First, a promised aside: how to view a representation \( \rho: A \to \text{End}(V) \) of a unital associative k-algebra as a functor.

An **enriched category** \( E \) is to replace the monoidal category \( \text{Set}, \times, 1 \) which was implicit in the definition of (locally small) category with another monoidal category \( M \).

So, instead of \( \text{id}_A \in \text{Mor}_E(A,A) \), we use a morphism of \( M \):

\[
\text{id}_A: 1 \to \text{Mor}_M(A,A)
\]

and now \( \text{Mor}_M(A,A) \) is just an object of \( M \).

Instead of \( \otimes \) in \( \text{Set} \):

\[
o: \text{Mor}_E(B,C) \times \text{Mor}_E(A,B) \to \text{Mor}_E(A,C)
\]

we use a morphism of \( M \):

\[
\otimes: \text{Mor}_M(B,C) \times \text{Mor}_M(A,B) \to \text{Mor}_M(A,C)
\]
such that appropriate commutative
 diagrams enforcing identity & associativity
 in \( E \) in terms of \( Id \) and \( o \) arrows.

- A locally small category (usual def) is enriched
  over \( \text{set, } x, u \).

- 2-categories are categories enriched over \( \text{cat, } x \)

A pre additive category is enriched over \( \text{Ab, } \circ \)

(can add diagrams)

A \( k \)-linear category is enriched over \( \text{Vec}_k \)

\[ \text{rep}(R-\text{linear}) \]

- commutative ring

An \( M \)-enriched functor \( F: E \to \text{Vec}_k \)

uses \( M \)-morphisms \( \text{Mor}_E(A, B) \to \text{Mor}_M(F(A), F(B)) \)

instead of set functions \( f \to F(f) \)

(and functional axioms replaced w/commuting diagrams)

So, as promised

- a \( k \)-algebra is a category enriched
  over \( \text{Vec}_k \) (with one object)

  \( k \)-linear cat

- A representation of a \( k \)-algebra is
  a \( k \)-linear functor \( \rho: A \to \text{Vec}_k \)

rep is a functor to \( \text{Vec}_k \)
As a bonus: we now know how to formalize things like linear combinations of diagrams or \( 5H + 3I1 \) or \( \text{new graphs} \).

\[ x^20 + i \chi, \quad \text{III} + \frac{1}{\chi} + \chi \quad - \chi_1 - 18 - \chi \quad \text{or} \quad x^20 + \chi \]

even before assigning interps/rops/values.
X-monoidal categories, Four applied examples

Recall that we defined a tensor network to be a diagram in the X-monoidal category \( \mathcal{fdVect} \)

That is, a diagram = equivalence class of words in the monoidal language \( T^{\otimes} \) over a (finite) tensor scheme \( T \)

\[ \text{obj variables A, B, ...} \]
\[ \text{mor variables \& \&} \]
\[ \text{cod, dom: Mor(T) \to Ob(T)} \]

We equate it with an interpretation/induction/representability:

\[ \text{M(T) \to \mathcal{fdVect}} \]

defined by fixing values of variables.

What about the \( X \)?

Idea: The \( X \) is some adjective which tells us what equivalence means.

So far we've mostly discussed the free monoidal category over a tensor scheme.

There are many other types: symmetric, traced, closed, compact closed, dagger compact, ...

Each adds axioms and sometimes special morphisms.

Each has its own graphical language (although not always as nice or proven-coherent as most like).

Each has its own (mostly wide open) complexity problems (word equivalence? normal form?)
So in pursuing the goal

"All diagrammatic languages arising in nature are diagrams in some X-monoidal category."

The game is to find the right tensor scheme, interpretation, and adjective $X$ to axiomatize the language.

Once we've done this, we can translate results (and even algorithms) among fields effortlessly, since we are using a common language.

Now we'll discuss some of the adjectives and some examples.
Let's start with a symmetric monoidal category.

One way to think about this is that we add special morphisms, satisfying additional axioms, to our category.

\[ \begin{array}{ccc} A & \xrightarrow{\circlearrowleft} & B \\ \downarrow & & \downarrow \\ A & \xleftarrow{e} & B \end{array} \]

The morphism of interest is in \( \text{Mor}(A \otimes B, B \otimes A) \).

We want this to be an isomorphism, really a choice of isomorphism for each ordered pair \( A \otimes B \).

\( \sigma : A \otimes B \to B \otimes A \)

called a braiding. We ask it to satisfy a hexagon axiom, i.e.

\[ A \otimes (B \otimes C) \xrightarrow{\sigma} (B \otimes C) \otimes A \]

\[ \xrightarrow{\bullet} \]

\[ (A \otimes B) \otimes C \xrightarrow{\sigma \otimes 1} B \otimes (A \otimes C) \]

It has a special symbol in the graphical language.

\[ \begin{array}{ccc} B & \xrightarrow{\circlearrowleft} & A \\ \downarrow & & \downarrow \\ A & \xleftarrow{e} & B \end{array} \]

This gives a braided monoidal category, and it has a coherent graphical language (see above).
useful for thinking about braids & knots.

we might get to it later.

If we also require that $\sigma_4 \sigma_3 = \text{id}$

we get a symmetric monoidal category.

we can pick a braiding and talk about free symmetric monoidal categories, symmetric monoidal functors, and so on.

Note: This is just a map that satisfies some axioms, it might not always do what you think it will.

Note: $\times \neq =$
An interpretation $\mathcal{M}(T) \rightarrow \mathcal{S}$ is given by assigning values to each object and morphism in $\mathcal{S}$. An interpretation $\mathcal{M}(T) \rightarrow \mathcal{S}$ is such that when we haven't told what the changes $C$ will be, a (symmetric) word $\mathcal{M}$ is an abstract description of a (symmetric) model $\mathcal{M}$. Consider the tensor scheme $\mathcal{S}$ which is nonhuman variable $\mathcal{S}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{I}$, $\mathcal{O}$. Choose a $2 \times 2$ matrix $C$ or better.
Interpret in NRel w/

\[ B \rightarrow \emptyset_{13} \]

so \( B^n \) is a length-\( n \) bitstring

\[ \cup_n : I \rightarrow B^n \text{ is } 1 \text{ on } (1,00\ldots0) \]
\[ \text{ and } (1,11\ldots1) \]

and 0 o/w

represents Boolean variable

\[ c : B^3 \rightarrow I \text{ is } 0 \text{ on } (000,I) \text{ and } 1 \text{ o/w} \]

represents OR clause.

Now determining which element of \( N \) is defined by a word \( w \in Mor(I,I) \), \( w \in T^{\emptyset_0} \)

is solving a monadically \#3SAT problem

(\#P complete)

\[ w : B \rightarrow B \text{ to remove monad} \]

So now a huge class of problems defined by \( T \), interp, adjective.

"Left circuit"

\[ \text{Problem } w : T^{\emptyset_0} \]

\[ \begin{array}{ccc}
E & \rightarrow & D \\
\text{Empty country} & \text{cont.} & \text{Hard country (Mor(I,I) word problem)}
\end{array} \]