associative and generally non-commutative algebra, called the Clifford algebra. The spin group is the defined to be a certain subgroup of the Clifford algebra.

The aim of this paper is to construct the Clifford algebra of an quadratic form and its spin group, and show that the spin group is the universal cover of SO_n. In §2, we define and examine the special orthogonal group. We show that SO_n is not simply connected. Using this fact and the classification of complex semi-simple Lie algebras, we motivate the construction of the spin group. Furthermore, we explicitly construct the spin group $\mathrm{Spin}_3 \cong SU(2)$. In §3, we define Clifford algebras and spin groups and give several examples of the low dimensional algebras and spin groups. In §4, we examine the topological properties of Spin_n and SO_n and show that Spin_n is the universal cover of SO_n. Finally, in §5, we construct some interesting isomorphisms between low dimensional spin groups and the classical groups. Also, it turns out that one can neatly classify the Clifford Algebras as matrix algebras over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ (the quaternions). While we do not explore this subject in its entirety, we show some examples and state the major theorem.

2. The Special Orthogonal Group

Definition 2.1. The special orthogonal group of a bilinear form Q on a vector space V is

$$SO(Q) = \{T \in SL(V) |\ T^tQT = Q\} \tag{2.1}$$

We will use special notation for two cases of particular interest. We write $SO(n)$ for the special orthogonal group on $\mathbb{R}^n$ with the symmetric form $Q(v,w) = v^t \cdot w$. For the complex special orthogonal group and trivial form, we write SO_n.

We think of $SO(n)$ as the group of rotations of $\mathbb{R}^n$. For example, it is easy to see that

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta \in (0, 2\pi] \right\} \tag{2.2}$$

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In order to motivate the construction of the spin group, we will use some facts from the theory of Lie algebras. This can certainly be skipped if the reader is not comfortable with these facts.

In Lie theory, groups which are simply connected play a key role. Every Lie algebra is the set of tangent vectors at the origin of a unique simply connected Lie group. There may be many Lie groups that give rise to a certain Lie algebra; however, only one is simply connected.

Generally, given a connected group, we can find a simply connected cover of it, called the universal cover.

Definition 2.2. The *universal covering group* of a connected topological group G is a simply connected topological group $\hat{G}$ and a continuous homomorphism $\rho : \hat{G} \to G$ which is surjective and locally injective.

Example 2.3. Let G be the circle S^1. Then $\hat{G} = \mathbb{R}$, and the homomorphism is $\rho : \mathbb{R} \to S^1$, $\rho(x) = e^{2\pi i x}$.

Among Lie algebras, those that are semisimple are fundamental, in the same way that simple groups are the basic building blocks of group theory. It turns out that complex semisimple Lie algebras have an exceedingly simple classification. They lie in one of 4 classes, called A_n, B_n, C_n, and D_n, (see Table 1), or are one of 5 exceptional Lie algebras called E_6, E_7, E_8, F_4, and G_2.

The non-exceptional Lie algebras are called the classical Lie algebras. They come from the classical Lie groups SL_n, Sp_{2n}, and SO_n (see Table 2). Although SL_n and Sp_{2n} are simply connected, SO_n isn't.

TABLE 1. Classical semisimple complex Lie algebras

Cartan Family	Classical name	Constraint
A_n	$\mathfrak{sl}(n+1)$	$\operatorname{Tr} X = 0$
B_n	$\mathfrak{so}(2n+1)$	$X^t + X = 0$
C_n	$\mathfrak{sp}(2n)$	$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$, $B^t = B$, $C^t = C$
D_n	$\mathfrak{so}(2n)$	$X^t + X = 0$

TABLE 2. The Classical Groups

name	Cartan Family	Constraint
SL_{n+1}	A_n	$\det X = 1$
SO_{2n+1}	B_n	Operators that preserve the standard form.
Sp_{2n}	C_n	preserves form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
SO_{2n}	D_n	preserves standard form.

Theorem 2.4. *SL_n and Sp_{2n} are simply connected.*

Proof. This proof is not particularly difficult; however, it requires a few tools from algebraic topology and would lead us off course. See, for example, [3, pp. 367-368]. □

However, this theorem is not true for SO_n.

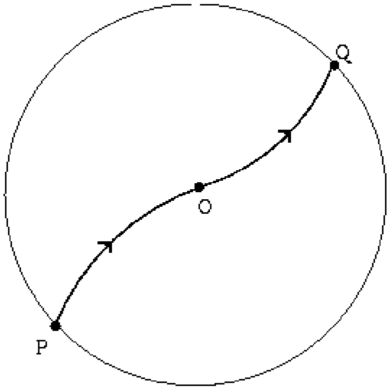
Lemma 2.5. *SO_n is not simply connected.*

Proof. The double covering is perhaps most readily visible for the real group $SO(3)$ of rotations. For a proof of the general case, again see [3, pp. 367-368].

Picture $SO(3)$ as a solid sphere of radius π in $\mathbb{R}^3$. We write a point P in this ball as (v, θ), where v is a vector pointing in the direction of P and θ is the distance from P to the origin. Given v and θ, we construct a rotation of $\mathbb{R}^3$ as the rotation around $\vec{v}$ of angle θ. Notice, however, that (v, θ) and $(-v, -\theta)$ produce the same rotation. Thus this picture is a double covering of $SO(3) \setminus \{I\}$, since every pair of antipodal points corresponds to a single element of $SO(3) \setminus \{I\}$. Furthermore, every element of $SO(3)$ is represented by one or two points in this picture. Locally, there is a one to one correspondence between points of the sphere minus the origin and $SO(3) \setminus \{I\}$. Thus this sphere is the desired double covering of $SO(3)$. We will construct this double cover more algebraically later, it turns out that these two constructions are the same, up to a choice of sign.

Now we can see that $SO(3)$ is not simply connected, since the path in Figure 1 is closed, but is not contractible.

FIGURE 1. A path in $SO(3)$ from 0 to $Q = P$ back to 0.



□

This theorem leads one to ask what the universal cover of SO_n actually is. It turns out that we can construct this group directly for $n = 3$. Unfortunately, it is not clear how to generalize this method to larger values of n.

We will construct the double cover of $SO(3)$ by thinking of rotations of $\mathbb{R}^3$ as rotations of the Riemann sphere, $\mathbb{C} \cup \{\infty\}$. To do this, consider the Riemann sphere as a subset of $\mathbb{R}^3$. Then the Möbius groupacts on $\mathbb{R}^3$. We will call the axes of $\mathbb{R}^3$ ξ, η, and ζ.

A rotation by γ around the ζ-axis is written as

$$U_\zeta(\gamma) : z \mapsto e^{i\gamma} z. \tag{2.3}$$

Under the standard identification of $SL_2(\mathbb{C})$ with the Möbius group

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}, \tag{2.4}$$

we can write

$$U_\zeta(\gamma) = \pm \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}. \tag{2.5}$$

The ambiguity in sign of $U_\zeta(\gamma)$ comes from the fact the the Möbius group is really $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$, that is, the elements I and $-I$ in SL_2 both generate the same Möbius transformation.

This gives us a correspondance between U_ζ and the element of $SO(3)$ that corresponds to rotation around the ζ axis, which we call R_ζ.

$$\pm \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \longmapsto \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_\zeta(\gamma). \tag{2.6}$$

Next we rotate the Riemann sphere around the ξ and η axes. To rotate by angle β around the η axis, we first rotate the η axis up to the ζ axis, then rotate by β, and then rotate back.

It is easy to check that the rotation that takes the η axis to the *zeta* axis is just

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{2.7}$$

with inverse

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{2.8}$$

Thus the entire rotation is given by

$$\begin{aligned} U_\eta(\gamma) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \end{aligned} \tag{2.9}$$

Similarly, for a rotation around the ξ axis, we take the ξ axis to the ζ axis via

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{2.10}$$

Performing the whole sequence gives

$$\begin{aligned} U_\xi(\gamma) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta/2 & i\sin\beta/2 \\ i\sin\beta/2 & \cos\beta/2 \end{pmatrix} \end{aligned} \tag{2.11}$$

Now we have found the $SL_2(\mathbb{C})$ matrices that map onto the generators of $SO(3)$

$$\begin{aligned} \pm U_\xi(\alpha) &\mapsto R_\xi(\alpha) \\ \pm U_\eta(\beta) &\mapsto R_\eta(\beta) \\ \pm U_\zeta(\gamma) &\mapsto R_\zeta(\gamma) \end{aligned} \tag{2.12}$$

These three matrices $U_\xi(\alpha)$, $U_\eta(\beta)$, and $U_\zeta(\gamma)$ generate $SU(2)$, the group of 2 by 2 unitary matrices with determinant 1. An arbitrary element of $SU(2)$ has the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{2.13}$$

If we write $\alpha = a+bi$, $\beta = c+di$, then $|\alpha|^2+|\beta|^2 = a^2+b^2+c^2+d^2 = 1$ and we see that $SU(2)$ is homeomorphic to the sphere S^3 in $\mathbb{R}^4$, and thus is simply connected. The map $\rho : SU(2) \to SO(3)$ that we have constructed is surjective and locally injective, thus $SU(2)$ is the universal cover of $SO(3)$.

It is immediately evident that this method does not generalize easily, since we don't have an analogue of the Möbius group in higher dimensions. It is possible to construct other covering groups of $SO(n)$ by hand, see [3, pp. 102-103] for an exercise which leads to a construction of the universal cover of $SO(4)$ (which we will later see is isomorphic to $SU(2) \times SU(2)$). However, it eventually becomes more practical to construct the spin groups.

3. CLIFFORD ALGEBRAS AND SPIN GROUPS

We give three definitions of the Clifford algebra. All are useful at times. It is easy, yet somewhat tedious to see that they are all equivalent.

Definition 3.1. let V be a vector space over a commutative field k with a symmetric bilinear form Q. The *Clifford Algebra* $C(Q)$ is the universal associative algebra with unit generated by V, with

$$v \cdot w + w \cdot v = 2Q(v, w) \tag{3.1}$$

for all $v, w \in V$.

Definition 3.2. Alternatively, let

$$T^*(V) = \bigoplus_{n \geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots. \tag{3.2}$$

Then

$$C(Q) = T^*(V)/I(Q), \tag{3.3}$$

where $I(Q)$ is the 2–sided ideal generated by $v \otimes v - Q(v, v) \cdot 1$.

Definition 3.3. $C(Q)$ is the associative algebra with basis

$$e_I := e_{i_1} \cdot e_{i_2} \ldots e_{i_k}, \tag{3.4}$$

for all choices of $I = \{i_1 < i_2 \cdots < i_k\}$. Multiplication is defined as follows.

$$(e_1^{\alpha_1} \cdot e_2^{\alpha_2} \ldots e_n^{\alpha_n}) \cdot (e_1^{\beta_1} \cdot e_2^{\beta_2} \ldots e_n^{\beta_n}) = (-1)^{\sum_{i \leq j} \alpha_i \beta_j} e_1^{\alpha_1+\beta_1} \ldots e_n^{\alpha_n+\beta_n}, \tag{3.5}$$

where all additions are taken to be modulo 2, and all α_i and β_i are 0 or 1.

Definition 3.3 is by far the least intuitive; however, it is convenient because it makes multiplication precise and avoids the necessity of proving that e_I is a basis for $C(Q)$. There are many other equivalent definitions of the Clifford algebra, for examples, see [1] or [4].

Often we will just consider the base field to be $\mathbb{R}$ and the bilinear form to be the usual form on $\mathbb{R}^n$.

Definition 3.4. $C_k := C(-Q_k)$ where Q_k is the standard form on $\mathbb{R}^k$.

Definition 3.5. We write $C(Q)^0$ for the even subalgebra of the Clifford algebra. That is, $C(Q)^0$ has as a basis e_I where I runs over all sets such that $|I| \equiv 0 \pmod 2$. Similarily, $C(Q)^1$ is the odd subalgebra, spanned by e_I for $|I| \equiv 1 \pmod 2$.

We say that the Clifford algebra is a mod 2 graded algebra because

$$C^i(Q) \cdot C^j(Q) \subset C^{i+j}(Q), \tag{3.6}$$

where the addition is modulo 2 and i and j are 0 or 1. This property is easy to verify using Definition 3.3, another advantage of the concrete approach.

Example 3.6. C_1 has basis $1, e_1$ with the relation $e_1 \cdot e_1 + e_1 \cdot e_1 = -2Q(e_1, e_1) = -2$, so $e_1 \cdot e_1 = -1$. Clearly, under the map $e_1 \mapsto i$, this gives an isomorphism of C_1 with $\mathbb{C}$.

Example 3.7. C_2 has basis $1, e_1, e_2, e_1 \cdot e_2$. The multiplication table is

	1	e_1	e_2	$e_1 \cdot e_2$
1	1	e_1	e_2	$e_1 \cdot e_2$
e_1	e_1	-1	$e_1 \cdot e_2$	$-e_2$
e_2	e_2	$-e_1 \cdot e_2$	-1	e_1
$e_1 \cdot e_2$	$e_1 \cdot e_2$	e_2	$-e_1$	-1

Now, it is easy to see that under the map

$$\begin{aligned} e_1 &\mapsto i \\ e_2 &\mapsto j \\ e_1 \cdot e_2 &\mapsto k \end{aligned} \tag{3.7}$$

C_2 is isomorphic to $\mathbb{H}$.

Example 3.8. C_2^0 has basis 1, $e_1 \cdot e_2$, with $e_1 \cdot e_2 \cdot e_1 \cdot e_2 = -1$, so $C_2^0 \cong C_1 \cong \mathbb{C}$. This example holds in general, we have $C_n^0 \cong C_{n-1}$.

We could continue writing down these isomorphisms, however, they grow cumbersome. In section 5 we state the full classification of Clifford algebras.

Definition 3.9. There is an anti-involution of the Clifford algebra, $x \mapsto x^*$, given by

$$(v_1 \cdot \dots \cdot v_r)^* = (-1)^r (v_r \cdot \dots \cdot v_1), \tag{3.8}$$

for $v_1, \dots, v_r \in V$. This operation is the composition of
the *main antiautomorphism* $\tau : C \to C$, given by

$$\tau(v_1 \cdot \dots \cdot v_r) = v_r \cdot \dots \cdot v_1. \tag{3.9}$$

and
the *main involution* α, which is the identity on C^0 and acts on C^1 by multiplication by -1.

$$\alpha(v_1 \dots v_r) = (-1)^r v_1 \dots v_r. \tag{3.10}$$

Notice that since $\tau(x \cdot y) = \tau(y) \cdot \tau(x)$ and $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$, we have $(x \cdot y)^* = y^* \cdot x^*$.

Definition 3.10. The *Spin Group* is

$$\operatorname{Spin}(Q) = \{x \in C(Q)^0 | \, x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}. \tag{3.11}$$

It will later be convienient to define a larger group, called the pin group.

$$\operatorname{Pin}(Q) = \{x \in C(Q) | \, x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}. \tag{3.12}$$

The pin group corresponds to the orthogonal group in the same way the spin group corresponds to the special orthogonal group. Its name seems to be the product of two jokes: it sounds like *pine groupe*, a vulgar expression in French, and its name is obtained from spin by removing an 's', just as one goes from SO to O.

Intuitively, this definition says that the spin group consists of those elements which are units in the Clifford algebra, with another condition added. We will see later that this condition ensures that elements of the spin group correspond to rotations. It turns out that the map

$$\begin{aligned} \rho_u :& \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ r \; & \longmapsto uru^*, \qquad u \in \operatorname{Spin}_n \end{aligned} \tag{3.13}$$

is a rotation of $\mathbb{R}^n$.

Example 3.11. We saw in Example 3.8 that $C_2^0 \cong \mathbb{C}$. Thus, we expect that the spin group Spin_2 would be the group of units in $\mathbb{C}$, namely, the unit circle. To be precise about this, C_2^0 has basis elements 1 and e_{12}. Write an element of C_2^0 as $u = a + be_{12}$, $a, b \in \mathbb{R}$. Then

$$\begin{aligned} u \cdot u^* &= (a + be_{12}) \cdot (a + be_{21}) \\ &= a^2 + abe_{21} + abe_{12} + b^2 e_{12} e_{21} \\ &= a^2 + b^2 \end{aligned} \tag{3.14}$$

Thus the first condition for membership in the spin group is that an element lies on the unit circle. Now for $r = ae_1 + be_2 \in \mathbb{R}^2$, we check that $u \cdot r \cdot u^* \in \mathbb{R}^2$. Notice that this is trivial for $u = 1$, so by linearity, we just have to check it for the other basis element, $u = e_{12}$.

$$\begin{aligned} u \cdot r \cdot u^* &= e_{12}(ae_1 + be_2)e_{21} \\ &= ae_{12}^2 \cdot e_1 + be_{12} \cdot e_2 \cdot e_{21} \\ &= ae_1 + be_2. \end{aligned} \tag{3.15}$$

Thus every element of C_2^0 satisfies the second requirement for the spin group, and we have that $\operatorname{Spin}_1 \cong S^1 \cong SO(2)$. Notice that in this degenerate case, the spin group is not simply connected.

Example 3.12. Spin_3 is the group of invertible elements that correspond to rotations in $C_3^0 \cong C_2$. Example 3.7 shows that $C_2 \cong \mathbb{H}$. We claim that the group of units in $\mathbb{H}$ is $SU(2)$, and that every element in $SU(2)$ satisfies the rotation condition $u \cdot \mathbb{R}^3 \cdot u^* \subset \mathbb{R}^3$. To see that the group of units in $\mathbb{H}$ is $SU(2)$, we recall that the norm on $\mathbb{H}$ is $|a + bi + cj + dk|^2 = a^2 + b^2 + c^2 + d^2$. Define a map

$$\begin{aligned} \pi : \mathbb{H} &\longrightarrow SU(2) \\ z = \alpha + \beta i + \gamma j + \delta k &\longmapsto \begin{pmatrix} \alpha + \beta i & \gamma + \delta i \\ -\gamma + \delta i & \alpha - \beta i \end{pmatrix} \end{aligned} \tag{3.16}$$

Now notice that

$$\det\begin{pmatrix}\alpha+\beta i & \gamma+\delta i\\ -\gamma+\delta i & \alpha-\beta i\end{pmatrix} = \alpha^2+\beta^2+\gamma^2+\delta^2, \tag{3.17}$$

and so $\pi(z) \in SU(2)$ if and only if $|z|^2 = 1$.

This takes care of the first condition of the spin group. The second one can be checked as above, but it is messy and presents no surprises.

4. TOPOLOGICAL PROPERTIES

Due to Example 3.12 and our construction in Section 2, we have seen that $\mathrm{Spin}_3 \cong SU(2)$ is the universal cover of $SO(3)$. In this section, we prove that the spin groups are the universal covers of $SO(n)$. Let $V = \mathbb{C}^n$ for $n \geq 3$, and Q be a form on V.

Theorem 4.1. *Spin(Q) is the universal covering group of SO(Q).*

Proof. This proof is from [3, pp. 308-309].

Begin by introducing the map

$$\begin{aligned}\rho &: \mathrm{Pin}(Q) \to \mathrm{O}(Q)\\ \rho(x)(v) &= \alpha(x)\cdot v\cdot x^*.\end{aligned} \tag{4.1}$$

We claim that this map is a surjective homomorphism with $\ker\rho = \pm 1$.

First, to see $\rho(x)$ preserves the form Q (and thus ρ is a homomorphism), note that multiplication in the Clifford Algebra agrees with Q on V, that is,

$$Q(v,v) = v\cdot v = -v\cdot v^*. \tag{4.2}$$

Then

$$\begin{aligned}Q(\rho(x)(v),\rho(x)(v)) &= -\alpha(x)\cdot v\cdot x^*\cdot(\alpha(x)\cdot v\cdot x^*)^*\\ &= -\alpha(x)\cdot v\cdot x^*\cdot x\cdot v^*\cdot\alpha(x)^*\\ &= -\alpha(x)\cdot v\cdot v^*\cdot\alpha(x^*)\\ &= Q(v,v)\alpha(x)\cdot\alpha(x^*)\\ &= Q(v,v)\alpha(x\cdot x^*) = Q(v,v).\end{aligned} \tag{4.3}$$

Where we have used the fact that $x\cdot x^* = 1$.

Next, we clain that ρ is surjective. Recall that $O(Q)$ is generated by reflections. Thus we have to show that every reflection lies in the image of ρ. Let R_w be the reflection in the hyperplane perpendicular to a vector w. Then it is easy to see that w is in $\mathrm{Pin}(Q)$ and $\rho(w) = R_w$. To be precise, normalize w so that $Q(w,w) = -1$. Then

$$w\cdot w^* = w\cdot(-w) = -Q(w,w) = 1, \tag{4.4}$$

and so

$$\rho(w)(w) = \alpha(w)\cdot w\cdot w^* = -w\cdot 1 = -w. \tag{4.5}$$

Also, if $Q(w,v) = 0$,

$$\rho(w)(v) = \alpha(w)\cdot v\cdot w^* = -w\cdot v\cdot w^* = v\cdot w\cdot w^* = v. \tag{4.6}$$

Since $\rho(w)$ takes w to $-w$ and fixes the hyperplane perpendicular to w, it is the reflection in this hyperplane.

Next, we claim that the kernel of ρ is ± 1. Suppose $x \in \mathrm{Pin}(Q)$ is in the kernel of ρ, and write $x = x_0 + x_1$ where $x_0 \in C(Q)^0$ and $x_1 \in C(Q)^1$. Now if $x_0 + x_1 \in \ker \rho$, then for any vector v,

$$\begin{aligned} \rho(x_0 + x_1)(v) &= \rho(x_0)(v) + \rho(x_1)(v) \\ &= \alpha(x_0) \cdot v \cdot x_0^* + \alpha(x_1) \cdot v \cdot x_1^* \\ &= x_0 \cdot v \cdot x_0^* - x_1 \cdot v \cdot x_0 = v. \end{aligned} \tag{4.7}$$

This can only hold if $x_0 \cdot v = v \cdot x_0$ and $x_1 \cdot v = -v \cdot x_1$ for any v. The first condition means that x_0 is in the center of $C(Q)$. By Lemma 4.2 this means that x_0 is in $\mathbb{C} \cdot 1$. The second condition, again by Lemma 4.2, shows that $x_1 = 0$. Thus $x = x_0$ is in $\mathbb{C}$ and $x \cdot x^* = x^2 = 1$, so $x = \pm 1$.

Now we want to show that $\rho^{-1}(SO(Q)) = \mathrm{Spin}(Q)$, which implies that the map ρ make $\mathrm{Spin}(Q)$ a covering of $SO(Q)$. This is immediate from the above discussion and the fact that $SO(Q)$ is generated by pairs of reflections.

Lastly, by Lemma 4.3, we know that $\mathrm{Spin}(Q)$ is connected, thus $\mathrm{Spin}(Q)$ is the desired connected two-sheeted covering of $SO(Q)$.

□

Lemma 4.2. *The intersection of the center of $C(Q)$ with the even subalgebra C^0 is the one-dimensional space of scalars. Furthermore, if $x \in C^1$ and $x \cdot v = -v \cdot x$ for all vectors v, then $x = 0$.*

Proof. Recall that the center of an algebra C is the subalgebra $Z(C)$ consisting of those elements of C which commute with all elements of C.

Suppose that $x = \sum c_I e_I \in Z(C)$. Then $x \cdot v_j = v_j \cdot x$. Notice that $e_I \cdot e_j = (-1)^{|I|} e_j \cdot e_I$ if $j \notin I$, since we have to swap elements $|I|$ times. If $j \in I$, then $e_I \cdot e_j = (-1)^{|I|-1} e_j \cdot e_I$, since here we don't have to swap e_j with itself.

Thus we see that $e_I \in Z(C)$ if $|I|$ is even and there is no $j \in I$, or if $|I|$ is odd, and every j is in I. If the dimension n of V is even, then the second condition is impossible, and the center consists of only scalars times e, which is just the scalars. If n is odd, then the center is the scalars and multiples of the element $e_{1\ldots n}$.

To prove the last assertion, note that if $|I|$ is odd, then $e_I \cdot e_j = -e_j \cdot e_I$ if $j \notin I$ and $e_I \cdot e_j = e_j \cdot e_I$ if $j \in I$. Thus if $x \in C^1$ satisfies $x \cdot v = -v \cdot x$ for all v, then $x = 0$. □

Lemma 4.3. Spin(Q) *is connected.*

Proof. It is enough to show that $+1$ and -1 are connected in $\mathrm{Spin}(Q)$, since the map $\phi : \mathrm{Spin}(Q) \to SO(Q)$ is locally trivial.

We claim that the element

$$(e_1 \cos t + e_2 \sin t) \cdot (e_1 \cos t - e_2 \sin t) \tag{4.8}$$

is in $\mathrm{Spin}(Q)$, since

$$\begin{aligned} E &:= (e_1 \cos t + e_2 \sin t) \cdot (e_1 \cos t - e_2 \sin t) \\ &= e_1 \cdot e_1 \cos^2 t - e_1 \cdot e_2 \cos t \sin t + e_2 \cdot e_1 \sin t \cos t - e_2 \cdot e_2 \sin^2 t \\ &= \cos^2 t - sin^2 t - 2 e_1 \cdot e_2 \cos t \sin t \\ &= -\cos 2t - e_1 \cdot e_2 \sin 2t \end{aligned} \tag{4.9}$$

We can now see that E lies in the Spin group, first compute

$$\begin{aligned} E \cdot E^* &= (-\cos 2t - e_1 \cdot e_2 \sin 2t) \cdot (-\cos 2t + e_1 \cdot e_2 \sin 2t) \\ &= \cos^2 2t - e_1 \cdot e_2 \cdot e_1 \cdot e_2 \sin^2 2t \\ &= \cos^2 2t + \sin^2 2t = 1, \end{aligned} \tag{4.10}$$

since $e_1 \cdot e_2 \cdot e_1 \cdot e_2 = -e_2 \cdot e_1 \cdot e_1 \cdot e_2 = -1$.

Another messy calculation verifies that $E \cdot V \cdot E^* \subset V$. We write a typical element of V as $\sum a_i e_i$, and compute

$$\begin{aligned} &(-\cos 2t - e_1 \cdot e_2 \sin 2t)(\sum a_i e_i)(-\cos 2t + e_1 \cdot e_2 \sin 2t) \\ &\quad = \cos^2 2t \sum a_i e_i + \cos 2t \sin 2t \sum a_i e_i \cdot e_1 \cdot e_2 \\ &\quad\quad - \cos 2t \sin 2t \sum a_i e_i \cdot e_1 \cdot e_2 \\ &\quad\quad - \sin^2 2t \sum a_i e_1 \cdot e_2 \cdot e_1 \cdot e_2 \cdot e_i \\ &\quad = \cos^2 2t \sum a_i e_i + \sin^2 2t \sum a_i e_i \\ &\quad = \sum a_i e_i. \end{aligned} \tag{4.11}$$

Therefore, $-\cos 2t - e_1 e_2 \sin 2t \in \mathrm{Spin}(Q)$. By letting t vary from 0 to $\pi/2$ we get a a path in $\mathrm{Spin}(Q)$ from -1 to $+1$.

□

5. CLASSIFICATION OF CLIFFORD ALGEBRAS

We will just outline some of the interesting results on this topic. For further reading, see [3] or [4].

Theorem 5.1 is the simplist case of what physists call "Bott periodicity", and it is one of the reasons that they are so enamoured with the number 8.

Theorem 5.1. *$C_{k+8} = C_k \otimes C_8$.*

Using this theorem, we just have to calculate the first eight Clifford algebras, and then we can classify all of them. Here are the first eight. We use the notation that $\mathbb{R}(n)$ is the set of n by n matrices over $\mathbb{R}$, etc.

TABLE 3. CA + SG

k	C_k
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$
8	$\mathbb{R}(16)$

We can extract a few more interesting facts from this table. Since C_3 is $\mathbb{H} \oplus \mathbb{H}$, C_4^0 must be the same thing. We saw above that $\mathrm{Spin}_3 \cong SU(2)$, so Spin_4 is just $SU(2) \times SU(2)$. There are other interesting isomorphisms between the low-dimensional spin groups and the classical groups, such as $\mathrm{Spin}_5 \cong Sp_2$, that can be obtained with a little more work.

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