Daggers, dungeons, and bases.

Iden. 1: Duads do not give us a chosen isomorphism
\[ A \cong A^* \]

2. Given \( f : A \rightarrow B \), we can use duads to construct \( f^* : B^* \rightarrow A^* \). You might ask students to say \( f^* : B^* \rightarrow A^* \) as \( f^* \circ f \rightarrow 1 \) or \( f^* \circ f \rightarrow 1 \). What might the

But we have no way of constructing a canonical map
\[ f^* : B \rightarrow A \]

In pictures, we want a reflection:

\[ \begin{array}{ccc}
A & \xrightarrow{f^*} & B \\
\downarrow f & \downarrow & \downarrow f^* \\
I & \xrightarrow{f^*} & A
\end{array} \]

As these pictures suggest, given a vector \( v \)

we can reflect it & obtain an "inner product"

\[ \begin{array}{ccc}
\left\langle v, v \right\rangle & \xrightarrow{f^*} & \langle \rangle \\
v & \xrightarrow{f^*} & v
\end{array} \]

So the dagger operation can also be viewed as the abstraction of a choice of inner product.

**Note:** 3 DIFFERENT THINGS:
- Vector space
- Vector space with inner product
- Vector space with basis
The idea of a dagger category arose in quantum physics and information as a way to axiomatize abstract algebra rules.

Baez, Abramsky, Coecke, Selinger ...

Still mostly used in this context.

Although also useful in talking about the "flow of time" in some (not necessarily quantum) process.

For a $\mathbf{C}$-matrix $m \in \mathbf{ON}$ basis, it is conjugate transpose / adjoint: $f^\dagger = f^*$

\[ \langle fu, w \rangle = \langle u, f^*w \rangle \]

Abstract version sometimes called duals for morphisms.

Define abstract: unitary $f^{*+} = f^+; f^* f = I$ and self-adjoint: $f^+ = f^*$; equipped with a

**Definition** A dagger category is a cat $\mathcal{C}$ with identity-on-obj contravariant functor $t: \mathcal{C} \to \mathcal{C}$ associ $t f t = f$.

\[ id_A^t = id_A \]

$\text{gof}^t = f^t g^t$

\[ f^{*+} = f \]

But we care about $t$ plus other structure & methods.

If $\mathcal{C}$ is also a symmetric monoidal, and

\[ (f \otimes g)^t = f^t \otimes g^t \]

structure morphisms are unitary \( \alpha^t = \alpha, \beta^t = \beta, \epsilon^t = \epsilon \), and $\sigma^t = \sigma$

Then a $t$-symmetric $\mathcal{C}$.
**Frobenius objects**

**Def** A monoid object \((A, m, u)\) in a m.c. \((E, \otimes, I)\) is an object \(A\) of \(E\) of multiplication

\[ m : A \otimes A \rightarrow A \]

and unit

\[ u : I \rightarrow A \]

such that

(i) \rightarrow associative

\[ \begin{array}{ccc}
A & \rightarrow & A \\
\otimes & \rightarrow & \otimes \\
\downarrow & \rightarrow & \downarrow \\
A & \rightarrow & A
\end{array} \]

(ii) units

\[ \begin{array}{ccc}
E & \rightarrow & E \\
\alpha & \rightarrow & \alpha \\
\downarrow & \rightarrow & \downarrow \\
E & \rightarrow & E
\end{array} = \begin{array}{ccc}
E & \rightarrow & E \\
\delta & \rightarrow & \delta \\
\downarrow & \rightarrow & \downarrow \\
E & \rightarrow & E
\end{array} \]

A comonoid object has the arrows reversed (same axioms)

\[ \sigma : A \rightarrow I \quad \text{"delete"} \]

\[ \Sigma : A \rightarrow A \otimes A \quad \text{"comultiplication or copy"} \]

If \(E\) is symmetric, can also require

the (co)monoid object be symmetric.

\[ \begin{array}{ccc}
E & \rightarrow & E \\
\alpha & \rightarrow & \alpha \\
\downarrow & \rightarrow & \downarrow \\
E & \rightarrow & E
\end{array} = \begin{array}{ccc}
E & \rightarrow & E \\
\delta & \rightarrow & \delta \\
\downarrow & \rightarrow & \downarrow \\
E & \rightarrow & E
\end{array} \]

Note: every object can be symmetric \(\alpha_{AA}\) w/o \(E\) being sure.
Don't want monoid/commutative home - usually too restrictive.

Example

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\[ (-E, \text{uniform distribution}, \Omega) \]
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\[ \text{or magnetization} \quad \Phi \]
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**Definition** A Frobenius object \((A, \eta, \mu, \delta, \varepsilon)\) in \(C\):

- \((A, \eta, \mu)\) monoid
- \((A, \delta, \varepsilon)\) comonoid
- Frobenius condition holds!

\[
\begin{align*}
\sum & = \text{coproduct} = \downarrow \downarrow \\
\end{align*}
\]

In this case, any connected morphism built from these ingredients is in the same equivalence class:

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\text{``directed spider''}
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A Frobenius object is **commutative** if

\[
\text{coproduct} = \text{product}
\]

and **special** if

\[
\text{coproduct} = \text{product}
\]
Aside: The Frobenius, monoid, & comonoid axioms are "topologically true."  
\[ u \quad e \quad \varepsilon \]

\[ \mu \quad \eta \quad \delta \quad \epsilon \]

making precise is the "folklore" claim of TACAT:
The 2-dim oriented bordism category is the free symmetric monoidal category of a single commutative Frobenius algebra object.

- A subject in TACAT = Frobenius algebra
- Dagger - Frobenius: \( \delta^* = m \quad \epsilon^* = u \)  \[ \Rightarrow \quad \neg \mathcal{C}^* = \mathcal{C}, \quad o^* = o \]

Thus (Coecke, Pavlovic, Vizny)

\[ \uparrow \text{SCFA} \iff \text{ON basis} \quad \text{axiomatization} \]

so now we know how to "abstractly"
choose a basis

eg given ON Basis (0), (1) get copy & consult dolls content of thin; can go other way. (Ex)
**Defn.** Spidered cat: strict SMC / SCFO stroke on each object 
\((\mathcal{A}, m, \mu, \delta, \varepsilon, \sigma^S)\)

- undirected spiders

**Defn.** Dungeon cat: cpt closed cat \((\mathcal{E}, \sigma^S, \iota, \varepsilon)\) s.t.

(i) each obj has SCF structure \(\sim / \sigma^S = \sigma^F\)
\((\mathcal{A}, m, \mu, \delta, \varepsilon, \sigma^F)\)

(ii) if \(f, g\) are (connected) morphisms built from \(\varepsilon, \delta, \mu, \sigma^S, \sigma^F\), and have same domain \(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}\) (\(\otimes\) of \(\mathcal{A} \otimes \mathcal{A}\)’s)
and codomain (\(\otimes\) of \(\mathcal{A} \otimes \mathcal{A}\)’s)
are equal.

**Note:**

1) This structure “induces forces a dagger.”

But, it might not be closed under (as motivation)
This dagger (eg stochastic matrices, in Bayesian networks)
\(\varepsilon\) transposes. It is closed to “Dungeon”
can complete, but not meaning triangles.

2) So, a dungeon cat is “cpt, closed. There
are spiders everywhere, and if you
are lucky you are equipped with
a dagger.”

Ex: Rel 3 t-dungeon
If you start w/ a dagger opt closed cat, introduce to $\mathcal{E}$ & $\mathcal{B}$, few axioms/simpler definition to get bases. (Bar, Ox)
Implementability

Another problem related to the word problem is the implementability problem:

A T, \( \mathcal{T} \) and interpretation \( \rho \)
geve a "library" of morphisms (e.g., tensors)
from which we can build other morphisms, when can we do it?

Implementability

In general, the problem: given a target morphism \( g \in \mathcal{R} \),
T, and interp \( \rho \), is there a word \( w \) s.t.
such that \( \rho(w) = g \)
\( \in \) in \( \mathcal{X} \cdot \mathcal{M} \).

B undecided. (TIN \#m)

\( \Rightarrow \) problem (toump or formal)

In the case of Boolean tensors \( \mathcal{X} \rightarrow \mathcal{R} \) in \( \mathcal{C} \),
the problem was solved by a Galois correspondence
\( \mathcal{C} \leftrightarrow \text{pp-definability} \)

Constructible = \( \text{Fun}(\text{Pol}(\text{Library})) \)

Bodnar, P. et al. 69
Gerg, 68
Creignou et al. 08 \( \Rightarrow \) quadratic algorithm.
Now we get the abstract version of fdHilb

(A Vector of inner products) on each object

A dagger compact closed category is a TSMC

which is opt closed and

\[ \mathbb{R} = \mathbb{C} \]

\[ \mathbb{I} \rightarrow \mathbb{A} \otimes \mathbb{A} \]

\[ \mathbb{I} \rightarrow \mathbb{A} \otimes \mathbb{A} \]

\[ \mathbb{I} \rightarrow \mathbb{A} \otimes \mathbb{A} \]

which commutes

Thm (Seliger 0.7)

A graphical language is coherent

Thm (Seliger 12)

FDHilb is complete for DCCC

i.e. now if holds in all

FDHilb-reps.

Ex: FdRel is DCCC

"Categorical quantum mechanics" do much of Q info

using piece together protocols, Alice & Bob...

A topological quantum field theory is a functor

\[ n\text{Cob} \rightarrow \text{fdHilb} \]
Bayesian networks

\[ x \rightarrow y \rightarrow z \rightarrow w \]

\[ x \rightarrow f(x) \rightarrow y \rightarrow w \]

`conultr = Hadamard product`  `eg RISM`

"Belief propagation in monoidal categories"
There are many interesting computational aspects:

"Categorifying algorithms" side benefit of popularity of informal network languages.

Having cast a diagram lang into CThy:
  rephrase algorithms as categorical,

get a widely applicable version: walks for any P in satisfying the axioms
not just analogous but equal algorithms
not BP in MC (ext abs)
many "analogous algo" = really one algo.
message ? not redes, just walks of Mor(I,A)
  (small ones hopefully)

message pass eq

define study pt (unit)
and propagation categorically

fixed pt of updates from
BP equations $\Rightarrow$ equations of Mor(I,A)