Homework II: Stat 246

Due, Thursday, April 17

1. Let $f(x;g)$ be an exponential family of the form

$$f(x;g) = h(x)\exp[g^t T(x) - B(g)],$$

with $g \in R^d, T(x) \in R^d$. Show that the matrix of second derivatives of $B(g)$ is the covariance matrix of the vector $T(X)$ where $X \sim f(x;g)$.

2. Let $f(x,y)$ be a joint distribution on $x \in R^d, y \in R^k$. We assume that $f_X(x) = N(\mu, \Sigma)$, with $\mu \in R^d$ and Σ a $d \times d$ symmetric positive definite. Assume that $f_{Y|X}(y|x) = N(Ax + b, Q)$ where $A \in R^{k \times d}$, $b \in R^k$ and Q a $k \times k$ symmetric positive definite matrix.

   (a) Show that $f(x,y)$ is jointly normal in (x,y) and write the mean and covariance.

   (b) Compute the marginal $f_Y(y)$ and the conditional $f_{X|Y}(x|y)$.

3. Let $f(x;\alpha,\beta)$ be a family of distributions.

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}\exp[-x/\beta],$$

where $\Gamma(\alpha)$ is the Γ-function: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Show that this is an exponential family. What are the natural parameters? Derive the MLE equations for the two parameters. Is there a closed form solution to these equations?

4. The Dirichlet distribution in R^d is defined on the simplex $T_d = \{x \in R^d : x_i \geq 0, \sum_{i=1}^d x_i = 1\}$, as

$$p(x|\alpha) \propto \prod_{i=1}^d x_i^{\alpha_i - 1}, \quad \alpha_i > 0, i = 1, \ldots, d.$$

   (a) Show that this is an exponential family and that it is the conjugate prior for the multinomial distribution.

   (b) Let $\alpha_0 = \sum_{i=1}^d \alpha_i$. Show that the normalizing constant $Z(\alpha) = \frac{\prod_{i=1}^d \Gamma(\alpha_i)}{\Gamma(\alpha_0)}$. [Hint: one way is by induction on d.]

   (c) Connection to Γ distribution. Let $Y_i \sim \Gamma(\alpha_i, 1), i = 1, \ldots, d$, and assume the Y_i are independent. Show that $Y = \sum_{i=1}^d Y_i \sim \Gamma(\alpha_0, 1)$.

   Show that $X_i = Y_i/Y$ are jointly $Dir(\alpha)$.

   (d) Show that if $X \sim Dir(\alpha)$ then X_1 is independent of $\frac{X_i}{1-X_1}, i = 2, \ldots, d$.

(e) Derive the following moments for the Dirichlet distribution:

$EX_i = \alpha_i/\alpha_0$.

$VarX_i = \frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)}$.

$Cov(X_i, X_j) = -\frac{\alpha_i\alpha_j}{\alpha_0^2(\alpha_0+1)}$.

(f) Define $\psi(\alpha) = \frac{d\Gamma(\alpha)}{d\alpha}$. Show that the maximum likelihood estimate of α from an i.i.d sample $X^{(k)}, k = 1, \ldots, n$ from $Dir(\alpha)$ is obtained by solving the system of equations:

$\psi(\alpha_i) - \psi(\alpha_0) = \frac{1}{n}\sum_{k=1}^{n} \log(X_i^{(k)}), \quad i = 1, \ldots, d$

5. Example of a family of distributions that is not exponential. Let $f(x;\theta)$ be uniform on the interval $[0,\theta]$.

(a) Explain why this is not an exponential family.

(b) Given an i.i.d. sample $X_1, \ldots, X_N$ from $f(x;\theta)$, what is the M.L.E for θ. Don't try to do this with differentiation.

(c) Let $M = M(X_1, \ldots, X_N)$ be the index at which the maximum is achieved:

$$M = \text{argmax}_{i=1,\ldots,N} X_i.$$

Compute

$$P(X_1 < z_1, \ldots, X_N < z_N | M = i \text{ and } X_i = z).$$

Show that $P(M = i) = 1/N$. Conclude that

$$P(X_1 < z_1, \ldots, X_N < z_N | X_M = z),$$

does not depend on θ, so that X_M is a sufficient statistic for θ.