The University of Chicago
CAAM/Stat 31240
Variational Methods in Image Processing
Spring 2024
Course Overview

Course description: This course discusses mathematical models arising in image processing. Topics covered will include an overview of tools from the calculus of variations and partial differential equations, applications to the design of numerical methods for image denoising, deblurring, and segmentation, and the study of convergence properties of the associated models. Students will gain an exposure to the theoretical basis for these methods as well as their practical application in numerical computations.

Prerequisites: The course will be self contained. Experience with real analysis, linear algebra and/or CAAM/Stat 31210 (Applied Functional Analysis) or CAAM/Stat 31220 (Partial Differential Equations) can be beneficial, but is not required.

Course website: Available through Canvas (`https://canvas.uchicago.edu/`).

Textbook: G. Aubert and P. Kornprobst, Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations (2nd edition).

Online copy available through the library at:
`http://proxy.uchicago.edu/login?qurl=http%3A%2F%2Fdx.doi.org%2F10.1007%2F978-0-387-44588-5`
(If a hard copy is desired, one relatively inexpensive option is available through the university's contract with Springer – look at the "MyCopy" link when viewing the online copy of the text as above.)

Learning Objectives

After successfully completing the course, students will be familiar with and able to reason about:

- basic models of images used in applied mathematics,
- total variation methods for denoising images,
- elements of numerical methods for image processing tasks such as deblurring and denoising, and
- an overview of how ideas from applied mathematics and nonlinear PDEs can lead to the formulation and analysis of techniques for image segmentation problems.

Students will also develop their skills in reading research papers in applied mathematics, and developing and expressing their mathematical ideas and arguments in writing.

Grading

Students are expected to attend class, complete assignments, and complete the reading project and presentation/writeup component of the course. The breakdown of points used in grading will be as follows:

Homework (40%): 4-5 assignments.
(The lowest homework score will be dropped.)
Quizzes:
Quiz 1 (10%): Assigned 4/17, due 4/19.
Quiz 2 (10%): Assigned 5/8, due 5/10.
Presentation and Writeup (associated to reading project) (40%):
Meet with instructor to discuss topic selection: Week of April 1.
Topics selected by: 4/10.
Outline of writeup due: 4/22.
Practice presentation due: By end of the day Monday before your presentation.
Presentation Dates (tentative): 5/13, 5/15.
Draft of writeup due: 5/6.

Writeups due: 5/17.

Letter grades will be assigned using the following cutoffs:

A, 93% or higher;
A-, 90% or higher;
B+, 87% or higher;
B, 83% or higher;
B-, 80% or higher;
C+, 77% or higher;
C, 60% or higher;
D, 50% or higher;
F, less than 50%.

We reserve the right to adjust this grading scale, but only in a fashion that would improve the grade earned by what is listed above.

More on the reading project, presentation, and writeup: Students will work with the instructor to select individual or small-group (no more than 3 students per group) reading projects. Each group will be assigned to work together to read a selected portion of textbook material or one or more research papers (chosen in consulation with the instructor, according to student background and interests). Associated to this reading project, each individual student will work to prepare a 20-25 minute presentation and writeup on a portion of this material. A list of sample starting points for possible topics will be distributed near the beginning of the quarter.

Note: Research papers and advanced textbooks in applied mathematics often leave many details to the reader, and reading and understanding these materials often take a substantial amount of time, even for experts. For this reason, it is a good idea to get started early (and to work together, and ask questions! – I am always happy to discuss during office hours or, when possible, by email). A schedule of deadlines is outlined above.

Presentation grading: The presentations will be graded out of 35 points, based on the following rubric:

Organization (logical presentation of ideas) and coordination with team members on content (if applicable): 15 points
Mathematical content: 15 points
Appropriate length (20-25 minutes): 5 points

Writeup grading: Grading of the writeup will be based on clarity of exposition, mathematical correctness, and organization of the material. The assignment will be graded out of 35 points.

Tentative Outline of Topics / Reading Assignments

- Introduction:
  - A selection of fundamental image processing problems: Denoising and Segmentation
  - An overview of some mathematical tools
  - A perspective on numerics
- Mathematical Tools: the direct method of the Calculus of Variations (reading: §2.1 of [AK])
  - Banach Spaces, strong and weak convergence, convexity and lower-semicontinuity.
  - Overview of Sobolev spaces. Derivation of Euler-Lagrange equations.
  - "Gaussian convolution": parabolic smoothing (reading: §3.3.1, pp. 95–98, §A.1, §A.3.1 of [AK])
- Denoising via Total Variation methods
  - Motivations and the Rudin-Osher-Fatemi (TV) functional
  - BV functions, existence and uniqueness of minimizers (reading: §2.2 of [AK])
  - Strong stability results
  - "Weak stability" results
  - Further Ideas: Modified Functionals
    - More general TV penalizations: existence and uniqueness of solutions for the relaxed problem. (Reading: §3.2.3 of [AK])
    - Numerical approaches. (Reading: §3.2.4–3.2.5 of [AK], Boyd-Parikh-Chu-Peleato-Eckstein on ADMM methods)
    - Approximate problems and the language of Gamma convergence.
    - "Exact reconstruction" results and L^1 fidelity (Chan-Esedoglu).
    - Nonlocal modifications (if time permits).
- Return to Nonlinear PDEs
  - Total Variation Flow
  - Perona-Malik
    (Reading: pp. 98–107 of [AK])
  - An axiomatic approach: Alvarez-Guichard-Lions-Morel (and some material on viscosity solutions). (Reading: §2.3, §3.3.1, pp. 107–113 of [AK])
- Other topics as time permits (and according to student interests). Some possible topics may include:
  - Variational models for image segementation – Mumford-Shah: §4.1–4.2 of [AK], [Morel-Solimini], [Vese-Le Guyadere, Chap. 6]
  - Level-set methods: §4.3 of [AK], [Osher-Fedkiw]
  - "Cartoon-Texture" decomposition : §5.2 of [AK], [Vese-Le Guyadere, Chap. 5]

Selected bibliography/additional references (not required)

Some of the material covered in lectures will be related to the following research papers:

- R. Acar and C.R. Vogel. Analysis of bounded variation penalty methods of ill- posed problems. Inverse Problems 10 (1994), no. 6, 1217-1229.
- A. Chambolle, P.L. Lions. Image recovery via total variation minimization and related problems. Numer. Math. 76 (1997), no. 2, 167-188.
- R. Choksi, I. Fonseca, B. Zwicknagl. A few remarks on variational models for denoising. Commun. Math. Sci. 12 (2014), no. 5, 843-857.

In addition to our assigned textbook, there are several good texts (and some survey papers) which cover the topics we will be discussing at a variety of levels. A short selection is listed below. Many of them are available through the library system (some in an electronic format).

- L. Vese, C. Le Guyader. Variational Methods in Image Processing.
- P. Christian Hansen, J. Nagy, D. P. O'Leary. Deblurring Images: Matrices, Spectra and Filtering
- S. Osher and R. Fedkiw. Level Set Methods and Dynamic Implicit Surfaces.
- A. Chambolle, V. Caselles, M. Novaga, D. Cremers, T. Pock. An introduction to Total Variation for Image Analysis.
- B. Kawohl. Variational versus PDE-based approaches in mathematical image processing. Singularities in PDE and the calculus of variations, CRM Proc. Lecture Notes 44, Amer. Math. Soc., Providence. RI, 2008, 113-126.

The following texts cover material related to our course, at times at a more in-depth level:

- F. Andreu-Vaillo, V. Caselles, and J. Mazón. Parabolic Quasilinear Equations Minimizing Linear Growth Functionals.
- T. Chan, J. Shen. Image Processing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods.
- J.M. Morel and S. Solimini. Variational Models in Image Segmentation.

From time to time, we may draw on portions of material from the references above (as well as others) in preparing the lecture and problem sets.

Lastly, some reference books related to Analysis and PDE topics are listed below. Our course will be self-contained and will not necessarily assume you are familiar with these materials. They are listed only as suggested references for interested students. Please don't hesitate to ask questions at any point.

- R. Wheeden and A. Zygmund. Measure and Integral.
- L. Evans. Partial Differential Equations.
- M. Lieb and M. Loss. Analysis (Second edition).
- S. Larsson and V. Thomeé. Partial Differential Equations with Numerical Methods.