# Stat 246, Midterm, April 24 2007

1. (40 pts.) Let $X_1, X_2, Y$ be Bernoulli variables. $P(Y=1) = \pi$, and conditional on $Y$ we assume $X_1, X_2$ independent with the following marginal probabilities:

$$P(X_1 = 1|Y=0) = P(X_2 = 1|Y=0) = p, \quad P(X_1 = 1|Y=1) = P(X_2 = 1|Y=1) = q.$$

   (a) The joint distribution of $X_1, X_2, Y$ can be written as

$$p(y, x_1, x_2) = \exp[y\eta_1 + y(x_1 + x_2)\eta_2 + (1-y)(x_1 + x_2)\eta_3 - b].$$

   Write $\eta_1, \eta_2, \eta_3, b$ in terms of $\pi, p, q$.

   (b) Write the conditional distribution $P(Y = 1|X_1, X_2)$.

   (c) Given an i.i.d sample $(X_{1,1}, X_{1,2}, Y_1), \ldots, (X_{N,1}, X_{N,2}, Y_N)$, from the same distribution, what are the MLE estimates for $\pi, p, q$. You can either derive the estimates formally from the likelihood, or explain them using known facts about MLE estimates in other settings.

   (d) If you do not observe the $Y_n$'s you want to estimate $\theta = (\pi, p, q)$ using the EM algorithm. Given current estimates $\theta^{(t)} = \pi^{(t)}, p^{(t)}, q^{(t)}$, write the formula for:
   (a) $q_n(y|X_n, \theta^{(t)})$. (b) $\pi^{(t+1)}, p^{(t+1)}, q^{(t+1)}$.

2. (40 pts.) Let $Y_1 \sim N(0, \sigma_y^2)$. Define

$$Y_2 = \beta Y_1 + U_1, \quad X_1 = \alpha Y_1 + Z_1, \quad X_2 = \alpha Y_2 + Z_2,$$

   with $Z_1, Z_2 \sim N(0, \sigma_z^2), U_1 \sim N(0, \sigma_u^2)$ and $Z_1, Z_2, U_1, Y_1$ are mutually independent.

   (a) Draw a the directed acyclic graph representing the dependency relations between $Y_1, Z_1, X_1, U_1, Y_2, Z_2, X_2$, and explain why these variables are jointly Gaussian.

   (b) Are $Y_2, Z_2$ independent? Are $Y_2, Z_2$ conditionally independent given $X_2$?

   (c) Compute the joint distribution of $X_1, X_2$.

   (d) Assume $\sigma_u \leq \sigma_y$. For what value of $\beta$ is the distribution of $Y_2$ the same as that of $Y_1$. Can this hold if $\sigma_u > \sigma_y$.

3. (30 pts.) Let $X_n, n = 1, \ldots, N$ be i.i.d $Poisson(\lambda) : p(x) = e^{-\lambda}\lambda^x/x!$.

   (a) Write the distribution of $X_n$ as an exponential family: $f(x) = h(x)\exp[\theta(\lambda)\psi(x) - B(\lambda)]$. What are $h, \theta(\lambda), \psi, B(\lambda)$.

   (b) What is the MLE of $\lambda$.

   (c) Show that the $\Gamma$ distribution $\Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}$, is a conjugate prior for $\lambda$.

   (d) What are the Maximum Posterior and Posterior mean estimates of $\lambda$. (Hint: The mean of the distribution is easy to compute knowing the normalizing constant $\beta^\alpha/\Gamma(\alpha)$. Recall that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.)