STATISTICS 244 **Problem Set 3** **9:30-10:50 TTh**

Due Thursday January 31

1. Assume the random variable X has the Bernoulli distribution (i.e., binomial with n = 1): $P(X = 1) = \theta = 1 - P(X = 0)$.
(a) Find $E(\sqrt{X})$. (b) Find $E(X^3)$. (c) Find $E(X^{10})$. (d) Find $Var(X^3)$. (e) Find $E(3X^3+4)$.
(f) Find $Var(3X^3+4)$.

2. Consider the following bivariate distribution:

		Y = 2	Y = 3	Y = 4	Y = 5
X	1	.1	.2	.0	.0
X	2	.0	.2	.1	.1
X	3	.0	.0	.1	.2

(a) Find the marginal distributions of X and Y. (b) Find the conditional distribution of Y given X = 1. Find the conditional distribution of Y given X = 2. (c) Find E(Y).
(d) Find the covariance of X and Y.

3. Suppose an urn contains 3 tickets numbered "1", 3 tickets numbered "2", 2 ticket numbered "3", and 1 ticket numbered "4". A student draws a ticket at random and notes the number, X. The student then returns the ticket to the urn, shakes it up, and draws again, noting the number, Y. Let Z = the minimum of X and Y.
(a) Find the probability distribution of X, the cumulative distribution of X, and graph both.
(b) Find the probability distribution of Z. (c) Find E(X), E(Y), E(Z). (d) Find the bivariate probability function p(x, z) of X and Z, and the covariance of X and Z.

4. Covariance matrices. (See Chapter 3, pp. 3-21, 3-22) Suppose that X_1, X_2, and X_3 are independent random variables with variances 3, 4, and 6, respectively.
Let $Y_1 = X_1 + 3X_2$, $Y_2 = X_3 - 2X_2$, and $Y_3 = X_1 + X_2 + X_3$. (a) Using the general relationship
Cov(W+X, Y+Z) = Cov(W,Y) + Cov(W, Z) + Cov(X, Y) + Cov(X, Z), find
$Cov(Y_i, Y_j)$ for all i, j. (b) Consider now the random vectors $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$. From the hypotheses and the results of part (a), write the 3x3 covariance matrices Cov(**X**) and Cov(**Y**) of **X** and **Y**. Verify that $\mathbf{Y} = \mathbf{A}^T \mathbf{X}$ and that $Cov(\mathbf{Y}) = \mathbf{A}^T Cov(\mathbf{X})\, \mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

5. Consider the bivariate density of X and Y,

$$f(x, y) = 6(x + y)^2/7 \text{ for } 0 < x < 1, 0 < y < 1$$
$$= 0 \quad \text{otherwise.}$$

(a) Verify that this is a bivariate density (that the total volume $\iint f(x,y)dxdy = 1$).
(b) Find the marginal density of Y. (c) Find the conditional density of X given Y = 0.5. (d) Find E(X), $E(X^2)$, Var(X), E(XY), and Cov(X,Y).
(e) Find $P(.1 \le X \le .5 \text{ and } .4 \le Y \le .8)$. (f) Find $P(X+Y \le 1)$

6. Suppose that the pair (X,Y) has the continuous distribution with bivariate density $f(x,y) = Cx(1+y)$ for $0 < x, y < 2$
(a) Find C. (b) Find E(XY) and Cov(X,Y), (c) Find P(X>1.5), (d) Find P(X > Y).

7. Consider the bivariate density of X and Y,

$$f(x, y) = \frac{2}{\pi} \quad \text{for } x^2 + y^2 \le 1 \text{ and } y > x$$
$$= 0 \quad \text{otherwise.}$$

(a) Verify that this is a bivariate density (that is, the total volume $\iint f(x,y)dxdy = 1$). [Hint: Draw a picture]
(b) Find the marginal density of Y. (c) Find the conditional density of X given Y = 0. (d) Find E(X), E(X^2), Var(X), E(XY), and Cov(X,Y). (e) Find P(X≤ 0).
(f) Find P(X+Y≤1)

8. The following is a model used in educational psychology. If a student is selected at random from among the high school seniors in suburban high schools and given a two-part test (Part 1 is "Verbal", Part 2 is "Mathematical"), the scores will be X (on Part 1) and Y (on Part 2). The model says that

$$X = S_1 + E_1$$
$$Y = S_2 + E_2$$

where you may think of S_1 and S_2 as the student's true Verbal and Math abilities, respectively. E_1 and E_2 are measurement errors, the idea being that the tests are not so accurate that they can discover the true abilities exactly, but they record them as if an error were added on. The Total score will then be T = X + Y. The model also says that

$$E(S_1) = E(S_2) = 60$$
$$Var(S_1) = Var(S_2) = 64$$
$$Cov(S_1, S_2) = 40$$
$$E(E_1) = E(E_2) = 0$$
$$Var(E_1) = Var(E_2) = 36$$
$$Cov(E_1, E_2) = 0$$

Furthermore, the S's and the E's are independent.
Find: (a) E(X); (b) σ_X ; (c) Cov(X,Y); (d) ρ_{XY}; (e) E(T); (f) Var(T); (g) Cov(X,T)

9. Here is a useful fact: for any two random variables W and V, E{E(W|V)} = E(W). Here is a possible use. In Problem 3, you have already found the expectation of X, E(X).
(a) Find the expectation of Z given X, E(Z|X=x) as a function of x (i.e. list the answers for x=1, 2, and 3).
(b) Then use the fact to find E(Z) (it should agree with what you found in Problem 3).
(c) Would the computation be any different if you had been asked to start with E(Y) and then find E(Z|Y)? [Hint: What is the distribution of Y?]