# Homework IV: Stat 246

Due, Thursday, May 8

1. Let $(a_1, a_2)$ be a unit vector i.e. $a_1^2 + a_2^2 = 1$, $a_1 \geq 0$. Let $X_1 = Sa_1 + Z_1, X_2 = Sa_2 + Z_2$, where $S, Z_1, Z_2$ are independent normal with $S \sim N(0, \alpha^2), Z_i = N(0, \sigma^2), i = 1, 2$. With $\alpha > 0, \sigma > 0$.

   (a) Draw the DAG representing the dependencies between $S, Z_1, Z_2, X_1, X_2$.

   (b) Show that $X_1, X_2$ are jointly Gaussian, write the mean $\mu(\theta)$ and covariance matrix $C(\theta)$ in terms of $\theta = (a_1, a_2, \alpha^2, \sigma^2)$.

   (c) Show that $a = (a_1, a_2), b = (-a_2, a_1)$ are the two eigenvectors of $C(\theta)$. Write the corresponding eigenvalues $\lambda > \mu > 0$ in terms of $\alpha^2, \sigma^2$.

   (d) Let $C$ be any symmetric positive definite $2 \times 2$ matrix, with distinct eigenvalues $\lambda > \mu > 0$. Let $a = (a_1, a_2), b = (-a_2, a_1)$ be the unit length eigenvectors of $C$, and set $a_1 \geq 0$. Show that in this case there exist unique $\alpha > 0, \sigma > 0$ such that $C$ can be written as $C(\theta)$ with $\theta = (a_1, a_2, \alpha^2, \sigma^2)$.

   Based on this, argue that there exists a one-to-one invertible map $\Phi$ between the set $\Sigma^*$ of $2 \times 2$ symmetric positive definite matrices with two distinct positive eigenvalues and the set $\Theta = \{(a_1, a_2, \alpha^2, \sigma^2); a_1^2 + a_2^2 = 1, a_1 \geq 0, \alpha^2 > 0, \sigma^2 > 0\}$.

   (e) Given $N$ i.i.d observations $X_n \sim N(0, C(\theta))$, let $\hat{C} = \frac{1}{N}\sum_{n=1}^{N} X_n X_n^t$ be the empirical covariance matrix. We know that $\hat{C}$ is the MLE of the covariance matrix $C$.

   Let $\hat{a}, \hat{b}$ be the two eigenvectors of $\hat{C}$ with eigenvalues $\lambda_a > \lambda_b > 0$.

   Show that $\hat{a}$ is the MLE of $a$, $\lambda_a - \lambda_b$ is the MLE of $\alpha$ and $\lambda_b$ is the MLE of $\sigma$.

2. Let $G = (V, E)$ be an undirected graph with $V = \{1, \ldots, K\}$. Let $C_1, \ldots, C_L$ be the set of maximal cliques in $G$. Let

$$p(x) = \frac{1}{Z} \exp\left[\sum_{\ell=1}^{L} \psi_\ell(x_{C_\ell})\right].$$

   Show that if $A, B, C \subset V$ are disjoint, and if $C$ separates $A$ from $B$, (all paths from $A$ to $B$ go through $C$) then $X_A$, $X_B$ are conditionally independent given $X_C$.

3. Let $Y_1, \ldots, Y_N$ be a sequence of random variables such that $Y_1 \sim N(0, \sigma_1^2)$, $Y_{n+1} = Y_n + U_n$, with $U_n \sim N(0, \sigma_u^2)$. Let $X_n = Y_n + Z_n, n = 1, \ldots, N$, with $Z_n \sim N(0, \sigma_z^2)$, Assume $Y_1, U_n, Z_n, n = 1, \ldots, N$ are mutually independent.

   (a) Draw the DAG expressing the dependencies between $X_n, Y_n, Z_n, U_n, n = 1, \ldots, N$. Explain why $(Y_1, X_1), \ldots, (Y_n, X_n)$ represents an HMM.

(b) Explain why the joint distribution of all the variables is joint Gaussian with mean 0.

(c) Let $C_n = \begin{pmatrix} V_n & b_n \\ b_n^t & a_n \end{pmatrix}$ be the covariance matrix of $X_1, \ldots, X_n, Y_n$, with $V_n$ the covariance matrix of $(X_1, \ldots, X_n)$.

(d) Compute $Cov(Y_n, Y_{n+1})$ and $Cov(X_s, Y_{n+1}), s = 1, \ldots, n$. Compute

$$Var(Y_{n+1}),\ Cov(X_{n+1}, Y_{n+1}),\ Var(X_{n+1}),\ Cov(X_n, X_{n+1}).$$

(None of these computations should be complicated.)

(e) Write the expression for $C_{n+1} = \begin{pmatrix} V_{n+1} & b_{n+1} \\ b_{n+1}^t & a_{n+1} \end{pmatrix}$ in terms of the components of $C_n$.

(f) Assume you observe $X_1, \ldots, X_n$ what would your prediction be for $Y_{n+1}$, for $X_{n+1}$. You should be computing $E(Y_{n+1}|X_1, \ldots, X_n)$, and $E(X_{n+1}|X_1, \ldots, X_n)$.