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ASYMPTOTIC PROPERTIES OF SPATIAL CROSS-PERIODOGRAMS USING FIXED-DOMAIN ASYMPTOTICS

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Asymptotic Properties of Spatial Cross-Periodograms using Fixed-Domain Asymptotics

Chae Young Lim and Michael Stein *

Abstract

Cross-periodograms can be used to study a multivariate spatial process observed on a lattice. For spatial data, it is often appropriate to study asymptotic properties of statistical procedures under fixed-domain asymptotics in which the number of observations increases in a fixed region while shrinking distances between neighboring observations. Using fixed-domain asymptotics, we prove relative asymptotic unbiasedness and relative consistency of a smoothed cross-periodogram after appropriate filtering of the data. In addition, we show smoothed cross-periodograms are asymptotically normal when the process is stationary multivariate Gaussian with appropriate assumptions on high frequency behavior of the spectral density.

Keywords: Multivariate Gaussian process, Infill asymptotics, Spectral Density, Fourier transform, Joint cumulant

1 Introduction

Spectral analysis of stationary processes is a powerful tool for analyzing spatial data sets on a grid. Properties of the spatial periodogram of lattice data have been studied by many authors [see e.g. Whittle (1954), Guyon (1982), Guyon (1995), Ripley (1981) and Rosenblatt (1985)]. Often, the processes of interest are defined in a continuous space. Then, the observed data can be regarded as a realization of a random field on a lattice. Consider a multivariate stationary random field $\mathbf{Z} = (Z_1, \cdots, Z_p)$ on $\mathbb{R}^d$ with $p \times p$ spectral density matrix $F = (f_{ab})$ and data observed at $\delta\mathbf{J}$ for $\mathbf{J} \in \prod_{j=1}^{d}\{1, \cdots, m_j\}$. Here δ is the distance between neighboring observations. When $\delta = 1$ independent of m_j, we have observations on the integer lattice, $\mathbb{Z}^d$. For simplicity, suppose that $m_1 = \cdots = m_d = m$. Guyon (1982) showed that standard results for periodograms in time series can be carried over to spatial periodograms for stationary random fields on $\mathbb{Z}^d$ as $m \to \infty$ if bias correction is applied

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to the periodogram. This approach, letting the number of observations go to infinity while fixing δ, is called increasing-domain asymptotics [Cressie (1993)].

When processes on $\mathbb{R}^d$ are observed on a lattice, it is natural to let δ vary with m. Fuentes (2002) studied asymptotic behavior of the periodogram as $m \to \infty$, $\delta \to 0$ and $\delta m \to \infty$ for both stationary and non-stationary Gaussian process on $\mathbb{R}^2$. In this approach, the number of observations goes to infinity and the distance between neighboring observations tends to zero but with slower speed so that the observation region grows without bound and is what she called shrinking asymptotics. Hall et al. (1994) introduced nonparametric estimators of the covariance function and variogram using the kernel method and the Fourier transform. Under shrinking asymptotics, they showed some asymptotic results of their nonparametric estimators when the observations are not necessarily evenly spaced. There have been many other studies regarding to this type of asymptotics [see e.g. Constantine and Hall (1994), Lahiri (1999), Fazekas and Chuprunov (2006), Zhu and Lahiri (2007) and the references therein].

If we are interested in processes on a given fixed region of $\mathbb{R}^d$, then we can take $\delta = bm^{-1}$ where b is a constant independent of m. Letting $m \to \infty$ under this condition is called fixed-domain asymptotics [Stein (1995)] or infill asymptotics [Cressie (1993)]. This approach, letting the number of observation increase in a given fixed domain, is often appropriate for spatial data [Stein (1999)]. In time or spatial domain, the fixed-domain approach has been considered by many authors [see e.g. Ying (1993), Chen et al. (2000), Furrer (2005), Zhang and Zimmerman (2005) and the references therein].

In the spectral domain, there is little asymptotic work from the fixed-domain perspective. Stein (1995) showed, under some assumptions on the process and the frequency of interest, that the spatial periodogram of an appropriately filtered version of the process is nearly unbiased for the spectral density of the filtered process on a lattice. He also showed that the periodogram values at different Fourier frequencies are asymptotically uncorrelated under some further appropriate conditions. In this paper, we extend some results of Stein (1995) to cross-periodograms. In addition, we show asymptotic normality of smoothed periodograms and cross-periodograms under fixed-domain asymptotics when the process is stationary multivariate Gaussian, under an appropriate assumption on high frequency behavior of the spectral density. To do so, we make use of assumptions similar to those in Stein (1995), but extended to cover the multivariate case.

Define a lattice process $\mathbf{Y}_\delta = (Y_{\delta,1}, Y_{\delta,2}, \cdots, Y_{\delta,p})$ on $\mathbb{Z}^d$ by $\mathbf{Y}_\delta(\mathbf{J}) = \mathbf{Z}(\delta \mathbf{J})$ for $\mathbf{J} \in \mathbb{Z}^d$. Then $\mathbf{Y}_\delta$ has spectral density matrix $\bar{F}_\delta$, whose (a, b) entry is

$$\bar{f}_{\delta,\,ab}(\boldsymbol{\omega}) = \delta^{-d} \sum_{\mathbf{Q} \in \mathbb{Z}^d} f_{ab}(\delta^{-1}(\boldsymbol{\omega} + 2\pi \mathbf{Q})),$$

for $\boldsymbol{\omega} \in (-\pi, \pi]^d$. We set $\bar{f}_{\delta,\,a} = \bar{f}_{\delta,\,aa}$. The function $\bar{f}_{\delta,\,ab}(\boldsymbol{\omega})$ has integral over $(-\pi, \pi]^d$ independent of δ, but more and more of its mass gets concentrated near the origin as $\delta \to 0$. This peakedness near the origin can cause problems for the periodogram that can be addressed by appropriate prewhitening. Stein (1995) proposed a possibly iterated discrete Laplacian operator to difference the data; the number of iterations required relates to the behavior of the spectral density at high-frequencies. Define the Laplacian operator

$$\Delta_\delta Z(\mathbf{x}) = \sum_{j=1}^{d} \{ Z(\mathbf{x} + \delta \mathbf{1}_j) - 2Z(\mathbf{x}) + Z(\mathbf{x} - \delta \mathbf{1}_j) \},$$

where $\mathbf{1}_j$ is the unit vector along the jth coordinate. Define $\mathbf{Z}_\delta^\tau(\mathbf{x}) = (\Delta_\delta)^\tau \mathbf{Z}(\mathbf{x})$ and $\mathbf{Y}_\delta^\tau(\mathbf{J}) = \mathbf{Z}_\delta^\tau(\delta\mathbf{J})$ where the Laplacian operator is applied to each component process. Then $\mathbf{Y}_\delta^\tau$ has spectral density matrix $\bar{F}_\delta^\tau$, whose (a,b) entry is

$$\bar{f}_{\delta,\,ab}^\tau(\boldsymbol{\omega}) = \left\{\sum_{j=1}^d 4\sin^2\left(\frac{\omega_j}{2}\right)\right\}^{2\tau} \bar{f}_{\delta,\,ab}(\boldsymbol{\omega}).$$

To obtain limiting results, we will need the following assumptions on the cross-spectral density. For positive functions a and b, $a(\boldsymbol{\omega}) \asymp b(\boldsymbol{\omega})$ means that there exist constants C_1 and C_2 such that $0 \le C_1 < a(\boldsymbol{\omega})/b(\boldsymbol{\omega}) \le C_2 < \infty$ for all possible $\boldsymbol{\omega}$. We assume that for $a = 1, \cdots, p$, the spectral density of Z_a satisfies

$$f_a(\boldsymbol{\omega}) \quad \asymp \quad (1 + |\boldsymbol{\omega}|)^{-\alpha_a}. \tag{1.1}$$

Throughout this work, we will write $\bar{\alpha}$ for $(\alpha_a + \alpha_b)/2$, supressing the dependence on a and b. We assume that for some c_{ab}, the cross-spectral density for Z_a and Z_b satisfies

$$f_{ab}(\boldsymbol{\omega})|\boldsymbol{\omega}|^{\bar{\alpha}} = c_{ab} \exp\left\{i\,\theta_{ab}(\boldsymbol{\upsilon})\right\} + o(1) \tag{1.2}$$

as $|\boldsymbol{\omega}| \to \infty$, where $\boldsymbol{\omega}/|\boldsymbol{\omega}| \to \boldsymbol{\upsilon}$ and $\theta_{ab}(\boldsymbol{\upsilon})$ is a continuous function on the unit sphere in $\mathbb{R}^d$. To have a valid cross-spectral density matrix in the limit, we assume that the matrix, $\bar{F}$ with $\bar{F}_{ab} = c_{ab} \exp\left\{i\,\theta_{ab}(\boldsymbol{\upsilon})\right\}$ is positive semi-definite. For $a = b$, we would always have $\theta_{ab}(\boldsymbol{\omega}) = 0$ and $c_a = c_{aa}$ is positive since the spectral density is positive. Note that c_{ab} can be zero, which includes the case that Z_a and Z_b are independent.

By assuming (1.2), for any fixed $\boldsymbol{\omega} \in (-\pi, \pi]^d$, $\boldsymbol{\omega} \ne \mathbf{0}$, we have as $\delta \to 0$,

$$\delta^{d-\bar{\alpha}} \bar{f}_{\delta,\,ab}^\tau(\boldsymbol{\omega}) \to c_{ab} \left\{\sum_{j=1}^d 4\sin^2\left(\frac{\omega_j}{2}\right)\right\}^{2\tau} \sum_{\mathbf{Q}\in\mathbb{Z}^d} |\boldsymbol{\omega} + 2\pi\mathbf{Q}|^{-\bar{\alpha}} \exp\left\{i\,\theta_{ab}\left(\frac{\boldsymbol{\omega} + 2\pi\mathbf{Q}}{|\boldsymbol{\omega} + 2\pi\mathbf{Q}|}\right)\right\}. \tag{1.3}$$

The absolute value of this limit behaves like $c_{ab}|\boldsymbol{\omega}|^{4\tau-\bar{\alpha}}$ near the origin. Thus selecting τ such that $4\tau - \bar{\alpha} > -d$ makes the limit integrable. Here we denote the right side of (1.3) by $g_{ab}(\boldsymbol{\omega})$.

In this study, we consider cross-periodograms of the data differenced τ times, including $\tau = 0$, which means no differencing. Suppose that we observe $\mathbf{Y}_\delta^\tau(\mathbf{J}) = \mathbf{Z}_\delta^\tau(\delta\mathbf{J})$ for $\mathbf{J} \in T_m = \{1, \cdots, m\}^d$. Define a discrete Fourier transform of the process $Y_{\delta,a}^\tau$,

$$D_a(\boldsymbol{\omega}) = \sum_{\mathbf{S}\in T_m} Y_{\delta,a}^\tau(\mathbf{S}) e^{-i\boldsymbol{\omega}^T\mathbf{S}}. \tag{1.4}$$

Then the cross-periodogram is defined as

$$I_{m,\delta}^\tau(\boldsymbol{\omega}; a, b) = (2\pi m)^{-d} D_a(\boldsymbol{\omega}) D_b(\boldsymbol{\omega})^*,$$

where * indicates complex conjugate. Here, we only consider $I_{m,\delta}^\tau(\boldsymbol{\omega}; a, b)$ at the Fourier frequencies $2\pi m^{-1}\mathbf{J}$, $\mathbf{J} \in \mathcal{T}_m = \{-\lfloor (m-1)/2 \rfloor, \cdots, m - \lfloor m/2 \rfloor\}^d$. We also study a smoothed cross-periodogram. Consider a symmetric continuous function K on $\mathbb{R}^d$ which satisfies $K(\mathbf{x}) \ge 0$ and $K(\mathbf{0}) > 0$. Let $K_h(\mathbf{x}) = K(\frac{\mathbf{x}}{h}) I_{\{\|\mathbf{x}\| \le h\}}$, where $\|\mathbf{x}\| = \max(|x_1|, \cdots, |x_d|)$, and

$$W_h(\mathbf{K}) = \frac{K_h(2\pi m^{-1}\mathbf{K})}{\sum_{\mathbf{L}\in\mathcal{T}_m} K_h(2\pi m^{-1}\mathbf{L})}.$$

Then we define the smoothed cross-periodogram by

$$\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}\right)=\sum_{\mathbf{K}\in\mathcal{T}_m}W_h(\mathbf{K})I^{\tau}_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}+\mathbf{K});a,b\right). \tag{1.5}$$

In addition, define the $p\times p$ matrix $\hat{F}^{\tau}_{\delta}$, the smoothed cross-periodogram matrix whose (a,b) entry is defined by (1.5).

In Section 2, we study asymptotic behavior of the mean and variance of smoothed and raw cross-periodograms. We find bounds for the expected value, variance and correlations of cross-periodograms that are similar to results for the periodogram in Stein (1995). Since the spectral density matrix $\bar{F}^{\tau}_{\delta}$ goes to zero componentwise as $m\to\infty$, we consider relative asymptotic unbiasedness and relative consistency. For O the matrix of zeros, we will say $\hat{F}^{\tau}_{\delta}$ is *relatively asymptotically unbiased* for $\bar{F}^{\tau}_{\delta}$ if $E\left\{\left(\mathrm{diag}\bar{F}^{\tau}_{\delta}\right)^{-1/2}\left[\hat{F}^{\tau}_{\delta}-\bar{F}^{\tau}_{\delta}\right]\left(\mathrm{diag}\bar{F}^{\tau}_{\delta}\right)^{-1/2}\right\}\to O$, where $\mathrm{diag}\bar{F}^{\tau}_{\delta}$ is the diagonal matrix of $\bar{F}^{\tau}_{\delta}$. $\hat{F}^{\tau}_{\delta}$ is *relatively consistent* if $\left(\mathrm{diag}\bar{F}^{\tau}_{\delta}\right)^{-1/2}\left[\hat{F}^{\tau}_{\delta}-\bar{F}^{\tau}_{\delta}\right]\left(\mathrm{diag}\bar{F}^{\tau}_{\delta}\right)^{-1/2}\to O$ componentwise, in probability.

Proposition 3 shows that for sufficiently large $\|\mathbf{J}\|$ and m, $E\left\{\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right\}$ is close to $\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)$ under appropriate assumptions on f_{ab} and τ. In other words, $\hat{F}^{\tau}_{\delta}$ is relatively asymptotically unbiased for $\bar{F}^{\tau}_{\delta}$, the spectral density matrix of $\mathbf{Y}^{\tau}_{\delta}$. The conditions for Proposition 3 together with (2.15) imply that, asymptotically, one can ignore the correlation between cross-periodograms at different Fourier frequencies in calculating the variance of the sum for a smoothed cross-periodogram. Using this result, Proposition 6 shows that the variance of each component in $\left(\mathrm{diag}\bar{F}^{\tau}_{\delta}\right)^{-1/2}\left[\hat{F}^{\tau}_{\delta}-\bar{F}^{\tau}_{\delta}\right]\left(\mathrm{diag}\bar{F}^{\tau}_{\delta}\right)^{-1/2}$ goes to 0 when $\sum_{\mathbf{K}\in\mathcal{T}_m}W_h(\mathbf{K})^2\to 0$ as $m\to\infty$. Thus $\hat{F}^{\tau}_{\delta}$ is relatively consistent for $\bar{F}^{\tau}_{\delta}$ by Propositions 3 and 6.

Section 3 studies the asymptotic normality of the cross-periodograms under fixed-domain asymptotics. Unlike the increasing domain case, Gaussian assumption for the process is essential for the asymptotic result here. Non-Gaussianity will likely lead to non-Gaussian limits under the fixed-domain approach, which is the case, for example, when estimating the fractal index for a class of non-Gaussian stationary processes [Chan and Wood (2004)]. Suppose that the process is stationary multivariate Gaussian on $\mathbb{R}^d$ and that its spectral density matrix F satisfies (1.1) and (1.2) with smoothness condition on high frequencies given by (2.3). Consider $h=Cm^{-\gamma}$, for $0<\gamma<1$. Then, the total number of neighboring Fourier frequencies summed over in the smoothed cross-periodogram defined in (1.5) increases more slowly than m^d as $m\to\infty$. Under this condition on h, Lemma 9 shows that the expected value of $m^{-(d-\bar{\alpha})}$ times the smoothed cross-periodogram at $2\pi m^{-1}\mathbf{J}^m$ converges to $g_{ab}(\boldsymbol{\mu})$ given in (1.3) if $\lim_{m\to\infty}2\pi\mathbf{J}^m/m=\boldsymbol{\mu}\neq\mathbf{0}$. Proposition 10 provides the covariance structure of smoothed and scaled cross-periodograms under fixed-domain asymptotics. To obtain the asymptotic covariance structure of smoothed cross-periodograms, we impose additional restrictions on γ when $d\geq 3$. Consider two Fourier frequencies, $2\pi m^{-1}\mathbf{J}^m_1$ and $2\pi m^{-1}\mathbf{J}^m_2$, for which $\lim_{m\to\infty}2\pi\mathbf{J}^m_s/m=\boldsymbol{\mu}_s\neq\mathbf{0}$ for $s=1,2$. In particular, Proposition 10 shows that if $\boldsymbol{\mu}_1\pm\boldsymbol{\mu}_2\neq 0$, then smoothed and scaled cross-periodograms at these two Fourier frequencies are asymptotically independent. Finally, we establish the asymptotic normality of cross-periodograms in Proposition 12. These results under fixed-domain asymptotics are the multivariate spatial version of standard results in multivariate spectral analysis of time series [see e.g. Brillinger (1981) and Brockwell and Davis (1987)]. Brillinger (1981) briefly considers multivariate spatial spectra for the tapered spatial process observed everywhere in a bounded region. Proofs of most Lemmas and Propositions are given in Section 4.

2 Mean and Variance of the Smoothed Cross-Periodogram

In this section, we approximate the mean and variance of the cross-periodogram and the smoothed cross-periodogram under fixed-domain asymptotics. Suppose that $\mathbf{Z}$ is a stationary multivariate process on $\mathbb{R}^d$. Define

$$a^{\tau}_{m,\delta}(\mathbf{J},\mathbf{K};a,b) = (2\pi m)^{-d}\int_{(-\pi,\pi]^d} \bar{f}^{\tau *}_{\delta,ab}(\boldsymbol{\omega})\prod_{j=1}^{d}\frac{\sin^2\left(\frac{m\omega_j}{2}\right)}{\sin\left(\frac{\omega_j}{2}+\frac{\pi J_j}{m}\right)\sin\left(\frac{\omega_j}{2}+\frac{\pi K_j}{m}\right)}d\boldsymbol{\omega}.$$

Then we have

$$E\left\{I^{\tau}_{m,\delta}\left(\frac{2\pi\mathbf{J}}{m};a,b\right)\right\} = a^{\tau}_{m,\delta}(\mathbf{J},\mathbf{J};a,b) \tag{2.1}$$

for $\mathbf{J}\in\mathcal{T}_m\setminus\{\mathbf{0}\}$ or $\mathbf{J}=\mathbf{0}$ and $\tau\geq 1$. When $\mathbf{Z}$ is Gaussian, for $\mathbf{J},\mathbf{K}\in\mathcal{T}_m\setminus\{\mathbf{0}\}$,

$$\begin{aligned}&\operatorname{cov}\left\{I^{\tau}_{m,\delta}\left(\frac{2\pi\mathbf{J}}{m};a,b\right),I^{\tau}_{m,\delta}\left(\frac{2\pi\mathbf{K}}{m};a',b'\right)\right\}\\ &\quad= a^{\tau}_{m,\delta}(\mathbf{J},\mathbf{K};a,a')a^{\tau}_{m,\delta}(\mathbf{J},\mathbf{K};b,b')^{*} + a^{\tau}_{m,\delta}(\mathbf{J},-\mathbf{K};a,b')a^{\tau}_{m,\delta}(\mathbf{J},-\mathbf{K};a',b)^{*},\end{aligned} \tag{2.2}$$

where $\operatorname{cov}(X,Y)=E\{(X-EX)(Y-EY)^*\}$.

In addition to (1.1) and (1.2), we make the following assumptions on the smoothness of the spectral density matrix F at high frequencies. For $a,b=1,\cdots,p$, f_{ab} is twice differentiable with

$$|f_{ab}(\boldsymbol{\omega})|\left(1+|\boldsymbol{\omega}|\right)^{\bar{\alpha}},\quad \left|\frac{\partial}{\partial\omega_j}f_{ab}(\boldsymbol{\omega})\right|\left(1+|\boldsymbol{\omega}|\right)^{\bar{\alpha}+1},\quad\text{and}\quad \left|\frac{\partial^2}{\partial\omega_j\partial\omega_k}f_{ab}(\boldsymbol{\omega})\right|\left(1+|\boldsymbol{\omega}|\right)^{\bar{\alpha}+2} \tag{2.3}$$

uniformly bounded for $j,k=1,\cdots,d$. Note that $\alpha_a>d,\ a=1,\cdots,p$ is required for f to be integrable and that uniform boundedness of f_{ab} when $a=b$ was already guaranteed by the assumption (1.2). For real-valued functions a and b, define $a(m,\mathbf{J}^m)\ll b(m,\mathbf{J}^m)$ to mean $b(m,\mathbf{J}^m)>0$ and $|a(m,\mathbf{J}^m)|/b(m,\mathbf{J}^m)$ is uniformly bounded in m and $\|\mathbf{J}^m\|$. Define $I_{\{A\}}=1$ if A is true, $I_{\{A\}}=0$, otherwise and $\langle m\rangle^q=m^q$ if $q\neq 0$ and $\langle m\rangle^0=\log m$. For simplicity, we assume that $\delta=m^{-1}$ throughout this study.

Proposition 2, which includes Proposition 1 in Stein (1995) as a special case with $a=b$, approximates $E\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m;a,b\right)\right\}$ as $m\to\infty$. Proposition 2 shows that $I^{\tau}_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m;a,b\right)$ is relatively asymptotically unbiased for $\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)$ if $\|\mathbf{J}^m\|\to\infty$ and $\langle m\rangle^{4\tau-\bar{\alpha}-1}/\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}\to 0$ as $m\to\infty$. The proof of the following Lemma is similar to the proof of Proposition 1 in Stein (1995) except that when $a\neq b$, $I^{\tau}_{m,\delta}$ is complex-valued. Thus, we state the Lemma without proof.

Lemma 1 *Suppose that f_{ab} satisfies (1.1) and (2.3), and that $4\tau>\max\{\alpha_a,\alpha_b\}-1$. Then, for any m and $\mathbf{J}^m\in\mathcal{T}_m\setminus\{\mathbf{0}\}$,*

$$\begin{aligned}&\left|E\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m;a,b\right)\right\}-\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right|\\ &\qquad\ll\ m^{d-4\tau}\left(\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1}+\|\mathbf{J}^m\|^{-1}+\langle m\rangle^{4\tau-\bar{\alpha}-1}\right).\end{aligned} \tag{2.4}$$

If $\|\mathbf{J}^m\| \asymp m$, the left hand side of (2.4) becomes $m^{d-\bar{\alpha}-1}(1+m^{-(4\tau-\bar{\alpha})}+\log m\, I_{\{4\tau=\bar{\alpha}+1\}}) = o(m^{d-\bar{\alpha}})$. Then

$$E\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m; a, b\right)\right\} = a^{\tau}_{m,\delta}(\mathbf{J}^m, \mathbf{J}^m; a, b) = \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right) + o(m^{d-\bar{\alpha}}). \qquad (2.5)$$

Furthermore, note that using (1.1) and (2.3), we have

$$\bar{f}^{\tau}_{\delta,a}\left(2\pi m^{-1}\mathbf{J}^m\right) \asymp m^{d-4\tau}\|\mathbf{J}^m\|^{4\tau-\alpha_a} \text{ and} \qquad (2.6)$$

$$|\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)| \ll m^{d-4\tau}\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}. \qquad (2.7)$$

Proposition 2 *Under the conditions of Lemma 1, for any m and $\mathbf{J}^m \in \mathcal{T}_m \setminus \{\mathbf{0}\}$,*

$$\left|\frac{E\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m; a, b\right)\right\} - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)}{\sqrt{\bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)}}\right| \ll \frac{\langle m\rangle^{4\tau-\bar{\alpha}-1}}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}} + \frac{1}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}+1}} + \frac{1}{\|\mathbf{J}^m\|}. \qquad (2.8)$$

Proof This result follows from Lemma 1 and (2.6). □

We can also get a bound for the smoothed cross-periodogram defined in (1.5). Consider $h = Cm^{-\gamma}$ for some $C > 0$ and $0 < \gamma < 1$. To make use of Proposition 2, we impose an additional condition on h to avoid including the zero frequency in the smoothing procedure.

Proposition 3 *With the conditions of Lemma 1, suppose that $h = \min\{Cm^{-\gamma}, 2\pi(\|\mathbf{J}^m\| - 1)/m)\}$ for some $C > 0$ and $0 < \gamma < 1$. Define $\mathbf{K}_{\min} = \operatorname{argmin}_{\mathbf{K}\in\mathcal{T}_m, \|2\pi\mathbf{K}/m\|\le h, \mathbf{K}\neq\mathbf{0}}\|\mathbf{J}^m + \mathbf{K}\|$. Then, for any $m \geq 2$ and $\mathbf{J}^m \in \mathcal{T}_m \setminus \{\mathbf{0}\}$,*

$$\left|\frac{E\left\{\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right\} - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)}{\sqrt{\bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)}}\right| \ll \frac{\langle m\rangle^{4\tau-\bar{\alpha}-1}}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}} + \frac{m^{1-\gamma}}{\|\mathbf{J}^m\|} + \frac{1}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}+1}} + \frac{1}{\|\mathbf{J}^m + \mathbf{K}_{\min}\|\,\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}} + \frac{\|\mathbf{J}^m + \mathbf{K}_{\min}\|^{4\tau-\bar{\alpha}-1}}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}} I_{\{4\tau\le\bar{\alpha}+1\}}. \qquad (2.9)$$

By Proposition 3, $\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)$, the smoothed cross-periodogram at a given non-zero Fourier frequency is relatively asymptotically unbiased for $\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)$ if $\|\mathbf{J}^m\| \to \infty$ in such a way that $\langle m\rangle^{4\tau-\bar{\alpha}-1}/\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}$ and $m^{1-\gamma}/\|\mathbf{J}^m\|$ tend to zero as $m \to \infty$,

Next, we develop the asymptotic uncorrelatedness of the cross-periodograms at distinct Fourier frequencies under the fixed-domain perspective. Lemma 4, which is a generalization of Proposition 3 in Stein (1995), is a key ingredient for showing the result. An analogous result to the variance of cross-periodogram under increasing-domain asymptotics is established in Proposition 5.

Lemma 4 *Suppose that $\mathbf{Z}$ is Gaussian, f_{ab} satisfies (1.1) and (2.3) and $4\tau > \max\{\alpha_a, \alpha_b\} - 1$. Then, for $\mathbf{J}^m, \mathbf{K}^m \in \mathcal{T}_m \setminus \{\mathbf{0}\}$ with $\mathbf{J}^m \neq \mathbf{K}^m$,*

$$\left|a^{\tau}_{m,\delta}(\mathbf{J}^m, \mathbf{K}^m; a, b)\right| \ll \beta_m^{4\tau-\bar{\alpha}-1} m^{d-4\tau}, \qquad (2.10)$$

where $\beta_m = \min(\|\mathbf{J}^m\|, \|\mathbf{K}^m\|)$.

Proof Similar to the proof of Proposition 3 in Stein (1995). □

Proposition 5 *With the conditions of Lemma 4, suppose that* $\mathbf{J}^m \in \mathcal{T}_m \setminus \{\mathbf{0}\}$. *Let* $\alpha_{\max} = \max\{\alpha_a, \alpha_b\}$ *and* $\alpha_{\min} = \min\{\alpha_a, \alpha_b\}$. *If* $2\mathbf{J}^m/m \notin \mathbb{Z}^d$, *then*

$$\frac{\operatorname{var}\left\{I^\tau_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m; a, b\right)\right\}}{\bar{f}^\tau_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^\tau_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)} - 1$$
$$\ll \prod_{j=a,b} \frac{\langle m\rangle^{4\tau-\alpha_j-1}}{\|\mathbf{J}^m\|^{4\tau-\alpha_j}} + \sum_{j=a,b} \frac{\langle m\rangle^{4\tau-\alpha_j-1}}{\|\mathbf{J}^m\|^{4\tau-\alpha_j}} + \frac{1}{\|\mathbf{J}^m\|^{4\tau-\alpha_{\max}+1}} + \frac{1}{\|\mathbf{J}^m\|}. \quad (2.11)$$

If $2\mathbf{J}^m/m \in \mathbb{Z}^d$, *then*

$$\frac{\operatorname{var}\left\{I^\tau_{m,\delta}\left(2\pi m^{-1}\mathbf{J}^m; a, b\right)\right\}}{\bar{f}^\tau_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^\tau_{\delta,b}(2\pi m^{-1}\mathbf{J}^m) + \left|\bar{f}^\tau_{\delta,ab}(2\pi m^{-1}\mathbf{J}^m)\right|^2} - 1$$
$$\ll \prod_{j=a,b} \frac{\langle m\rangle^{4\tau-\alpha_j-1}}{\|\mathbf{J}^m\|^{4\tau-\alpha_j}} + \sum_{j=a,b} \frac{\langle m\rangle^{4\tau-\alpha_j-1}}{\|\mathbf{J}^m\|^{4\tau-\alpha_j}} + \frac{\langle m\rangle^{4\tau-\bar{\alpha}-1}}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}} + \frac{1}{\|\mathbf{J}^m\|^{4\tau-\alpha_{\max}+1}} + \frac{1}{\|\mathbf{J}^m\|}. \quad (2.12)$$

By (2.2) and Proposition 5, Corollary 1 gives a bound on the correlation between cross-periodograms at distinct Fourier frequencies that tends to 0 when $\beta_m \to \infty$ as $m \to \infty$.

Corollary 1 *Under the conditions of Lemma 4 , if* $\beta_m \asymp m$, *then for* $\mathbf{J}^m, \mathbf{K}^m$ *with* $\mathbf{J}^m \pm \mathbf{K}^m \neq \mathbf{0}$,

$$\left|\operatorname{corr}\left\{I^\tau_{m,\delta}\left(\frac{2\pi\mathbf{J}^m}{m}; a, b\right), I^\tau_{m,\delta}\left(\frac{2\pi\mathbf{K}^m}{m}; a, b\right)\right\}\right| \ll \beta_m^{-2}. \quad (2.13)$$

Using Corollary 1, we can seek a condition similar to (14) in Stein (1995) that will make

$$\operatorname{var}\left\{\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right\} \sim \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \operatorname{var}\left\{I^\tau_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K}); a, b\right)\right\}; \quad (2.14)$$

that is, asymptotically, the correlation between cross-periodograms can be ignored in calculating the variance of a smoothed cross-periodogram. Once (2.14) is satisfied, Proposition 6 together with Proposition 3 provides that a smoothed cross-periodogram is relatively consistent to the corresponding cross-spectral density on a lattice when $\sum W_h(\mathbf{K})^2 \to 0$ as $m \to \infty$. Assume that h is as in Proposition 3 and $\|\mathbf{J}^m\| \asymp m$. Let Γ_m be the subset of $\mathcal{T}_m$ for which $W_h(\mathbf{K}) \neq 0$ and let L_m be the number of elements of Γ_m. By our assumptions, we can apply Corollary 1 to $\hat{f}_{h,ab}$ and obtain

$$
\begin{aligned}
&\left| \operatorname{var}\left\{ \hat{f}_{h,ab}\left(\frac{2\pi \mathbf{J}^m}{m}\right)\right\} - \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \operatorname{var}\left\{ I^{\tau}_{m,\delta}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}; a,b\right)\right\}\right| \\
&\le 2 \sum_{\substack{\mathbf{K},\mathbf{K}'\in\Gamma_m,\\ \mathbf{K}\neq\mathbf{K}'}} W_h(\mathbf{K})W_h(\mathbf{K}') \left| \operatorname{cov}\left\{ I^{\tau}_{m,\delta}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}; a,b\right), I^{\tau}_{m,\delta}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K}')}{m}; a,b\right)\right\}\right| \\
&\ll \sum_{\substack{\mathbf{K},\mathbf{K}'\in\Gamma_m,\\ \mathbf{K}\neq\mathbf{K}'}} W_h(\mathbf{K})W_h(\mathbf{K}')\, m^{2d-8\tau} \left(\|\mathbf{J}^m+\mathbf{K}\|\,\|\mathbf{J}^m+\mathbf{K}'\|\right)^{4\tau-\bar{\alpha}} \min\{\|\mathbf{J}^m+\mathbf{K}\|, \|\mathbf{J}^m+\mathbf{K}'\|\}^{-2} \\
&\ll m^{2d-8\tau} \sum_{\mathbf{K}\in\Gamma_m} W_h(\mathbf{K})\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\bar{\alpha}-2} \sum_{\mathbf{K}'\in\Gamma_m} W_h(\mathbf{K}')\|\mathbf{J}^m+\mathbf{K}'\|^{4\tau-\bar{\alpha}}, \\
&\ll m^{2d-8\tau} \sum_{\mathbf{K}\in\Gamma_m} W_h(\mathbf{K})^2\|\mathbf{J}^m+\mathbf{K}\|^{2(4\tau-\bar{\alpha})} \sqrt{L_m \sum_{\mathbf{K}\in\Gamma_m} \|\mathbf{J}^m+\mathbf{K}\|^{-4}},
\end{aligned}
$$

the last step using the Cauchy-Schwarz inequality.

Since we have

$$
\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \operatorname{var}\left\{ I^{\tau}_{m,\delta}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}; a,b\right)\right\} \asymp m^{2d-8\tau} \sum_{\mathbf{K}\in\Gamma_m} W_h(\mathbf{K})^2\|\mathbf{J}^m+\mathbf{K}\|^{2(4\tau-\bar{\alpha})},
$$

(2.14) holds if

$$
L_m \sum_{\mathbf{K}\in\Gamma_m} \|\mathbf{J}^m+\mathbf{K}\|^{-4} \to 0. \tag{2.15}
$$

Proposition 6 *Suppose that* $\mathbf{Z}$ *is Gaussian and that* f_{ab} *satisfies (1.1) and (2.3). Also suppose that* $4\tau > \max\{\alpha_a, \alpha_b\} - 1$, $h = \min\{Cm^{-\gamma}, 2\pi(\|\mathbf{J}^m\| - 1)/m)\}$ *for some* $C > 0$ *and* $0 < \gamma < 1$ *and (2.15). Then, for* $\mathbf{J}^m$ *such that* $\|\mathbf{J}^m\| \asymp m$ *and* $2m^{-1}\mathbf{J}^m \in (-1,1)^d$,

$$
\operatorname{var}\left\{\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right\} \sim \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m). \tag{2.16}
$$

3 Asymptotic distribution of Smoothed Cross-Periodograms

In this section, we show the limiting distribution of smoothed cross-periodograms under fixed-domain asymptotics is Gaussian after appropriate normalization if the process is stationary multivariate Gaussian and its spectral density matrix satisfies some regularity conditions at high frequency. We prove this by showing that cumulants of order 3 or higher go to zero as $\delta \to 0$. First, we introduce the definition of the joint cumulant and some necessary terminology and results in Brillinger (1981) and Leonov and Shiryaev (1959). Define the rth order joint cumulant, $\operatorname{cum}(Y_1, \cdots, Y_r)$, of $(Y_1, \cdots, Y_r)$, where Y_j are real or complex with $E|Y_j| < \infty, j = 1, \cdots, r$, by

$$
\operatorname{cum}(Y_1, \cdots, Y_r) = \sum (-1)^{p-1}(p-1)!(E\prod_{j\in\nu_1} Y_j)\ldots(E\prod_{j\in\nu_p} Y_j),
$$

where the summation extends over all partitions $(\nu_1, \cdots, \nu_p)$, $p = 1, \cdots, r$ of $(1, \cdots, r)$.

Consider a two-way table

$$\begin{array}{ccc} (1,1) & \dots & (1,J_1) \\ \vdots & & \vdots \\ (I,1) & \dots & (I,J_I) \end{array} \tag{3.1}$$

and a partition $P_1 \cup P_2 \cup \dots P_M$ of its entries. Two sets, $P_{m'}$ and $P_{m''}$ of the partition are said to *hook* if there exist $(i_1, j_1) \in P_{m'}$ and $(i_2, j_2) \in P_{m''}$ such that $i_1 = i_2$. Two sets, $P_{m'}$ and $P_{m''}$ of the partition are said to *communicate* if there exists a sequence of sets $P_{m_1} = P_{m'}, P_{m_2}, \cdots, P_{m_N} = P_{m''}$ such that P_{m_s} and $P_{m_{s+1}}$ hook for $s = 1, \cdots, N-1$. A partition is said to be *indecomposable* if all sets communicate.

Theorem 7 *(Brillinger 1981) Consider a two-way array of random variables* $X_{i\,j}$, $j = 1, \cdots, J_i$, $i = 1, \cdots I$. *Consider* I *random variables*

$$Y_i = \prod_{j=1}^{J_i} X_{i\,j}, i = 1, \cdots I.$$

The joint cumulant $\mathrm{cum}\,(Y_1, \cdots, Y_I)$ *is then given by*

$$\sum_{\boldsymbol{\nu}} \mathrm{cum}\,(X_{i\,j}; ij \in \nu_1) \dots \mathrm{cum}\,(X_{i\,j}; ij \in \nu_p)$$

where the summation is over all indecomposable partitions $\boldsymbol{\nu} = \nu_1 \cup \ldots \cup \nu_p$ *of the Table (3.1).*

Consider a special case that will be used later. Suppose that $J_i = 2$ for all $i = 1, \cdots r$ and X_{ij} are Gaussian with $E(X_{ij}) = 0$. Then joint cumulants of order 3 or higher are zero. Let e_i and $\bar{e}_i$ equal either 1 or 2, while $e_i \neq \bar{e}_i$. Then the partition $\boldsymbol{\nu} = \cup_{q=1}^{p} \nu_q$ is indecomposable if and only if

$$\nu_1 = \{(i_1, \bar{e}_1), (i_2, e_2)\}, \nu_2 = \{(i_2, \bar{e}_2), (i_3, e_3)\}, \cdots,$$
$$\nu_{r-1} = \{(i_{r-1}, \bar{e}_{r-1}), (i_r, e_r)\}, \nu_r = \{(i_r, \bar{e}_r), (i_1, e_1)\},$$

where $(i_1, \cdots, i_r)$ is some permutation of the numbers $1, \cdots, r$. Without loss of generality, we may always set $i_1 = 1$ and $\bar{e}_1 = 1$. Then indecomposable partitions correspond to pairs of collections $\{(i_2, \cdots, i_r), (e_2, \cdots, e_r)\}$, where $(i_2, \cdots, i_r)$ is a permutation of the numbers $2, 3, \cdots r$ and $(e_2, \cdots, e_r)$ is a collection of 1's and 2's. Let $\mathbf{i}_2 = (i_2, \cdots, i_r)$ and $\mathbf{e}_2 = (e_2, \cdots, e_r)$. Then we have

$$\mathrm{cum}\,(Y_1, \cdots, Y_r) = \sum_{\mathbf{i}_2, \mathbf{e}_2} \prod_{j=2}^{r} \mathrm{cum}\,(X_{i_{j-1}\bar{e}_{j-1}}, X_{i_j e_j})\, \mathrm{cum}\,(X_{i_j \bar{e}_j}, X_{i_{j+1} e_{j+1}}), \tag{3.2}$$

where $i_{r+1} = i_1 = 1$ and $e_{r+1} = e_1 = 2$. Lemma 8 shows that the second order joint cumulant of Discrete Fourier Transforms defined in (1.4) can be expressed in terms of $a^{\tau}_{m,\delta}$.

Lemma 8 *For* $\mathbf{J}, \mathbf{K} \in \mathcal{T}_m$*, the second order joint cumulant of Discrete Fourier Transforms is given by*

$$\begin{aligned} &\mathrm{cum}\,(D_a(2\pi m^{-1}\mathbf{J}), D_b(2\pi m^{-1}\mathbf{K})) \\ &\qquad = \exp\Big\{ -i\frac{m+1}{m}\pi \sum_{j=1}^{d} (J_j + K_j) \Big\}\, (2\pi m)^d a^{\tau}_{m,\delta}(\mathbf{J}, -\mathbf{K}; a, b) \end{aligned} \tag{3.3}$$

Proof Similar to the proof of (2.1) in Section 4. □

Consider $X_{ij} = D_{a_i}(2\pi m^{-1}(\mathbf{J}_i+\mathbf{K}_i))$ if $j = 1$ and $X_{ij} = D_{b_i}(-2\pi m^{-1}(\mathbf{J}_i+\mathbf{K}_i))$ if $j = 2$. Then we can apply (3.2) and Lemma 8 to have an expression for the rth order joint cumulant of smoothed cross-periodograms, since $D_a(2\pi m^{-1}\mathbf{J})$ is Gaussian with $E\left(D_a(2\pi m^{-1}\mathbf{J})\right) = 0$ for $\mathbf{J} \neq \mathbf{0}$:

$$\operatorname{cum}\left(\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}_1\right), \cdots, \hat{f}_{h,a_rb_r}\left(2\pi m^{-1}\mathbf{J}_r\right)\right) = \sum_{\mathbf{K}_1,\cdots,\mathbf{K}_r} \left\{ \prod_{j=1}^{r} W_h(\mathbf{K}_j) \times \right.$$
$$\left. \sum_{\mathbf{i}_2,\mathbf{e}_2} \prod_{j=1}^{r} a^{\tau}_{m,\delta}\big((-1)^{\bar{e}_j-1}(\mathbf{J}_{i_j}+\mathbf{K}_{i_j}), (-1)^{e_{j+1}}(\mathbf{J}_{i_{j+1}}+\mathbf{K}_{i_{j+1}}); u_{i_j\bar{e}_j}, u_{i_{j+1}e_{j+1}}\big)\right\}, \qquad (3.4)$$

where $u_{ie} = a_i$ if $e = 1$ and $u_{ie} = b_i$ if $e = 2$.

Lemma 9 and Proposition 10 provide the limit of the expected value and covariance of smoothed cross-periodograms. These limits depend on the behavior of the spectral density at high frequency through the assumption. Let $\bar{K}_r = \int_{[-1,1]^d} K(\mathbf{x})^r d\mathbf{x}$ that will be used in Proposition 10 and later.

Lemma 9 *Suppose that f_{ab} satisfies (1.1), (1.2) and (2.3), $4\tau > \max\{\alpha_a, \alpha_b\}-1$, $\lim_{m\to\infty} 2\pi\mathbf{J}^m/m = \boldsymbol{\mu} \neq \mathbf{0}$ and $h = Cm^{-\gamma}$ for some $C > 0$ and $0 < \gamma < 1$. Then we have*

$$\lim_{m\to\infty} m^{-(d-\bar{\alpha})} E\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right) = g_{ab}(\boldsymbol{\mu}). \qquad (3.5)$$

Proposition 10 *Suppose that $\mathbf{Z}$ is Gaussian, each entry of F satisfies (1.1), (1.2) and (2.3), $4\tau > \max_{s=1,\cdots,p}\{\alpha_s\}-1$, $h = Cm^{-\gamma}$ for $\max\{(d-2)/d, 0\} < \gamma < 1$ and $\lim_{m\to\infty} 2\pi\mathbf{J}^m_s/m = \boldsymbol{\mu}_s \neq \mathbf{0}$ for $s = 1, 2$. Let $\eta = d(1-\gamma)/2$. Then, for a_1, b_1, a_2 and $b_2 \in \{1, \cdots, p\}$,*

$$\lim_{m\to\infty} m^{2\eta}\operatorname{cov}\Big\{m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}^m_1\right), m^{-(d-\bar{\alpha}_2)}\hat{f}_{h,a_2b_2}\left(2\pi m^{-1}\mathbf{J}^m_2\right)\Big\}$$
$$= \begin{cases} \left\{(2\pi/C)^d\bar{K}_2/\bar{K}_1^2\right\} g_{a_1a_2}(\boldsymbol{\mu}_1)\, g_{b_2b_1}(\boldsymbol{\mu}_1) & , \mathbf{J}^m_1 = \mathbf{J}^m_2 \\ \left\{(2\pi/C)^d\bar{K}_2/\bar{K}_1^2\right\} g_{a_1b_2}(\boldsymbol{\mu}_1)\, g_{a_2b_1}(\boldsymbol{\mu}_1) & , \mathbf{J}^m_1 = -\mathbf{J}^m_2 \\ 0 & , \boldsymbol{\mu}_1 \neq \pm\boldsymbol{\mu}_2. \end{cases}$$

Consider $\mathbf{J}^m_1 \pm \mathbf{J}^m_2 \neq 0$ but $\boldsymbol{\mu}_1 \pm \boldsymbol{\mu}_2 = 0$. With similar arguments as in the proof of Proposition 10, we can prove that the asymptotic covariance is 0 if $\|\mathbf{J}^m_1 \pm \mathbf{J}^m_2\| = O(m^\rho)$ for $1-\gamma < \rho < 1$. If $0 < \rho \le 1-\gamma$ and $\mathbf{c}^{\pm}_u = \lim_{m\to\infty} 2\pi(\mathbf{J}^m_1 \pm \mathbf{J}^m_2)/mh$ exists, we can obtain a limiting expression for the covariance. For example, suppose that $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \boldsymbol{\mu}$. Then we can show that

$$\lim_{m\to\infty} m^{2\eta}\operatorname{cov}\Big\{m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}^m_1\right), m^{-(d-\bar{\alpha}_2)}\hat{f}_{h,a_2b_2}\left(2\pi m^{-1}\mathbf{J}^m_2\right)\Big\}$$
$$= \frac{(2\pi/C)^d}{\bar{K}_1^2}\int_{[-1,1]^d\cap\{[-1,1]^d-\mathbf{c}^-_u\}} K(\mathbf{x})K(\mathbf{c}^-_{\mathbf{u}}+\mathbf{x})d\mathbf{x}\; g_{a_1a_2}(\boldsymbol{\mu})\, g_{b_2b_1}(\boldsymbol{\mu}).$$

Thus, if $0 < \rho < 1-\gamma$, then $\mathbf{c}^{\pm}_u = \mathbf{0}$ so that we have the same limit as in Proposition 10. If $\rho = 1-\gamma$ and $[-1,1]^d \cap \{[-1,1]^d - \mathbf{c}^-_u\}$ is empty, the asymptotic covariance is 0. Therefore, with the asymptotic normality result in Proposition 12, the smoothed and scaled cross-periodograms at $2\pi\mathbf{J}_1^m/m$ and $2\pi\mathbf{J}_2^m/m$ are asymptotically independent if $\boldsymbol{\mu}_1 \pm \boldsymbol{\mu}_2 \neq 0$ or even if two Fourier frequencies have the same (or negative) limit when the difference of the two Fourier frequencies decreases slower than the bandwidth in the smoothing procedure.

For $\|\mathbf{J}\|, \|\mathbf{K}\| \asymp m$, we have $|a^{\tau}_{m,\delta}(\mathbf{J},\mathbf{K};a,b)| = O(m^{d-\bar{\alpha}})$ if $\mathbf{J}=\mathbf{K}$ and $|a^{\tau}_{m,\delta}(\mathbf{J},\mathbf{K};a,b)| = O(m^{d-\bar{\alpha}-1})$ if $\mathbf{J} \neq \mathbf{K}$ by (2.5) and Lemma 4. Then, the sum over $\mathbf{K}_1, \cdots, \mathbf{K}_r$ in (3.4) can be divided into several groups based on the number of sets in the partitions of $\{1, \cdots r\}$ so that each group is bounded by some order of m that is small enough to make all third and higher order joint cumulants with appropriate normalization go to 0 as $m \to \infty$. More specifically, we have the following result:

Lemma 11 *Under the conditions of Proposition 10, for* $r \geq 3$ *and* $a_1, b_1, \cdots, a_r, b_r \in \{1, \cdots, p\}$,

$$\Big|\text{cum}\Big(m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}_1\right), \cdots, m^{-(d-\bar{\alpha}_r)}\hat{f}_{h,a_rb_r}\left(2\pi m^{-1}\mathbf{J}_r\right)\Big)\Big| = o(m^{-\eta r}). \quad (3.6)$$

Proposition 12 *Suppose that* $\lim_{m\to\infty} 2\pi\mathbf{J}_s^m/m = \boldsymbol{\mu}_s \neq \mathbf{0}$ *for* $s = 1, \cdots, n$. *Let* $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n)$ *and* $\mathbf{g}(\tilde{\boldsymbol{\mu}}) = (g_{a_1b_1}(\boldsymbol{\mu}_1), \cdots, g_{a_nb_n}(\boldsymbol{\mu}_n))$. *Under the conditions of Proposition 10, for* $a_1, b_1, \cdots, a_n, b_n \in \{1, \cdots, p\}$,

$$m^{\eta}\Big\{\Big(m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}_1\right), \cdots, m^{-(d-\bar{\alpha}_n)}\hat{f}_{h,a_nb_n}\left(2\pi m^{-1}\mathbf{J}_n\right)\Big) - \mathbf{g}(\tilde{\boldsymbol{\mu}})\Big\} \longrightarrow N_n^c(0,\Sigma),$$

where $\eta = d(1-\gamma)/2$, N_n^c *is complex* n*-variate normal distribution and* Σ *is given by Proposition 10.*

We have let α_a, which controls the high frequency behavior of the spectral density of Z_a vary with a. Thus, it is natural to allow τ, the number of times to difference data, to vary with each component process. That is, consider $\mathbf{Y}_{\delta}^{\boldsymbol{\tau}}(\mathbf{J}) = ((\Delta_\delta)^{\tau_1}Z_1(\delta\mathbf{J}), \cdots, (\Delta_\delta)^{\tau_p}Z_p(\delta\mathbf{J}))$. Then, (a,b) entry of the spectral density matrix $\bar{F}_{\delta}^{\boldsymbol{\tau}}$ for $\mathbf{Y}_{\delta}^{\boldsymbol{\tau}}$ is

$$\bar{f}^{\bar{\tau}}_{\delta,\,ab}(\boldsymbol{\omega}) = \left\{\sum_{j=1}^{d} 4\sin^2\left(\frac{\omega_j}{2}\right)\right\}^{2\bar{\tau}} \bar{f}_{\delta,\,ab}(\boldsymbol{\omega}),$$

where $\bar{\tau} = (\tau_a + \tau_b)/2$. By replacing τ with $\bar{\tau}$ and assuming $4\tau_s > \alpha_s - 1$, $s = 1, \cdots, p$, we can extend Proposition 12 to include this case as well.

4 Proofs

Proof of (2.1)

For $\mathbf{J} \in \mathcal{T}_m \setminus \{\mathbf{0}\}$ or $\mathbf{J} = \mathbf{0}$ and $\tau \geq 1$, we have

$$\begin{aligned}
&E\left\{I^{\tau}_{m,\delta}\left\{2\pi m^{-1}\mathbf{J}; a, b\right)\right\} \\
&= (2\pi m)^{-d} E\left(D_a(2\pi m^{-1}\mathbf{J}) D_b(2\pi m^{-1}\mathbf{J})^*\right) \\
&= (2\pi m)^{-d} \sum_{\mathbf{s}} \sum_{\mathbf{u}} e^{-i2\pi m^{-1}\mathbf{J}^T(\mathbf{s}-\mathbf{u})} E\left(Y^{\tau}_{\delta,a}(\mathbf{s}) Y^{\tau}_{\delta,b}(\mathbf{u})\right) \\
&= (2\pi m)^{-d} \sum_{\mathbf{s}} \sum_{\mathbf{u}} e^{-i2\pi m^{-1}\mathbf{J}^T(\mathbf{s}-\mathbf{u})} \left\{ \int e^{i\boldsymbol{\omega}^T(\mathbf{s}-\mathbf{u})} \bar{f}^{\tau}_{\delta,ab}(\boldsymbol{\omega}) d\boldsymbol{\omega} + E\Big(Y^{\tau}_{\delta,a}(\mathbf{s})\Big) E\left(Y^{\tau}_{\delta,b}(\mathbf{u})\right) \right\} \\
&= (2\pi m)^{-d} \int \sum_{\mathbf{s}} e^{-i(2\pi m^{-1}\mathbf{J}-\boldsymbol{\omega})^T\mathbf{s}} \sum_{\mathbf{u}} e^{i(2\pi m^{-1}\mathbf{J}-\boldsymbol{\omega})^T\mathbf{u}} \bar{f}^{\tau}_{\delta,ab}(\boldsymbol{\omega}) d\boldsymbol{\omega} \\
&\quad + (2\pi m)^{-d} \sum_{\mathbf{s}} \sum_{\mathbf{u}} e^{-i2\pi m^{-1}\mathbf{J}^T(\mathbf{s}-\mathbf{u})} E\left(Y^{\tau}_{\delta,a}(\mathbf{s})\right) E\left(Y^{\tau}_{\delta,b}(\mathbf{u})\right) \\
&= (2\pi m)^{-d} \int \prod_{j=1}^{d} \frac{\sin^2\left(\frac{m\omega_j}{2}\right)}{\sin^2\left(-\frac{\omega_j}{2} + \frac{\pi J_j}{m}\right)} \bar{f}^{\tau}_{\delta,ab}(\boldsymbol{\omega}) d\boldsymbol{\omega} \qquad (4.1) \\
&= (2\pi m)^{-d} \int \prod_{j=1}^{d} \frac{\sin^2\left(\frac{m\omega_j}{2}\right)}{\sin^2\left(\frac{\omega_j}{2} + \frac{\pi J_j}{m}\right)} \bar{f}^{\tau *}_{\delta,ab}(\boldsymbol{\omega}) d\boldsymbol{\omega},
\end{aligned}$$

the equality in (4.1) following from the fact that $E\big(Y^{\tau}_{\delta,a}(\mathbf{s})\big) = 0$ if $\tau \geq 1$ and $E\big(Y^{\tau}_{\delta,a}(\mathbf{s})\big) =$ constant if $\tau = 0$.

Proof of Proposition 3

Note that we have

$$|\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K})\right) - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)| \ll m^{d-4\tau} \|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} \|\mathbf{K}\|, \qquad (4.2)$$

by (2.3) and $\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) = \sum_{\mathbf{K}\in\mathcal{T}_m, \|2\pi\mathbf{K}/m\|\leq h} W_h(\mathbf{K}) = 1$.

We use similar results as in Proposition 1 of Stein (1995). By the assumption on h, $\mathbf{J}^m + \mathbf{K} \neq \mathbf{0}$ so that we can apply Lemma 1 to $E\big\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K}); a, b\right)\big\} - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K})\right)$ as follows:

$$\begin{aligned}
&\left|E\big(\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\big) - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right| \\
&\leq \left| \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) \Big[E\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K}); a, b\right)\right\} - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K})\right) \Big] \right| \\
&\quad + \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) \left|\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}(\mathbf{J}^m + \mathbf{K})\right) - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right| \\
&\ll m^{d-4\tau} \Bigg\{ \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) \|\mathbf{J}^m + \mathbf{K}\|^{4\tau-\bar{\alpha}-1} + \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) \frac{1}{\|\mathbf{J}^m + \mathbf{K}\|} \\
&\qquad + \langle m \rangle^{4\tau-\bar{\alpha}-1} + \|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) \|\mathbf{K}\| \Bigg\}.
\end{aligned}$$

Note that

$$\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K}) \|\mathbf{K}\|^s = \sum_{\mathbf{K}\in\mathcal{T}_m, \|2\pi\mathbf{K}/m\|\leq h} W_h(\mathbf{K}) \|\mathbf{K}\|^s \ll m^{s(1-\gamma)}, \qquad (4.3)$$

for $s > 0$. Also we have for $s \le 0$,

$$\|\mathbf{J}^m + \mathbf{K}\|^s \quad \ll \quad \|\mathbf{J}^m + \mathbf{K}_{\min}\|^s, \tag{4.4}$$

when $\mathbf{K} \neq \mathbf{0}$ and $\|2\pi \mathbf{K}/m\| \le h$.

Thus, for $4\tau - \bar{\alpha} - 1 > 0$,

$$\begin{aligned} &\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})\|\mathbf{J}^m + \mathbf{K}\|^{4\tau-\bar{\alpha}-1} \\ &\qquad \ll \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} + \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})\|\mathbf{K}\|^{4\tau-\bar{\alpha}-1} \\ &\qquad \ll \|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} + m^{(4\tau-\bar{\alpha}-1)(1-\gamma)}, \end{aligned} \tag{4.5}$$

and for $4\tau - \bar{\alpha} - 1 \le 0$,

$$\begin{aligned} &\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})\|\mathbf{J}^m + \mathbf{K}\|^{4\tau-\bar{\alpha}-1} \\ &\qquad = W_h(\mathbf{0})\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} + \sum_{\mathbf{K}\in\mathcal{T}_m, \mathbf{K}\neq\mathbf{0}} W_h(\mathbf{K})\|\mathbf{J}^m + \mathbf{K}\|^{4\tau-\bar{\alpha}-1} \\ &\qquad \ll \|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} + \|\mathbf{J}^m + \mathbf{K}_{\min}\|^{4\tau-\bar{\alpha}-1}. \end{aligned} \tag{4.6}$$

By (4.3),

$$\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})\|\mathbf{K}\| \ll m^{1-\gamma}, \tag{4.7}$$

and, similar to (4.6),

$$\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})\frac{1}{\|\mathbf{J}^m + \mathbf{K}\|} \ll \frac{1}{\|\mathbf{J}^m\|} + \frac{1}{\|\mathbf{J}^m + \mathbf{K}_{\min}\|}. \tag{4.8}$$

From (4.5) - (4.8),

$$\begin{aligned} &\left| E\left\{\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right\} - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right| \\ &\qquad \ll m^{d-4\tau}\Big\{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} + m^{(4\tau-\bar{\alpha}-1)(1-\gamma)} I_{\{4\tau-\bar{\alpha}-1>0\}} \\ &\qquad\quad + \|\mathbf{J}^m + \mathbf{K}_{\min}\|^{4\tau-\bar{\alpha}-1} I_{\{4\tau-\bar{\alpha}-1\le 0\}} + \frac{1}{\|\mathbf{J}^m\|} \\ &\qquad\quad + \frac{1}{\|\mathbf{J}^m + \mathbf{K}_{\min}\|} + \langle m\rangle^{4\tau-\bar{\alpha}-1} + \|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} m^{1-\gamma}\Big\} \\ &\qquad \ll m^{d-4\tau}\Big\{\|\mathbf{J}^m + \mathbf{K}_{\min}\|^{4\tau-\bar{\alpha}-1} I_{\{4\tau-\bar{\alpha}-1\le 0\}} + \frac{1}{\|\mathbf{J}^m\|} \\ &\qquad\quad + \frac{1}{\|\mathbf{J}^m + \mathbf{K}_{\min}\|} + \langle m\rangle^{4\tau-\bar{\alpha}-1} + \|\mathbf{J}^m\|^{4\tau-\bar{\alpha}-1} m^{1-\gamma}\Big\}. \end{aligned}$$

With $\sqrt{\bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)} \asymp m^{d-4\tau}\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}}$, Proposition 3 follows.

Proof of Proposition 5

When $2\mathbf{J}^m/m \notin \mathbb{Z}^d$,

$$\begin{aligned}
&\mathrm{var}\left\{I^\tau_{m,\delta}\right\} - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b} \\
&\quad = a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a)a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) + a^\tau_{m,\delta}(\mathbf{J}^m,-\mathbf{J}^m;a,b)a^\tau_{m,\delta}(\mathbf{J}^m,-\mathbf{J}^m;a,b)^* - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b} \\
&\quad = \left\{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}\right\}\left\{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}\right\} \\
&\qquad + \left\{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}\right\}\bar{f}^\tau_{\delta,b} + \left\{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}\right\}\bar{f}^\tau_{\delta,a} \\
&\qquad + \left|a^\tau_{m,\delta}(\mathbf{J}^m,-\mathbf{J}^m;a,b)\right|^2.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\mathrm{var}\left\{I^\tau_{m,\delta}\right\} - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b}}{\bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b}} \quad = \quad & \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}}{\bar{f}^\tau_{\delta,a}} \times \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}}{\bar{f}^\tau_{\delta,b}} \\
& + \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}}{\bar{f}^\tau_{\delta,a}} + \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}}{\bar{f}^\tau_{\delta,b}} \\
& + \frac{\left|a^\tau_{m,\delta}(\mathbf{J}^m,-\mathbf{J}^m;a,b)\right|^2}{\bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b}}.
\end{aligned}$$

Note that we have

$$\frac{1}{\|\mathbf{J}^m\|^{4\tau-\alpha_k+1}} \ll \frac{1}{\|\mathbf{J}^m\|^{4\tau-\alpha_{\max}+1}} \quad \text{for } j = a, b, \tag{4.9}$$

and

$$\frac{1}{\|\mathbf{J}^m\|^{4\tau-\bar{\alpha}+1}} \ll \frac{1}{\|\mathbf{J}^m\|^{4\tau-\alpha_{\max}+1}}. \tag{4.10}$$

Then, by Proposition 1 in Stein (1995), Lemma 4 and (4.9), equation (2.11) follows.

When $2\mathbf{J}^m/m \in \mathbb{Z}^d$,

$$\begin{aligned}
&\mathrm{var}\left\{I^\tau_{m,\delta}\right\} - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b} - \left|\bar{f}^\tau_{\delta,ab}\right|^2 \\
&\quad = a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a)a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) + a^\tau_{m,\delta}(\mathbf{J}^m,-\mathbf{J}^m;a,b)a^\tau_{m,\delta}(\mathbf{J}^m,-\mathbf{J}^m;a,b)^* \\
&\qquad - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b} - \left|\bar{f}^\tau_{\delta,ab}\right|^2 \\
&\quad = a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a)a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) + a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,b)a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,b)^* \\
&\qquad - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b} - \left|\bar{f}^\tau_{\delta,ab}\right|^2 \\
&\quad = \left(a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}\right)\left(a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}\right) \\
&\qquad + \left(a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}\right)\bar{f}^\tau_{\delta,b} + \left(a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}\right)\bar{f}^\tau_{\delta,a} \\
&\qquad + \left(\left|a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,b)\right| - \left|\bar{f}^\tau_{\delta,ab}\right|\right)\left(\left|a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,b)\right| + \left|\bar{f}^\tau_{\delta,ab}\right|\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\mathrm{var}\left\{I^\tau_{m,\delta}\right\} - \bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b} - \left|\bar{f}^\tau_{\delta,ab}\right|^2}{\bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b}} \quad = \quad & \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}}{\bar{f}^\tau_{\delta,a}} \times \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}}{\bar{f}^\tau_{\delta,b}} \\
& + \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,a) - \bar{f}^\tau_{\delta,a}}{\bar{f}^\tau_{\delta,a}} + \frac{a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;b,b) - \bar{f}^\tau_{\delta,b}}{\bar{f}^\tau_{\delta,b}} \\
& + \frac{\left|a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,b)\right| - \left|\bar{f}^\tau_{\delta,ab}\right|}{\sqrt{\bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b}}} \times \frac{\left|a^\tau_{m,\delta}(\mathbf{J}^m,\mathbf{J}^m;a,b)\right| + \left|\bar{f}^\tau_{\delta,ab}\right|}{\sqrt{\bar{f}^\tau_{\delta,a}\bar{f}^\tau_{\delta,b}}}.
\end{aligned}$$

By Proposition 1 in Stein (1995), Proposition 2, Lemma 4, (4.9), (4.10) and

$$\frac{\bar{f}^{\tau}_{\delta,a}\bar{f}^{\tau}_{\delta,b}}{\bar{f}^{\tau}_{\delta,a}\bar{f}^{\tau}_{\delta,b} + \left|\bar{f}^{\tau}_{\delta,ab}\right|^2} \ll 1,$$

equation (2.12) follows.

Proof of Proposition 6

Since (2.14) holds under our assumption, it is enough to show

$$\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \mathrm{var}\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K}); a, b\right)\right\} \sim \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m). \tag{4.11}$$

We have

$$\begin{aligned}
&\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \mathrm{var}\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K}); a, b\right)\right\} - \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)\\
&= \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \Big[\mathrm{var}\left\{I^{\tau}_{m,\delta}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K}); a, b\right)\right\}\\
&\qquad - \bar{f}^{\tau}_{\delta,a}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right)\bar{f}^{\tau}_{\delta,b}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right)\Big]\\
&\quad + \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \Big[\bar{f}^{\tau}_{\delta,a}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right)\bar{f}^{\tau}_{\delta,b}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right)\\
&\qquad - \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\bar{f}^{\tau}_{\delta,a}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right)\bar{f}^{\tau}_{\delta,b}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right) - \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)\\
&= \Big\{\bar{f}^{\tau}_{\delta,a}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right) - \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\Big\}\Big\{\bar{f}^{\tau}_{\delta,b}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right) - \bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)\Big\}\\
&\quad + \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\Big\{\bar{f}^{\tau}_{\delta,b}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right) - \bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)\Big\}\\
&\quad + \bar{f}^{\tau}_{\delta,b}(2\pi m^{-1}\mathbf{J}^m)\Big\{\bar{f}^{\tau}_{\delta,a}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right) - \bar{f}^{\tau}_{\delta,a}(2\pi m^{-1}\mathbf{J}^m)\Big\}.
\end{aligned}$$

Define

$$\lambda_{h,ab}(\mathbf{J}^m) = \frac{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2\Big[\mathrm{var}\Big\{I^{\tau}_{m,\delta}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}; a, b\Big)\Big\} - \bar{f}^{\tau}_{\delta,a}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\Big)\bar{f}^{\tau}_{\delta,b}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\Big)\Big]}{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}(\frac{2\pi\mathbf{J}^m}{m})\bar{f}^{\tau}_{\delta,b}(\frac{2\pi\mathbf{J}^m}{m})}$$

and

$$\begin{aligned}
\phi_{h,ab}(\mathbf{J}^m) &= \frac{1}{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2}\Bigg\{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \frac{\bar{f}^{\tau}_{\delta,a}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\Big) - \bar{f}^{\tau}_{\delta,a}\big(\frac{2\pi\mathbf{J}^m}{m}\big)}{\bar{f}^{\tau}_{\delta,a}\big(\frac{2\pi\mathbf{J}^m}{m}\big)} \times\\
&\quad \frac{\bar{f}^{\tau}_{\delta,b}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\Big) - \bar{f}^{\tau}_{\delta,b}\big(\frac{2\pi\mathbf{J}^m}{m}\big)}{\bar{f}^{\tau}_{\delta,b}\big(\frac{2\pi\mathbf{J}^m}{m}\big)} + \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \frac{\bar{f}^{\tau}_{\delta,a}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\Big) - \bar{f}^{\tau}_{\delta,a}\big(\frac{2\pi\mathbf{J}^m}{m}\big)}{\bar{f}^{\tau}_{\delta,a}\big(\frac{2\pi\mathbf{J}^m}{m}\big)}\\
&\quad + \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \frac{\bar{f}^{\tau}_{\delta,b}\Big(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\Big) - \bar{f}^{\tau}_{\delta,b}\big(\frac{2\pi\mathbf{J}^m}{m}\big)}{\bar{f}^{\tau}_{\delta,b}\big(\frac{2\pi\mathbf{J}^m}{m}\big)}\Bigg\}.
\end{aligned}$$

Then

$$\frac{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \mathrm{var}\left\{ I^{\tau}_{m,\delta}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}; a, b\right)\right\}}{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}(\frac{2\pi\mathbf{J}^m}{m})\bar{f}^{\tau}_{\delta,b}(\frac{2\pi\mathbf{J}^m}{m})} - 1 = \lambda_{h,ab}(\mathbf{J}^m) + \phi_{h,ab}(\mathbf{J}^m).$$

Since $\|\mathbf{K}\|/m \ll m^{-\gamma}$, $2(\mathbf{J}^m+\mathbf{K})/m \notin \mathbb{Z}^d$ for sufficiently large m. Thus, by Proposition 5, we have

$$\begin{aligned}\lambda_{h,ab}(\mathbf{J}^m) \;\ll\; & \frac{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\right)\bar{f}^{\tau}_{\delta,b}\left(\frac{2\pi(\mathbf{J}^m+\mathbf{K})}{m}\right)}{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a}(\frac{2\pi\mathbf{J}^m}{m})\bar{f}^{\tau}_{\delta,b}(\frac{2\pi\mathbf{J}^m}{m})}\left[\prod_{j=a,b}\frac{\langle m\rangle^{4\tau-\alpha_j-1}}{\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\alpha_j}}\right.\\ &\left.+\sum_{j=a,b}\frac{\langle m\rangle^{4\tau-\alpha_j-1}}{\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\alpha_j}} + \frac{1}{\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\alpha_{\max}+1}} + \frac{1}{\|\mathbf{J}^m+\mathbf{K}\|}\right].\end{aligned}$$

By (4.4) and the assumption (1.1),

$$\begin{aligned}\lambda_{h,ab}(\mathbf{J}^m) \;\ll\; & \frac{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2}{\left(\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 \|\mathbf{J}^m\|^{2(4\tau-\bar{\alpha})}\right)}\left[\prod_{j=a,b}\langle m\rangle^{2(4\tau-\alpha_j-1)}\right.\\ &+\langle m\rangle^{4\tau-\alpha_{\min}-1}\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\alpha_{\max}} + \langle m\rangle^{4\tau-\alpha_{\max}-1}\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\alpha_{\min}}\\ &\left.+\|\mathbf{J}^m+\mathbf{K}\|^{4\tau-\alpha_{\min}-1} + \|\mathbf{J}^m+\mathbf{K}\|^{2(4\tau-\bar{\alpha})-1}\right]\\ \ll\; & \frac{1}{\|\mathbf{J}^m\|^{2(4\tau-\bar{\alpha})}}\left[\prod_{j=a,b}\langle m\rangle^{4\tau-\alpha_j-1} + \langle m\rangle^{4\tau-\alpha_{\min}-1}\left\{\left(\|\mathbf{J}^m\|^{4\tau-\alpha_{\max}}\right.\right.\right.\\ &\left.\left.+m^{(4\tau-\alpha_{\max})(1-\gamma)}\right)I_{\{4\tau>\alpha_{\max}\}} + \|\mathbf{J}^m+\mathbf{K}_{\min}\|^{4\tau-\alpha_{\max}}I_{\{4\tau\le\alpha_{\max}\}}\right\}\\ &+\langle m\rangle^{4\tau-\alpha_{\max}-1}\left\{\left(\|\mathbf{J}^m\|^{4\tau-\alpha_{\min}} + m^{(4\tau-\alpha_{\min})(1-\gamma)}\right)I_{\{4\tau>\alpha_{\min}\}}\right.\\ &\left.+\|\mathbf{J}^m+\mathbf{K}_{\min}\|^{4\tau-\alpha_{\min}}I_{\{4\tau\le\alpha_{\min}\}}\right\}\\ &+\left\{\|\mathbf{J}^m\|^{4\tau-\alpha_{\min}-1} + m^{(4\tau-\alpha_{\min}-1)(1-\gamma)}\right\}I_{\{4\tau>\alpha_{\min}+1\}}\\ &+\|\mathbf{J}^m+\mathbf{K}_{\min}\|^{4\tau-\alpha_{\min}-1}I_{\{4\tau\le\alpha_{\min}+1\}}\\ &+\left\{\|\mathbf{J}^m\|^{2(4\tau-\bar{\alpha})-1} + m^{(2(4\tau-\bar{\alpha})-1)(1-\gamma)}\right\}I_{\{2(4\tau-\bar{\alpha})>1\}}\\ &\left.+\|\mathbf{J}^m+\mathbf{K}_{\min}\|^{2(4\tau-\bar{\alpha})-1}I_{\{2(4\tau-\bar{\alpha})\le 1\}}\right].\end{aligned} \tag{4.12}$$

Since $\|\mathbf{J}^m\| \asymp m$ implies $\|\mathbf{J}^m+\mathbf{K}_{\min}\| \asymp m$, (4.12) becomes

$$\begin{aligned}\lambda_{h,ab}(\mathbf{J}^m) \;\ll\; & m^{-2}\left(1+\log m I_{\{4\tau=\alpha_a+1\}}\right)\left(1+\log m I_{\{4\tau=\alpha_b+1\}}\right)\\ &+\langle m\rangle^{4\tau-\alpha_{\min}-1}m^{-(4\tau-\alpha_{\min})} + \langle m\rangle^{4\tau-\alpha_{\max}-1}m^{-(4\tau-\alpha_{\max})}\\ &+m^{-(4\tau-\alpha_{\max}+1)} + m^{-1}\\ \ll\; & \left(\frac{\log m}{m}\right)^2 + \frac{\log m}{m} + m^{-(4\tau-\alpha_{\max}+1)} + m^{-1}\\ \longrightarrow\; & 0 \quad \text{as} \quad m\to\infty.\end{aligned} \tag{4.13}$$

By (4.2) and the assumption (1.1),

$$\begin{aligned}\phi_{h,ab}(\mathbf{J}^m) &\ll \frac{1}{\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2}\left\{\|\mathbf{J}^m\|^{-2}\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2\|\mathbf{K}\|^2+\|\mathbf{J}^m\|^{-1}\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2\|\mathbf{K}\|\right\}\\ &\ll m^{-\gamma}\\ &\longrightarrow 0 \qquad \text{as} \ \ m\to\infty. \end{aligned}\tag{4.14}$$

The last steps follows from

$$\begin{aligned}\sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2\|\mathbf{K}\|^s &= \sum_{\mathbf{K}\in\mathcal{T}_m,\|2\pi\mathbf{K}/m\|\le h} W_h(\mathbf{K})^2\|\mathbf{K}\|^s\\ &\ll \sum_{\mathbf{K}\in\mathcal{T}_m,\|2\pi\mathbf{K}/m\|\le h} W_h(\mathbf{K})^2 m^{s(1-\gamma)}\\ &= \sum_{\mathbf{K}\in\mathcal{T}_m} W_h(\mathbf{K})^2 m^{s(1-\gamma)},\end{aligned}$$

for $s>0$. Finally, by (4.13) and (4.14), (4.11) holds.

Proof of Lemma 9

By assuming (1.2) and (2.3), we have

$$m^{-(d-\bar{\alpha})}\bar{f}^{\tau}_{\delta,\,ab}(2\pi m^{-1}\mathbf{J}^m)\longrightarrow$$

$$c_{ab}\left\{\sum_{j=1}^{d}4\sin^2\left(\frac{\mu_j}{2}\right)\right\}^{2\tau}\sum_{\mathbf{Q}\in\mathbb{Z}^d}|\boldsymbol{\mu}+2\pi\mathbf{Q}|^{-\bar{\alpha}}\exp\left\{i\,\theta_{ab}\left(\frac{\boldsymbol{\mu}+2\pi\mathbf{Q}}{|\boldsymbol{\mu}+2\pi\mathbf{Q}|}\right)\right\},\tag{4.15}$$

because $m^{-(d-\bar{\alpha})}|\bar{f}^{\tau}_{\delta,\,ab}(2\pi m^{-1}\mathbf{J}^m)-\bar{f}^{\tau}_{\delta,\,ab}(\boldsymbol{\mu})|\to 0$ as $m\to\infty$. Also, note that we have $m^{-(d-\bar{\alpha})}|E\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)-\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)|\to 0$ by Proposition 3. Thus, as $m\to\infty$,

$$\begin{aligned}&\left|m^{-(d-\bar{\alpha})}E\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)-g_{ab}(\boldsymbol{\mu})\right|\\ &\qquad\le\ m^{-(d-\bar{\alpha})}\left|E\hat{f}_{h,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)-\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right|\\ &\qquad\quad+\ \left|m^{-(d-\bar{\alpha})}\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)-g_{ab}(\boldsymbol{\mu})\right|\\ &\qquad\longrightarrow\ 0.\end{aligned}$$

Proof of Proposition 10

Consider the following normalized covariance between two smoothed cross-periodograms:

$$\psi_h(\mathbf{J}_1^m,\mathbf{J}_2^m):=m^{d(1-\gamma)}\mathrm{cov}\Big\{m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}_1^m\right),m^{-(d-\bar{\alpha}_2)}\hat{f}_{h,a_2b_2}\left(2\pi m^{-1}\mathbf{J}_2^m\right)\Big\}.$$

By (3.2) and Lemma 8,

$$
\begin{aligned}
&\psi_h(\mathbf{J}_1^m, \mathbf{J}_2^m) \\
&= m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} \mathrm{cum}\left\{\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}_1^m\right), \hat{f}^*_{h,a_2b_2}\left(2\pi m^{-1}\mathbf{J}_2^m\right)\right\} \\
&= m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)}(2\pi m)^{-2d} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2) \times \\
&\quad \mathrm{cum}\Big\{D_{a_1}(2\pi m^{-1}(\mathbf{J}_1^m+\mathbf{K}_1))D_{b_1}(-2\pi m^{-1}(\mathbf{J}_1^m+\mathbf{K}_1)), \\
&\qquad D_{a_2}(-2\pi m^{-1}(\mathbf{J}_2^m+\mathbf{K}_2))D_{b_2}(2\pi m^{-1}(\mathbf{J}_2^m+\mathbf{K}_2))\Big\} \\
&= m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)}(2\pi m)^{-2d} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2) \times \\
&\quad \Big[\mathrm{cum}\Big\{D_{a_1}(2\pi m^{-1}(\mathbf{J}_1^m+\mathbf{K}_1)), D_{b_2}(2\pi m^{-1}(\mathbf{J}_2^m+\mathbf{K}_2))\Big\} \times \\
&\qquad \mathrm{cum}\Big\{D_{a_2}(-2\pi m^{-1}(\mathbf{J}_2^m+\mathbf{K}_2)), D_{b_1}(-2\pi m^{-1}(\mathbf{J}_1^m+\mathbf{K}_1))\Big\} \\
&\quad + \mathrm{cum}\Big\{D_{a_1}(2\pi m^{-1}(\mathbf{J}_1^m+\mathbf{K}_1)), D_{a_2}(-2\pi m^{-1}(\mathbf{J}_2^m+\mathbf{K}_2))\Big\} \times \\
&\qquad \mathrm{cum}\Big\{D_{b_2}(2\pi m^{-1}(\mathbf{J}_2^m+\mathbf{K}_2)), D_{b_1}(-2\pi m^{-1}(\mathbf{J}_1^m+\mathbf{K}_1))\Big\}\Big] \\
&= m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2) \times \\
&\quad \Big\{a^\tau_{m,\delta}(\mathbf{J}_1^m+\mathbf{K}_1, -(\mathbf{J}_2^m+\mathbf{K}_2); a_1, b_2) a^\tau_{m,\delta}(-(\mathbf{J}_2^m+\mathbf{K}_2), \mathbf{J}_1^m+\mathbf{K}_1; a_2, b_1) \\
&\quad + a^\tau_{m,\delta}(\mathbf{J}_1^m+\mathbf{K}_1, \mathbf{J}_2^m+\mathbf{K}_2; a_1, a_2) a^\tau_{m,\delta}(\mathbf{J}_2^m+\mathbf{K}_2, \mathbf{J}_1^m+\mathbf{K}_1; b_2, b_1)\Big\} \\
&= m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} \Big\{A(\mathbf{J}_1^m, \mathbf{J}_2^m) + B(\mathbf{J}_1^m, \mathbf{J}_2^m)\Big\}, \qquad (4.16)
\end{aligned}
$$

where

$$
A(\mathbf{J}_1^m, \mathbf{J}_2^m) = \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2)\, a^\tau_{m,\delta}(\mathbf{J}_1^m+\mathbf{K}_1, -(\mathbf{J}_2^m+\mathbf{K}_2); a_1, b_2) \times a^\tau_{m,\delta}(-(\mathbf{J}_2^m+\mathbf{K}_2), \mathbf{J}_1^m+\mathbf{K}_1; a_2, b_1),
$$

and

$$
B(\mathbf{J}_1^m, \mathbf{J}_2^m) = \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2)\, a^\tau_{m,\delta}(\mathbf{J}_1^m+\mathbf{K}_1, \mathbf{J}_2^m+\mathbf{K}_2; a_1, a_2) \times a^\tau_{m,\delta}(\mathbf{J}_2^m+\mathbf{K}_2, \mathbf{J}_1^m+\mathbf{K}_1; b_2, b_1).
$$

Next, write $B(\mathbf{J}_1^m, \mathbf{J}_2^m) = B_1(\mathbf{J}_1^m, \mathbf{J}_2^m) + B_2(\mathbf{J}_1^m, \mathbf{J}_2^m)$, where B_1 is the sum over terms in B with $\mathbf{K}_1 = \mathbf{K}_2$ and B_2 is the sum over terms in B with $\mathbf{K}_1 \neq \mathbf{K}_2$.

First, consider $\mathbf{J}_1^m = \mathbf{J}_2^m = \mathbf{J}^m$. From (4.16), we have

$$
\psi_h(\mathbf{J}^m, \mathbf{J}^m) = m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} \Big\{A(\mathbf{J}^m, \mathbf{J}^m) + B_1(\mathbf{J}^m, \mathbf{J}^m) + B_2(\mathbf{J}^m, \mathbf{J}^m)\Big\}.
$$

Since $\mathbf{J}^m + \mathbf{K}_1 \neq -\mathbf{J}^m + \mathbf{K}_2$ for large enough m,

$$\begin{aligned}
& m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)}\left|A(\mathbf{J}^m,\mathbf{J}^m)+B_2(\mathbf{J}^m,\mathbf{J}^m)\right| \\
&\quad \ll\ m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2)m^{-(\alpha_1+\alpha_2)+2d-2} \\
&\quad \leq\ m^{d(1-\gamma)-2} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2) \\
&\quad \longrightarrow\ 0, \qquad (4.17)
\end{aligned}$$

the last step following from $\frac{d-2}{d} < \gamma$ and $\sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2) < \infty$, which follows from (4.22).

Then, to prove the Proposition for $\mathbf{J}_1^m = \mathbf{J}_2^m = \mathbf{J}^m$, it is enough to show that

$$\left| m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} B_1(\mathbf{J}^m,\mathbf{J}^m) \ - \ \left\{(2\pi/C)^d \bar{K}_2/\bar{K}_1^2\right\} g_{a_1a_2}(\boldsymbol{\mu}_1)\, g_{b_2b_1}(\boldsymbol{\mu}_1)\right| \ \longrightarrow\ 0.$$

From Lemma 1, we have

$$\left|a^{\tau}_{m,\delta}(\mathbf{J}^m+\mathbf{K},\mathbf{J}^m+\mathbf{K};a,b) - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right)\right| = o(m^{d-\bar{\alpha}}).$$

Since $2\pi m^{-1}(\mathbf{J}^m+\mathbf{K}) = 2\pi m^{-1}\mathbf{J}^m + O(m^{-\gamma})$ for $\mathbf{K}\in\Gamma_m$,

$$|\bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}(\mathbf{J}^m+\mathbf{K})\right) - \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)| \ll \sum_j \left|\frac{\partial \bar{f}^{\tau}_{\delta,ab}(2\pi m^{-1}\mathbf{J}^m)}{\partial\omega_j}\right| m^{-\gamma} = o(m^{d-\bar{\alpha}}),$$

the last step following from (2.3). Thus, we have

$$m^{-(d-\bar{\alpha})}\left|a^{\tau}_{m,\delta}(\mathbf{J}^m+\mathbf{K},\mathbf{J}^m+\mathbf{K};a,b) \ - \ \bar{f}^{\tau}_{\delta,ab}\left(2\pi m^{-1}\mathbf{J}^m\right)\right| \ = \ o(1). \qquad (4.18)$$

By (4.22), we also have

$$\left| m^{d(1-\gamma)}\sum_{\mathbf{K}} W_h(\mathbf{K})^2 \ - \ \left\{(2\pi/C)^d\bar{K}_2/\bar{K}_1^2\right\}\right| \ = \ o(1). \qquad (4.19)$$

Finally,

$$\begin{aligned}
&\left| m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)} B_1(\mathbf{J}^m,\mathbf{J}^m) \ - \ \left\{(2\pi/C)^d\bar{K}_2/\bar{K}_1^2\right\} g_{a_1a_2}(\boldsymbol{\mu}_1)\, g_{b_2b_1}(\boldsymbol{\mu}_1)\right| \\
&\quad \leq\ m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)}\sum_{\mathbf{K}} W_h(\mathbf{K})^2 \Big| a^{\tau}_{m,\delta}(\mathbf{J}^m+\mathbf{K},\mathbf{J}^m+\mathbf{K};a_1,a_2)\,\times \\
&\qquad a^{\tau}_{m,\delta}(\mathbf{J}^m+\mathbf{K},\mathbf{J}^m+\mathbf{K};b_2,b_1) \ - \bar{f}^{\tau}_{\delta,a_1a_2}\left(\frac{2\pi\mathbf{J}^m}{m}\right)\bar{f}^{\tau}_{\delta,b_2b_1}\left(\frac{2\pi\mathbf{J}^m}{m}\right)\Big| \\
&\quad +\ \Big| m^{\bar{\alpha}_1+\bar{\alpha}_2-d(1+\gamma)}\sum_{\mathbf{K}} W_h(\mathbf{K})^2 \bar{f}^{\tau}_{\delta,a_1a_2}\left(\frac{2\pi\mathbf{J}^m}{m}\right)\times \\
&\qquad \bar{f}^{\tau}_{\delta,b_2b_1}\left(\frac{2\pi\mathbf{J}^m}{m}\right) - \left\{(2\pi/C)^d\bar{K}_2/\bar{K}_1^2\right\} g_{a_1a_2}(\boldsymbol{\mu}_1)\, g_{b_2b_1}(\boldsymbol{\mu}_1)\Big|. \qquad (4.20)
\end{aligned}$$

By (4.15) and (4.17) - (4.19), the right side of (4.20) goes to zero as $m\to\infty$. Using a similar argument as above, we can also show that

$$\begin{aligned}
& m^{d(1-\gamma)}\mathrm{cov}\Big\{m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(2\pi m^{-1}\mathbf{J}_1^m\right), m^{-(d-\bar{\alpha}_2)}\hat{f}_{h,a_2b_2}\left(2\pi m^{-1}\mathbf{J}_2^m\right)\Big\} \\
&\qquad\qquad \longrightarrow\ \left\{(2\pi/C)^d\bar{K}_2/\bar{K}_1^2\right\} g_{a_1b_2}(\boldsymbol{\mu}_1)\, g_{a_2b_1}(\boldsymbol{\mu}_1),
\end{aligned}$$

when $\mathbf{J}_1^m = -\mathbf{J}_2^m = \mathbf{J}^m$.

Now consider $\boldsymbol{\mu}_1 \neq \pm\boldsymbol{\mu}_2$. For large enough m, all indices of $a^{\tau}_{m,\delta}$ in (4.16) are different. Thus,

$$\begin{aligned}
&\Big| m^{d(1-\gamma)} \mathrm{cov}\big(m^{-(d-\bar\alpha_1)} \hat{f}_{h,a_1 b_1}\left(2\pi m^{-1}\mathbf{J}_1^m\right), m^{-(d-\bar\alpha_2)} \hat{f}_{h,a_2 b_2}\left(2\pi m^{-1}\mathbf{J}_2^m\right)\big)\Big| \\
&\ll m^{\bar\alpha_1+\bar\alpha_2-d(1+\gamma)} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2)\, m^{\sum_{s=1,2}(d-\bar\alpha_s)-2} \\
&\leq m^{d(1-\gamma)-2} \sum_{\mathbf{K}_1,\mathbf{K}_2\in\Gamma_m} W_h(\mathbf{K}_1)W_h(\mathbf{K}_2) \\
&\longrightarrow 0,
\end{aligned}$$

the last step following from the same argument as (4.17).

Proof of Lemma 11

Since a, b in (3.4) play no essential role in this proof except that each one appears only once in an indecomposable set, we ignore these subscripts. For each indecomposable set, we can reorder the indices so that $i_k = k$, so we can write down one summand in (3.4) as follows:

$$\sum_{\mathbf{K}_1,\cdots,\mathbf{K}_r} \prod_{i=1}^{r} W_h(\mathbf{K}_i) \prod_{j=1}^{r} a^{\tau}_{m,\delta}\big((-1)^{\bar e_j-1}(\mathbf{J}_j+\mathbf{K}_j), (-1)^{e_{j+1}}(\mathbf{J}_{j+1}+\mathbf{K}_{j+1})\big),$$

where we assume that $r+1=1$.

Since $\|\mathbf{J}_i\| \asymp m$ and $\mathbf{K}_i \in \Gamma_m$, we have $\|\mathbf{J}_i+\mathbf{K}_i\| \asymp m$ for $i = 1, \cdots, r$. Thus, by Lemma 1, $|a^{\tau}_{m,\delta}(\mathbf{J}_i+\mathbf{K}_i, \mathbf{J}_j+\mathbf{K}_j)| \ll m^{d-\bar\alpha}$ if $\mathbf{J}_i+\mathbf{K}_i = \mathbf{J}_j+\mathbf{K}_j$ and by Proposition 4, $|a^{\tau}_{m,\delta}(\mathbf{J}_i+\mathbf{K}_i, \mathbf{J}_j+\mathbf{K}_j)| \ll m^{d-\bar\alpha-1}$ if $\mathbf{J}_i+\mathbf{K}_i \neq \mathbf{J}_j+\mathbf{K}_j$. Note that, for large enough m, $\mathbf{J}_i+\mathbf{K}_i \neq -(\mathbf{J}_j+\mathbf{K}_j)$ when $\mathbf{J}_i = \mathbf{J}_j$. Thus, the largest possible order of m comes with the case

$$\sum_{\mathbf{K}_1,\cdots,\mathbf{K}_r} \prod_{i=1}^{r} W_h(\mathbf{K}_i) \prod_{j=1}^{r} a^{\tau}_{m,\delta}\big(\mathbf{J}+\mathbf{K}_j, \mathbf{J}+\mathbf{K}_{j+1}\big).$$

Define

$$\beta_{h,r}(\mathbf{J}) = m^{-\sum_{s=1,\cdots,r}(d-\bar\alpha_s)} \sum_{\mathbf{K}_1,\cdots,\mathbf{K}_r} \prod_{i=1}^{r} W_h(\mathbf{K}_i) \prod_{j=1}^{r} a^{\tau}_{m,\delta}\big(\mathbf{J}+\mathbf{K}_j, \mathbf{J}+\mathbf{K}_{j+1}\big).$$

To prove the Lemma, it is enough to show that

$$\big|\ m^{\eta r}\beta_{h,r}(\mathbf{J})\ \big| \ \rightarrow\ 0\,, \quad \text{as} \quad m \rightarrow 0\,,$$

because the total number of indecomposable sets is finite.

Now, define a subset of $\{\mathbf{K}_1, \cdots, \mathbf{K}_r\}$, $G_s = \{\mathbf{K}_1, \cdots, \mathbf{K}_r :$ only s distinct groups$\}$, For example, $G_1 = \{\mathbf{K}_1, \cdots, \mathbf{K}_r : \mathbf{K}_1 = \cdots = \mathbf{K}_r\}$ and $G_r = \{\mathbf{K}_1, \cdots, \mathbf{K}_r :$ all $\mathbf{K}_i$ are different $\}$. Then, $\{\mathbf{K}_1, \cdots, \mathbf{K}_r\} = G_1 \cup \cdots \cup G_r$, G_s are disjoint and the number of elements in G_s, $|G_s| = O(L_m^{s\,d}) = O(m^{sd(1-\gamma)})$. In each group, G_s , for $s = 2, \cdots, r$, there are at least s

cases of $\mathbf{J}+\mathbf{K}_j \neq \mathbf{J}+\mathbf{K}_{j+1}$ for $j=1,\cdots,r$. Thus,

$$
\begin{aligned}
\mid m^{\eta r}\beta_{h,r}(\mathbf{J})\mid &\leq m^{\eta r-\sum_{s=1,\cdots,r}(d-\bar{\alpha}_s)}\Big\{\sum_{G_1}\prod_{i=1}^{r}W_h(\mathbf{K}_i)\prod_{j=1}^{r}\Big|a^{\tau}_{m,\delta}\big(\mathbf{J}+\mathbf{K}_j,\mathbf{J}+\mathbf{K}_{j+1}\big)\Big| \\
&\qquad \cdots+\sum_{G_r}\prod_{i=1}^{r}W_h(\mathbf{K}_i)\prod_{j=1}^{r}\Big|a^{\tau}_{m,\delta}\big(\mathbf{J}+\mathbf{K}_j,\mathbf{J}+\mathbf{K}_{j+1}\big)\Big|\Big\} \\
&\ll m^{d(1-\gamma)r/2-\sum_{s=1,\cdots,r}(d-\bar{\alpha}_s)}\Big\{\sum_{G_1}\prod_{i=1}^{r}W_h(\mathbf{K}_i)m^{\sum_{s=1,\cdots,r}(d-\bar{\alpha}_s)} \\
&\qquad +\sum_{G_2}\prod_{i=1}^{r}W_h(\mathbf{K}_i)m^{\sum_{s=1,\cdots,r}(d-\bar{\alpha}_s)}m^{-2}+ \\
&\qquad \cdots+\sum_{G_r}\prod_{i=1}^{r}W_h(\mathbf{K}_i)m^{\sum_{s=1,\cdots,r}(d-\bar{\alpha}_s)}m^{-r}\Big\} \\
&\ll m^{d(1-\gamma)r/2}\Big\{\sum_{\mathbf{K}\in\Gamma_m}W_h(\mathbf{K})^r+m^{-2}\sum_{\mathbf{K}_1\neq\mathbf{K}_2}\sum_{s_2=1}^{r-1}W_h(\mathbf{K}_1)^{s_2}W_h(\mathbf{K}_2)^{r-s_2}+ \\
&\qquad \cdots+m^{-r}\sum_{\mathbf{K}_1\neq\cdots\neq\mathbf{K}_r}W_h(\mathbf{K}_1)\cdots W_h(\mathbf{K}_r)\Big\}
\end{aligned}
\tag{4.21}
$$

Note that $\sum_{\mathbf{K}\in\Gamma_m}K(\frac{2\pi\mathbf{K}}{mh})^s(\frac{2\pi}{mh})^d=\int_{(-1,1]^d}K(\mathbf{x})^s d\mathbf{x}+O((mh)^{-1})$, since $K(\mathbf{x})$ is continuous. Then, for $r_1+\cdots+r_s=r$, $r_i\geq 1$ and $s=1,\cdots,r$,

$$
\begin{aligned}
&\sum_{\mathbf{K}_1,\cdots,\mathbf{K}_s\in\Gamma_m}W_h(\mathbf{K}_1)^{r_1}\cdots W_h(\mathbf{K}_s)^{r_s} \\
&\qquad =\left(\frac{2\pi}{mh}\right)^{d(r-s)}\frac{\sum_{\mathbf{K}_1}K(\frac{2\pi\mathbf{K}_1}{mh})^{r_1}(\frac{2\pi}{mh})^d\cdots\sum_{\mathbf{K}_s}K(\frac{2\pi\mathbf{K}_s}{mh})^{r_s}(\frac{2\pi}{mh})^d}{\left[\sum_{\mathbf{K}}K(\frac{2\pi\mathbf{K}}{mh})(\frac{2\pi}{mh})^d\right]^r} \\
&\qquad =\left(\frac{2\pi}{mh}\right)^{d(r-s)}\left\{\frac{\bar{K}_{r_1}\cdots\bar{K}_{r_s}}{\bar{K}_1^r}+O\left(\frac{1}{mh}\right)\right\} \\
&\qquad =O\left(m^{-d(1-\gamma)(r-s)}\right),
\end{aligned}
\tag{4.22}
$$

because we have $\int_{(-1,1]^d}K(\mathbf{x})^s d\mathbf{x}<\infty$ for each positive integer s. Thus, by (4.22), (4.21) becomes

$$
\begin{aligned}
\mid m^{\eta r}\beta_{h,r}(\mathbf{J})\mid &\ll m^{\frac{1}{2}d(1-\gamma)r}\Big\{m^{-d(1-\gamma)(r-1)}+m^{-2-d(1-\gamma)(r-2)}+ \\
&\qquad \cdots+m^{-r-d(1-\gamma)(r-r)}\Big\} \\
&\ll m^{-d(1-\gamma)r/2+d(1-\gamma)}+\sum_{j=2}^{r}m^{-d(1-\gamma)r/2+d(1-\gamma)j-j} \\
&\longrightarrow\ 0.
\end{aligned}
\tag{4.23}
$$

The last step in (4.23) follows from $-d(1-\gamma)r/2+d(1-\gamma)<0$ and $-d(1-\gamma)r/2+d(1-\gamma)j-j<0$ for $j=2,\cdots,r$ because $r\geq 3$ and $(d-2)/d<\gamma<1$.

Proof of Proposition 12

To prove the fixed-domain asymptotic distribution is Gaussian, it is enough to show that the rth order joint cumulant of $m^{\eta}\left(m^{-(d-\bar{\alpha}_1)}\hat{f}_{h,a_1b_1}\left(\pm 2\pi m^{-1}\mathbf{J}_1^m\right), \cdots, m^{-(d-\bar{\alpha}_n)}\hat{f}_{h,a_nb_n}\left(\pm 2\pi m^{-1}\mathbf{J}_n^m\right)\right)$ goes to 0 as $m \to \infty$, for $r \geq 3$. (see Lemma P4.5 in Brillinger (1981)). This result follows from Lemma 11, and Proposition 12 follows from Lemma 9 and Proposition 10.

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