STATISTICS 244 **Problem Set 2** **9:30-10:50 TTh**

Due Thursday January 24

(1) Here are three possible probability density functions for a continuous random variable X;

(i) $f(x) = 20x^3(1-x)$ for $0 \le x \le 1$ (= 0 otherwise)

(ii) $f(x) = e^{-(x - 0.1)}$ for $x \ge 0.1$ (= 0 for $x < 0.1$)

(iii) $f(x) = ½ \sin(x)$ for $0 \le x \le \pi$ (= 0 otherwise)

In each case, (a) find the cdf of X; (b) find $\Pr\{0.2 \le X \le 0.5\}$; (c) find E(X);
(d) find the median of the distribution of X;
(e) find the probability density function of $Y = (X + 3)^{1/2}$.

(2) This problem concerns a European style roulette wheel, meaning a wheel with 37 slots numbered 0 to 36. The slots with odd numbers are colored Red, the positive even numbered slots are colored Black, and the Zero slot is colored Green. The order of the numbers (clockwise) is: 0-32-15-19-4-21-2-25-17-34-6-27-13-36-11-30-8-23-10-5-24-16-33-1-20-14-31-9-22-18-29-7-28-12-35-3-26.
The wheel is spun and a single ball is allowed to drop onto the wheel, and is supposed to be equally likely to land in any of the 37 slots, with each spin being independent of the others.
(a) What is the probability the ball lands in a Red slot numbered 20 or larger?
(b) What is the probability the ball lands in a slot that with a higher number than either of its neighbors? (c) If you win \$1 when a Red slot occurs, \$2 when a Black slot occurs, and \$0 when a Green slot occurs, what is the fair price to play the game?
(d) Suppose the wheel is spun twice; let the number from the first spin be X and the number from the second spin be Y. Find $\Pr\{X+Y \le 3\}$.

(3) The Pareto distribution. The Pareto distributions are a family of distributions of a continuous random variable X with probability density function given by

$$f(x;\, \alpha,\, \theta) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}} \text{ for } x \ge \theta$$
$$= 0 \quad \text{for } x < \theta,$$

where $\alpha > 0$ and $\theta > 0$ are the parameters of the family. The Pareto distribution arises as a model for the distribution of sizes of some measured quantity, such as personal or corporate income, or city population, or size of firm, given that it exceeds the threshold θ.
(a) Verify that this is the formula for a density (that is, that the function is non-negative and the area under it is 1.0).
(b) Find the formula for the cumulative distribution function.
(c) Assuming $\alpha > 1$, find E(X). What is E(X) if $0 < \alpha \le 1$?
(d) Show that if $\alpha > 2$, $\mathrm{Var}(X) = \dfrac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$. (e) Find (in terms of α and θ) the median of the distribution of X.
(f) Suppose $\alpha = 3$ and $\theta = 1$. Find $P(1 < X < 3)$ and $P(4 < X < 6)$.

(4) The random laser. Consider a two-ended laser spinner; that is, a pen-like laser acting as the arrow is mounted on a pin at the center of a spinner. Suppose the center of the disk is one meter away from a wall of infinite extent marked with a linear scale, with zero at the point closest the center of the spinner and negative numbers to left, positive to the right. The laser is spun and comes to a rest projecting for one of its ends at a point Y on the scale (with probability zero the laser will stop parallel to the wall and miss it; we ignore that possibility). Suppose that the angle X the laser makes to the perpendicular to the wall is uniformly distributed over $-\pi/2$ to $\pi/2$. Find the probability density of Y.

(5) Consider the roulette wheel of Problem (2). Suppose the wheel is spun 10 times, and let X = the number of times the ball lands in a slot with an even number ≤ 20. What is the probability distribution of X?

(6) Suppose that in a post-election examination of the 2016 Florida Presidential ballots, in those counties that used punched card ballots, a social science researcher wishes to study the relationship between a voter's intent and the voter's tendency to miss-punch the ballot in different ways. In a pilot study, the researcher selects 45 ballots at random from one county's votes, and of these, 30 are suitable for the study (i.e. the voter's intent is clear). The data, once gathered, will form a two-way table:

	Clear punch	Hanging chad	Dimpled chad	Other
Prefers Trump	X_1	X_4	X_7	X_{10}
Prefers Clinton	X_2	X_5	X_8	X_{11}
Other	X_3	X_6	X_9	X_{12}

The total of all 12 entries will be 30. Let $W = X_1 + X_2 + X_3$ be the total number that are clear punched. Suppose (hypothetically) that among all ballots in the county (many thousands): exactly 70% are Clear Punched, exactly 10% have Hanging Chads, exactly 50% Prefer Trump, and exactly 40% Prefer Clinton).
(a) What is the probability distribution of W? What is P{ W = 15 }?
(b) Suppose that in fact voter intent and type of punching are independent of one another. What is the probability distribution of X_1? What is P { X_1 = 10 }? What is the probability distribution of X_3? What is P { X_3 = 2 }? What is the probability distribution of $X_4 + X_5$? What is P { $X_4 + X_5 = 3$ }?
(c) Suppose a different researcher adopts a different sampling scheme, planning instead to choose ballots at random and inspect those with some clear intent until exactly 15 that are clear punched have been studied. Let T be the total number of ballots the second researcher inspects. What is the probability distribution of T, and what is P { T = 20 }?
(d) For the plan of the researcher in (c)(assuming voter preference and type of punching are independent of one another), what is the probability distribution of X_1 and what is P { X_1 = 6 }?

(7) Suppose that the waiting time for the CTA Campus bus at the Reynolds Club stop is a continuous random variable Z (in hours) with an exponential distribution, with density $f(z) = 6e^{-6z}$ for $z \geq 0$; $f(z) = 0$ for $z < 0$.
(a) What is the expected waiting time in minutes (the expected value of Z)? (b) Suppose you have been waiting exactly ½ hour. What is the expected additional waiting time E(W), where W = Z – ½ ? [Hint: For a > ½ , what is the conditional probability Z > a, given Z > ½ ? What is the conditional probability Z < a, given Z > ½ ? What is the conditional probability W < b=a–0.5, given Z > ½ ? What is the conditional density of W?]

(8) Consider the St. Petersburg game, where the coin is biased in favor of heads; in fact, P(H) = 2/3 and P(T) = 1/3, but you are still to receive $\$2^{X-1}$. What is the expected value now? What if P(H) = 1/3 and P(T) = 2/3?