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THE COMPUTATIONAL COMPLEXITY OF DUALITY*

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Abstract. We show that for any given norm ball or proper cone, weak membership in its dual ball or dual cone is polynomial-time reducible to weak membership in the given ball or cone. A consequence is that the weak membership or membership problem for a ball or cone is NP-hard if and only if the corresponding problem for the dual ball or cone is NP-hard. In a similar vein, we show that computation of the dual norm of a given norm is polynomial-time reducible to computation of the given norm. This extends to convex functions satisfying a polynomial growth condition: for such a given function, computation of its Fenchel dual/conjugate is polynomial-time reducible to computation of the given function. Hence the computation of a norm or a convex function of polynomial-growth is NP-hard if and only if the computation of its dual norm or Fenchel dual is NP-hard. We discuss implications of these results on the weak membership problem for a symmetric convex body and its polar dual, the polynomial approximability of Mahler volume, and the weak membership problem for the epigraph of a convex function with polynomial growth and that of its Fenchel dual.

Key words. dual norm, dual cone, Fenchel dual, NP-hard, weak membership, approximation

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1. Introduction. In convex optimization, we often encounter problems that involve one of the following notions of duality: for convex sets, (i) norm balls and their polar duals and (ii) proper cones and their dual cones; for convex functions, (iii) norms and their dual norms and (iv) functions and their Fenchel duals. The main goal of this article is to establish the equivalence between the polynomial-time computability or NP-hardness of these objects and their duals.

We will first show in section 3 that the weak membership problem for a norm ball is NP-hard (resp., is polynomial-time) if and only if the weak membership problem for its dual norm ball is NP-hard (resp., is polynomial-time). For readers unfamiliar with the notion, NP-hardness of *weak* membership is a *stronger* statement than NP-hardness of membership, i.e., the latter is implied by the former. Since every symmetric convex compact set with nonempty interior is a norm ball, the result applies to such objects and their polar duals as well.

In section 4 we show that the approximation of a norm to arbitrary precision is NP-hard (resp., is polynomial-time) if and only if weak membership in the unit ball of the norm is NP-hard (resp., is polynomial-time). A consequence is that if the weak membership problem for a norm ball is polynomial-time decidable, then its Mahler volume is polynomial-time approximable. In fact, computation of Mahler volume is polynomial-time reducible to the weak membership problem for a norm ball.

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In section 5, we establish an analogue of our norm ball result for proper cones, showing that the weak membership problem for such a cone is NP-hard (resp., is polynomial-time) if and only if the weak membership problem for its dual cone is NP-hard (resp., is polynomial-time).

We conclude by showing in section 6 that for convex functions that satisfy a polynomial-growth condition, its Fenchel dual must also satisfy the same condition with possibly different constants. A consequence of this is that such a function is polynomial-time approximable to arbitrary precision if and only if its Fenchel dual is also polynomial-time approximable to arbitrary precision. On the other hand, such a function is NP-hard to approximate if and only if its Fenchel dual is NP-hard to approximate.

2. Weak membership, weak validity, and polynomial-time reducibility.

We introduce some basic terminologies based on [6, Chapter 2] with some natural extensions for our context. Let $B(x,\delta)$ denote the closed Euclidean norm ball of radius $\delta > 0$ centered at x in $\mathbb{R}^n$. For any $\delta > 0$ and any $K \subseteq \mathbb{R}^n$, we define respectively a "thickened" K and a "shrunken" K by

$$S(K,\delta) \coloneqq \bigcup\nolimits_{x\in K} B(x,\delta) \quad \text{and} \quad S(K,-\delta) \coloneqq \{x \in K : B(x,\delta) \subseteq K\}. \tag{1}$$

Note that if K has no interior point, then $S(K,-\delta) = \varnothing$.

Definition 1. *Let $K \subseteq \mathbb{R}^n$ be a convex set with nonempty interior.*

(i) *The* membership problem *(*MEM*) for K is as follows: Given $x \in \mathbb{Q}^n$, determine if x is in K.*

(ii) *The* weak membership problem *(*WMEM*) for K is as follows: Given $x \in \mathbb{Q}^n$ and a rational $\delta > 0$, assert that $x \in S(K,\delta)$ or $x \notin S(K,-\delta)$.*

(iii) *The* weak violation problem *(*WVIOL*) problem for K is as follows: Given $c \in \mathbb{Q}^n$ and rational $\gamma, \varepsilon > 0$, either assert that $c^{\mathsf{T}}x \le \gamma + \varepsilon$ for all $x \in S(K,-\varepsilon)$ or find $y \in S(K,\varepsilon)$ with $c^{\mathsf{T}}y \ge \gamma - \varepsilon$.*

(iv) *The* weak validity problem *(*WVAL*) problem for K is as follows: Given $c \in \mathbb{Q}^n$ and rational $\gamma, \varepsilon > 0$, either assert that $c^{\mathsf{T}}x \le \gamma + \varepsilon$ for all $x \in S(K,-\varepsilon)$ or assert that $c^{\mathsf{T}}x \ge \gamma - \varepsilon$ for some $x \in S(K,\varepsilon)$.*

(v) *The* weak optimization problem *(*WOPT*) problem for K is as follows: Given $c \in \mathbb{Q}^n$ and a rational $\varepsilon > 0$, either find $y \in \mathbb{Q}^n$ such that $y \in S(K,\varepsilon)$ and $c^{\mathsf{T}}x \le c^{\mathsf{T}}y + \varepsilon$ for all $x \in S(K,-\varepsilon)$ or assert that $S(K,\varepsilon) = \varnothing$.*

For the benefit of readers unfamiliar with these notions, we highlight that in our weak membership problem, there are x's that satisfy both $x \in S(K,\delta)$ and $x \notin S(K,-\delta)$ simultaneously. So if we can ascertain MEM, we can ascertain WMEM, but not conversely. A consequence is that if the WMEM problem for K is NP-hard, then MEM for K is also NP-hard.

There will be occasions, particularly in section 5, when we have to discuss weak membership and weak validity of a convex set $K \subseteq \mathbb{R}^n$ of positive codimension, i.e., contained in an affine subspace of dimension less than n. As a subset of $\mathbb{R}^n$, K will have no interior points and WMEM and WVAL as defined above would make little sense as $S(K,-\delta) = \varnothing$. With this in mind, we introduce the following variant of Definition 1 that makes use of the interior of K relative to H, an affine subspace of minimal dimension that contains K, i.e., H is the affine hull of K. We start by defining

$$S_H(K,-\delta) \coloneqq \{x \in K : B(x,\delta) \cap H \subseteq K\} \quad \text{and} \quad S_H(K,\delta) \coloneqq S(K,\delta) \cap H.$$

Note that if $K \neq \varnothing$, then there exists $\varepsilon > 0$ such that $S_H(K, -\delta) \neq \varnothing$ for each $\delta \in (0, \varepsilon)$, even if K has no interior point. If K has nonempty interior, then $H = \mathbb{R}^n$ and $S_H(K, -\delta) = S(K, -\delta)$.

DEFINITION 2. *Let $K \subseteq \mathbb{R}^n$ be a convex set and let $H = \operatorname{aff}(K)$ be its affine hull.*

(i) *The weak membership problem (*WMEM*) for K relative to H is as follows: Given $x \in \mathbb{Q}^n$ and a rational number $\delta > 0$, assert that $x \in S_H(K, \delta)$ or $x \notin S_H(K, -\delta)$.*

(ii) *The weak validity problem (*WVAL*) problem for K relative to H is as follows: Given $c \in \mathbb{Q}^n$ and rational numbers $\gamma, \varepsilon > 0$, either assert that $c^{\mathsf{T}} x \leq \gamma + \varepsilon$ for all $x \in S_H(K, -\varepsilon)$ or assert that $c^{\mathsf{T}} x \geq \gamma - \varepsilon$ for some $x \in S_H(K, \varepsilon)$.*

An implicit assumption throughout this article is that when we study the computational complexity of WMEM and WVAL problems for a convex set $K \subseteq \mathbb{R}^n$ with nonempty interior, we assume that we know a point $a \in \mathbb{Q}^n$ and a rational $r > 0$ such that the Euclidean norm ball $B(a, r) \subseteq K$. This mild *centering assumption* guarantees that K is "centered" in the sense of [6, Definition 2.1.16] and is needed whenever we invoke the Yudin–Nemirovski theorem [13] and [6, Theorem 4.3.2].

Recall that a problem $\mathscr{P}$ is said to be *polynomial-time reducible* [6, p. 28] to a problem $\mathscr{Q}$ if there is a polynomial-time algorithm $A_{\mathscr{P}}$ for solving $\mathscr{P}$ by making a polynomial number of oracle calls to an algorithm $A_{\mathscr{Q}}$ for solving $\mathscr{Q}$. This notion of polynomial-time reducibility is also called Cook or Turing reducibility and will be the one used throughout our article. There is also a more restrictive notion of polynomial-time reducibility that allows only a single oracle call to $A_{\mathscr{Q}}$ called Karp or many-one reducibility.

Note that if $A_{\mathscr{Q}}$ is a polynomial-time algorithm for $\mathscr{Q}$, then $A_{\mathscr{P}}$ is a polynomial-time algorithm for $\mathscr{P}$. Consequently, if $\mathscr{Q}$ is computable in polynomial-time, then so is $\mathscr{P}$. On the other hand, if $\mathscr{P}$ is NP-hard, then so is $\mathscr{Q}$.

We say that $\mathscr{P}$ and $\mathscr{Q}$ are *polynomial-time interreducible* if $\mathscr{P}$ is polynomial-time reducible to $\mathscr{Q}$ and $\mathscr{Q}$ is polynomial-time reducible to $\mathscr{P}$. The polynomial-time interreducibility of two problems $\mathscr{P}$ and $\mathscr{Q}$ implies that they are in the same time-complexity class[1] whatever it may be. Nevertheless, in this article we will restrict ourselves to just polynomial-time computability and NP-hardness, the two most often used cases in optimization.

3. Weak membership in dual norm balls. Our technique for this section relies on tools introduced in [6, Chapter 4] and is inspired by [7, section 6.1]. While our discussion below is over $\mathbb{R}$, it is easy to extend it to $\mathbb{C}$ since $\mathbb{C}^n$ may be identified with $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$, where $z = x + \sqrt{-1}y \in \mathbb{C}^n$ is identified with $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. A norm $\nu : \mathbb{C}^n \to [0, \infty)$ induces a norm $\tilde{\nu} : \mathbb{R}^{2n} \to [0, \infty)$ via $\tilde{\nu}\big((x, y)\big) \coloneqq \nu(x + \sqrt{-1}y)$ and we may identify ν with $\tilde{\nu}$. In particular, the Hermitian norm on $\mathbb{C}^n$ gives exactly the Euclidean norm on $\mathbb{R}^{2n}$. Hence for the purpose of this article, it suffices to consider norms over real vector spaces.

Let $\nu : \mathbb{R}^n \to [0, \infty)$ be a norm and denote the closed ball and open ball centered at $a \in \mathbb{R}^n$ of radius $r > 0$ with respect to the norm ν by

$$B_\nu(a, r) \coloneqq \{x \in \mathbb{R}^n : \nu(x - a) \leq r\} \quad \text{and} \quad B_\nu^\circ(a, r) \coloneqq \{x \in \mathbb{R}^n : \nu(x - a) < r\},$$

respectively. For the special case $a = 0$ and $r = 1$, we write $B_\nu \coloneqq B_\nu(0, 1)$ and $B_\nu^\circ \coloneqq B_\nu^\circ(0, 1)$ for the closed and open unit balls. For the special case $\nu = \|\cdot\|$,

[1] Assuming that the complexity class is defined by polynomial-time interreducibility.

the Euclidean norm on $\mathbb{R}^n$, we write $B(a,r) \coloneqq B_{\|\cdot\|}(a,r)$ and $B^\circ(a,r) \coloneqq B^\circ_{\|\cdot\|}(a,r)$, dropping the subscript. Since all norms on $\mathbb{R}^n$ are equivalent, it follows that there exist constants $K_\nu \geq k_\nu > 0$ such that

$$k_\nu \|x\| \leq \nu(x) \leq K_\nu \|x\| \quad \text{for all } x \in \mathbb{R}^n. \tag{2}$$

There is no loss of generality in assuming that k_ν and K_ν are rational[2] and we may denote the number of bits required to specify them by $\langle k_\nu \rangle$ and $\langle K_\nu \rangle$, respectively.

Recall that the *dual norm* of ν, denoted ν^*, is given by

$$\nu^*(x) = \max\{|y^\mathsf{T} x| : \nu(y) \leq 1\}$$

for every $x \in \mathbb{R}^n$. Hence

$$\frac{1}{K_\nu}\|x\| \leq \nu^*(x) \leq \frac{1}{k_\nu}\|x\| \quad \text{for all } x \in \mathbb{R}^n. \tag{3}$$

Observe first that $B(0, 1/K_\nu) \subseteq B_\nu \subseteq B(0, 1/k_\nu)$ and $B(0, k_\nu) \subseteq B_{\nu^*} \subseteq B(0, K_\nu)$. So B_ν and B_{ν^*} satisfy the centering assumption after Definition 2 with $a = 0$. Hence

$$\langle B_\nu \rangle \coloneqq \langle n \rangle + \langle k_\nu \rangle + \langle K_\nu \rangle$$

may be regarded as the encoding length of B_ν in number of bits. A norm or unit-norm ball may therefore be encoded (for a Turing machine) in finitely many bits as $(n, k_\nu, K_\nu) \in \mathbb{Q}^3$. Whenever we discuss the computation of a norm, we implicitly assume knowledge of (n, k_ν, K_ν), i.e., an algorithm would have access to their values.

The main result of this section is the polynomial-time interreducibility between a norm and its dual.

THEOREM 3. *Let ν be a norm and ν^* be its dual norm. The* WMEM *problem for the unit ball of ν^* is polynomial-time reducible to the* WMEM *problem for the unit ball of ν.*

We will prove this result via two intermediate lemmas. A key step in our proof depends on the Yudin–Nemirovski theorem [13], which may be stated as follows [6, Theorem 4.3.2].

THEOREM 4 (Yudin–Nemirovski). *The* WVAL *problem for B_ν is polynomial-time reducible to the* WMEM *problem for B_ν. More generally this holds for any convex set with nonempty interior $K \subseteq \mathbb{R}^n$ for which we have knowledge of $a \in \mathbb{Q}^n$ and $0 < r \leq R \in \mathbb{Q}$ such that $B(a, r) \subseteq K \subseteq B(0, R)$.*

The original Yudin–Nemirovski theorem is in fact stronger than the version stated here, allowing the weak violation problem WVIOL to be reduced to WMEM. Nevertheless in this article we will only require the weaker result with WVAL in place of WVIOL.

For a compact set $K \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$, the *support function* of K at c is

$$\max(K, c) \coloneqq \max\{c^\mathsf{T} x : x \in K\}.$$

In particular, observe that

$$\nu(x) = \max(B_{\nu^*}, x).$$

[2] If not just pick a smaller k_ν or a larger K_ν that is rational.

LEMMA 5. *Let ν be a norm on $\mathbb{R}^n$ and $\delta > 0$. Then we have inclusions*

$$(1 + k_\nu\delta)B_\nu \subseteq S(B_\nu, \delta) \subseteq (1 + K_\nu\delta)B_\nu, \tag{4}$$

$$(1 - K_\nu\delta)B_\nu \subseteq S(B_\nu, -\delta) \subseteq (1 - k_\nu\delta)B_\nu, \tag{5}$$

whenever $K_\nu\delta < 1$, and the inequalities

$$\left(1 - \frac{\delta}{k_\nu}\right)\nu(x) \le \max\bigl(S(B_{\nu^*}, -\delta), x\bigr) \le \left(1 - \frac{\delta}{K_\nu}\right)\nu(x), \tag{6}$$

$$\left(1 + \frac{\delta}{K_\nu}\right)\nu(x) \le \max\bigl(S(B_{\nu^*}, \delta), x\bigr) \le \left(1 + \frac{\delta}{k_\nu}\right)\nu(x), \tag{7}$$

whenever $\delta/k_\nu < 1$.

Proof. To prove (4), observe that

$$k_\nu B_\nu \subseteq B(0,1) \subseteq K_\nu B_\nu, \qquad k_\nu B_\nu^\circ \subseteq B^\circ(0,1) \subseteq K_\nu B_\nu^\circ,$$

and thus

$$B_\nu(x, k_\nu\delta) \subseteq B(x,\delta) \subseteq B_\nu(x, K_\nu\delta), \qquad B_\nu^\circ(x, k_\nu\delta) \subseteq B^\circ(x,\delta) \subseteq B_\nu^\circ(x, K_\nu\delta).$$

Also, $\bigcup_{x\in B_\nu} B_\nu(x, r) = B_\nu(0, 1 + r)$ by the defining properties of a norm. Hence

$$S(B_\nu, \delta) = \bigcup_{x\in B_\nu} B(x,\delta) \subseteq \bigcup_{x\in B_\nu} B_\nu(x, K_\nu\delta) = B_\nu(0, 1 + K_\nu\delta).$$

On the other hand,

$$S(B_\nu, \delta) = \bigcup_{x\in B_\nu} B(x,\delta) \supseteq \bigcup_{x\in B_\nu} B_\nu(x, k_\nu\delta) = B_\nu(0, 1 + k_\nu\delta).$$

To prove (5), let $T = \bigcup_{x\,:\,\nu(x)=1} B^\circ(x,\delta)$ and so $S(B_\nu, -\delta) = B_\nu \setminus T$. Let

$$T_1 = \bigcup_{x\,:\,\nu(x)=1} B_\nu^\circ(x, K_\nu\delta), \qquad T_2 = \bigcup_{x\,:\,\nu(x)=1} B_\nu^\circ(x, k_\nu\delta).$$

Since $T_1 \supseteq T$ and $T_2 \subseteq T$, we obtain

$$S(B_\nu, -\delta) \supseteq B_\nu \setminus T_1 = (1 - K_\nu\delta)B_\nu, \qquad S(B_\nu, -\delta) \subseteq B_\nu \setminus T_2 = (1 - k_\nu\delta)B_\nu.$$

The last two inequalities follow from the first two inclusions and (3). ∎

LEMMA 6. *Let $k_\nu \ge 2$. Then the solution to the* WVAL *problem for B_{ν^*} gives the solution to the* WMEM *problem for B_ν.*

Proof. Let $x \in \mathbb{Q}^n$ and $\delta \in (0, \frac{1}{2}) \cap \mathbb{Q}$. We choose $\gamma = 1$. Suppose that $x^\mathsf{T} y \le 1 + \delta$ for all $y \in S(B_{\nu^*}, -\delta)$. Then $\max\bigl(S(B_{\nu^*}, -\delta), x\bigr) \le 1 + \delta$ and by (6) we have

$$\nu(x) \le \frac{1 + \delta}{1 - \delta/k_\nu}.$$

Since $k_\nu \ge 2$, it follows that

$$\frac{1 + \delta}{1 - \delta/k_\nu} \le 1 + k_\nu\delta.$$

It follows from (4) that $x \in S(B_\nu, \delta)$.

Suppose that $x^\mathsf{T} y > 1-\delta$ for some $y \in S(B_{\nu^*}, \delta)$. Then $\max\bigl(S(B_{\nu^*}, \delta), x\bigr) > 1-\delta$ and we deduce from (7) that

$$\nu(x) > \frac{1-\delta}{1+\delta/k_\nu}.$$

A straightforward calculation shows that

$$\frac{1-\delta}{1+\delta/k_\nu} \geq 1 - k_\nu \delta.$$

It follows from (5) that $x \notin S(B_\nu, -\delta)$. ◻

Proof of Theorem 3. We observe that the assumption $k_\nu \geq 2$ in Lemma 6 is not restrictive. Let $r \geq 2/k_\nu$. Then a new norm defined by $\nu_r(x) = r\nu(x)$ would satisfy the assumption. Now note that $x \in B_\nu$ if and only if $\frac{1}{r}x \in B_{\nu_r}$. With this observation, Theorem 3 follows from

WMEM for $B_{\nu^*} \Rightarrow$ WVAL for $B_{\nu^*} \Rightarrow$ WMEM for $B_\nu \Rightarrow$ WVAL for $B_\nu \Rightarrow$ WMEM for B_{ν^*}.

Here $\mathscr{P} \Rightarrow \mathscr{Q}$ means that $\mathscr{Q}$ is polynomial-time reducible to $\mathscr{P}$. The Yudin–Nemirovski theorem gives the first and third reductions, whereas Lemma 6 gives the second and last reductions. ◻

Since taking the dual of a dual norm gives us back the original norm, we have the following corollary.

COROLLARY 7. *The* WMEM *problem for the unit ball of a norm ν is polynomial-time decidable (resp., NP-hard) if and only if the* WMEM *problem for the unit ball of the dual norm ν^* is polynomial-time decidable (resp., NP-hard).*

Since every centrally symmetric compact convex set with nonempty interior is a norm ball for some norm and its *polar dual* is exactly the norm ball for the corresponding dual norm, we immediately have the following.

COROLLARY 8. *Let C be a centrally symmetric compact convex set with nonempty interior in $\mathbb{R}^n$ and*

$$C^* = \{x \in \mathbb{R}^n : x^\mathsf{T} y \leq 1\}$$

be its polar dual. Then WMEM *in C is polynomial-time interreducible to* WMEM *in C^*. In particular, if one is polynomial-time decidable (resp., NP-hard), then so is the other.*

4. Approximation of dual norms. In this section we show that for a given norm $\nu : \mathbb{R}^n \to [0, \infty)$ satisfying (2) for $k_\nu, K_\nu \in \mathbb{Q}$, WMEM in B_ν with respect to $\delta \in \mathbb{Q}$ is polynomial-time interreducible with a δ-approximation of the norm ν.

DEFINITION 9. *Let $\nu : \mathbb{R}^n \to [0, \infty)$ be a norm satisfying* (2) *for $k_\nu, K_\nu \in \mathbb{Q}$. The* approximation problem *(*APPROX*) for ν is as follows: Let $\delta \in \mathbb{Q}$ and $\delta > 0$. Given any $x \in \mathbb{Q}^n$ with $1/2 < \|x\| < 3/2$, compute an approximation $\omega(x) \in \mathbb{Q}$ such that*

$$\nu(x) - \delta < \omega(x) < \nu(x) + \delta. \tag{8}$$

*We call ω a δ-*approximation *of ν.*

The annulus $1/2 < \|x\| < 3/2$, where x has rational coordinates, is intended as a rational approximation of the unit sphere $\|x\| = 1$ in $\mathbb{R}^n$—points on the unit sphere that do not have rational coordinates can be approximated by rational points in the annulus. The requirement that $1/2 < \|x\| < 3/2$ is not restrictive since we may always scale any given x to meet this condition in polynomial-time. Note that an approximation problem has $n + \langle\delta\rangle + \langle K_\nu\rangle + \langle k_\nu\rangle$ input bits. If we say that such a problem can be solved in polynomial-time, we mean time polynomial in this number of input bits.

THEOREM 10. *Let $\nu : \mathbb{R}^n \to [0, \infty)$ be a norm satisfying* (2) *for $k_\nu, K_\nu \in \mathbb{Q}$. Then the following problems are polynomial-time interreducible:*
(i) *The approximation problem for ν.*
(ii) *The weak membership problem for B_ν.*

Proof. Let us use (i) as an oracle and solve (ii). Let $x \in \mathbb{Q}^n$ and a let rational $\delta > 0$ be given. If $\|x\| \le 1/K_\nu$, then $\nu(x) \le 1$, and so $x \in S(B_\nu, \delta)$. If $\|x\| \ge 1/k_\nu$, then $\nu(x) \ge 1$, and so $x \notin S(B_\nu, -\delta)$.

It remains to check the case $\|x\| \in (1/K_\nu, 1/k_\nu)$. Let $r \in (2\|x\|/3, 2\|x\|) \cap \mathbb{Q}$ and let $y := x/r$. Observe that $\nu(y) \in (k_\nu/2, 3K_\nu/2)$. Now let $\varepsilon = k_\nu^2\delta/4$ and let $\omega(y)$ be an ε-approximation of $\nu(y)$. Assume first that

$$r\omega(y) \le 1 + k_\nu\delta - \frac{2\varepsilon}{k_\nu} = 1 + \frac{k_\nu\delta}{2}.$$

Then

$$\nu(x) = r\nu(y) < r(\omega(y) + \varepsilon) < r\omega(y) + \frac{2}{k_\nu}\varepsilon \le 1 + k_\nu\delta,$$

and (4) yields that $x \in S(B_\nu, \delta)$. Assume now that

$$r\omega(y) > 1 + \frac{k_\nu\delta}{2}.$$

Then

$$\nu(x) > r(\omega(y) - \varepsilon) \ge r\omega(y) - \frac{2\varepsilon}{k_\nu} > 1 + \frac{k_\nu\delta}{2} - \frac{k_\nu\delta}{2} = 1$$

and so $x \notin S(B_\nu, -\delta)$. This shows that we may decide weak membership in B_ν with a δ-approximation to ν. In fact we just need one oracle call to APPROX.

Let us use (ii) as an oracle and solve (i). Let $x \in \mathbb{Q}^n$ where $\|x\| \in (1/2, 3/2)$ and let a rational $\delta > 0$ be given. Again, observe that $\nu(x) \in [a_1, b_1]$, where $a_1 = k_\nu/2$ and $b_1 = 3K_\nu/2$. Suppose that for an integer $i \ge 1$ we showed that $\nu(x) \in [a_i, b_i]$. Let

$$r = \frac{a_i + b_i}{2}, \qquad \varepsilon = \frac{b_i - a_i}{2K_\nu(b_i + a_i)}, \tag{9}$$

and consider $y = x/r$. Assume first that $y \in S(B_\nu, \varepsilon)$. Then the right inclusion in (4) yields $\nu(y) \le 1 + K_\nu\varepsilon$ and thus

$$\nu(x) = r\nu(y) \le \frac{a_i + b_i}{2}(1 + K_\nu\varepsilon) = \frac{3}{4}b_i + \frac{1}{4}a_i.$$

In this case we set $a_{i+1} = a_i$ and $b_{i+1} = 3b_i/4 + a_i/4$. Assume now that $y \notin S(B_\nu, -\varepsilon)$. Then the left inclusion in (5) yields

$$\nu(x) > r(1 - K_\nu\varepsilon) = \frac{1}{4}b_i + \frac{3}{4}a_i.$$

In this case we set $a_{i+1} = b_i/4 + 3a_i/4$ and $b_{i+1} = b_i$.

In either case, we obtain that $\nu(x) \in [a_{i+1}, b_{i+1}]$. Clearly, the sequence of intervals $\{[a_i, b_i] : i \in \mathbb{N}\}$ is nested and their successive lengths decrease by a factor of $3/4$. Let m be the smallest integer such that

$$m > 1 + \frac{\log_2 b_1 - \log_2 2\delta}{2 - \log_2 3} \quad \text{if } 2\delta b^{-1} \le 1,$$

and otherwise set $m = 1$. Then

$$b_m - a_m = \left(\frac{3}{4}\right)^{m-1} (b_1 - a_1) < \left(\frac{3}{4}\right)^{m-1} b_1 < 2\delta.$$

Clearly m is polynomial, in fact linear, in $\langle K_\nu \rangle + \langle k_\nu \rangle + \langle \delta \rangle$. Setting $\omega(x) \coloneqq (a_m + b_m)/2$, we obtain a δ-approximation of $\nu(x)$. This shows that we may determine a δ-approximation to ν with m oracle calls to WMEM in B_ν. ∎

COROLLARY 11. *A norm is polynomial-time approximable (resp., NP-hard to approximate) if and only if its dual norm is polynomial-time approximable (resp., NP-hard to approximate).*

We end this section with a word about *Mahler volume* [2]. For any norm $\nu : \mathbb{R}^n \to [0, \infty)$, let $\mathrm{Vol}_n(B_\nu)$ denote the volume of its unit ball B_ν. The Mahler volume of ν is defined as

$$M(\nu) \coloneqq \mathrm{Vol}_n(B_\nu)\,\mathrm{Vol}_n(B_{\nu^*}).$$

A particularly nice property of the Mahler volume is that it is invariant under *any* invertible linear transformation, regardless of whether it is volume-preserving.

COROLLARY 12. *If the weak membership problem in B_ν is polynomial-time decidable, then $M(\nu)$ is polynomial-time approximable.*

Proof. If WMEM in B_ν is polynomial-time decidable, then it follows from [4] that there exist polynomial-time algorithms to approximate $\mathrm{Vol}_n(B_\nu)$ to any given error $\varepsilon > 0$. By Corollary 11, WMEM in B_{ν^*} is also polynomial-time decidable and thus the same holds for $\mathrm{Vol}_n(B_{\nu^*})$. ∎

Mahler volume is more commonly defined for a centrally symmetric compact convex set but as we mentioned before Corollary 8, this is equal to a unit norm ball for an appropriate choice of norm.

5. Weak membership in dual cones.

In this section, we move our discussion from balls to cones. While every ball is, by definition, a norm ball, a (proper) cone may not be a norm cone, i.e., of the form $\{x \in \mathbb{R}^n : \|Ax\| \le c^\mathsf{T} x\}$ for some norm $\|\cdot\|$ and $A \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$. So the results in this section would not in general follow from the previous sections.

Let $K \subset \mathbb{R}^n$ be a *proper cone* in $\mathbb{R}^n$, i.e., K is a closed convex pointed[3] cone with nonempty interior. Then its *dual cone*,

$$K^* \coloneqq \{x \in \mathbb{R}^n : y^\mathsf{T} x \ge 0 \text{ for every } y \in K\},$$

is also a proper cone [12]. The main result of this section is an analogue of Theorem 3 for such cones: The weak membership problem for K^* is polynomial-time reducible to the weak membership problem for K.

[3]By pointed, we mean that $K \cap (-K) = \{0\}$.

It is well-known that deciding MEM for the cone of copositive matrices is NP-hard [11]. This result has recently been extended [3]: WMEM in the cone of copositive matrices and WMEM in its dual cone, the cone of completely positive matrices, are both NP-hard problems. Our result in this section generalizes this to arbitrary proper cones.

We first recall a well-known result regarding the interior points of K^*.

LEMMA 13. *Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Let b be an interior point of K^*, i.e., $b + z \in K^*$ for all $z \in B(0, \varepsilon_b)$ for some $\varepsilon_b > 0$. Then*

$$b^{\mathsf{T}} x \geq \varepsilon_b \|x\| \tag{10}$$

for every $x \in K$.

Proof. Let $x \in K \setminus \{0\}$. Then $c := b - \varepsilon_b x/\|x\| \in K^*$. Hence $c^{\mathsf{T}} x \geq 0$, which implies (10). ∎

We now discuss the notion of WMEM in K. Recall that $x \in K \setminus \{0\}$ if and only if $tx \in K$ for each $t > 0$. Hence it suffices to define WMEM in K for x with Euclidean norm $\|x\| = 1$, but as such an x may be not have rational coordinates, we instead define a WMEM problem for $x \in \mathbb{Q}^n$ that satisfies $\frac{1}{2} < \|x\| < 1$.

Let $a \in \mathbb{Q}^n$ and $b \in \mathbb{Q}^n$ be in the interior of K and K^*, respectively. By Lemma 13,

$$P_b := \{x \in K : b^{\mathsf{T}} x = 1\}, \quad P_a^* = \{y \in K^* : a^{\mathsf{T}} y = 1\} \tag{11}$$

are compact convex sets of dimension $n - 1$. Hence the sets $P_b - (b^{\mathsf{T}} a)^{-1} a$ and $P_a^* - (a^{\mathsf{T}} b)^{-1} b$ are full-dimensional compact convex sets in the orthogonal complements of span(b) and span(a), respectively. In fact P_b and P_a^* are compact convex sets of maximal dimension in the affine hyperplanes

$$H_b := \{z \in \mathbb{R}^n : b^{\mathsf{T}} z = 1\}, \quad H_a := \{z \in \mathbb{R}^n : a^{\mathsf{T}} z = 1\},$$

respectively. We may also view H_b and H_a as the affine hulls of P_b and P_a^*, respectively.

As the cones K and K^* are noncompact, these hyperplane sections P_b and P_a^* serve as their compact proxies, allowing us to encode K and K^* (for a Turing machine). We will assume knowledge of four positive rational numbers $\rho_a' < \rho_a$ and $\rho_b' < \rho_b$ such that

$$B(0, \rho_a') \cap H_a \subseteq P_a^* - (a^{\mathsf{T}} b)^{-1} b \subseteq B(0, \rho_a) \cap H_a,$$
$$B(0, \rho_b') \cap H_b \subseteq P_b - (b^{\mathsf{T}} a)^{-1} a \subseteq B(0, \rho_b) \cap H_b.$$

K will be encoded as $(n, a, b, \rho_a', \rho_a) \in \mathbb{Q}^{2n+3}$ and K^* as $(n, a, b, \rho_b', \rho_b) \in \mathbb{Q}^{2n+3}$. So

$$\langle K \rangle := \langle n \rangle + \langle a \rangle + \langle b \rangle + \langle \rho_a' \rangle + \langle \rho_a \rangle, \quad \langle K^* \rangle := \langle n \rangle + \langle a \rangle + \langle b \rangle + \langle \rho_b' \rangle + \langle \rho_b \rangle.$$

While the numbers $\rho_a, \rho_a', \rho_b, \rho_b'$ do not appear explicitly in our proofs, they are needed implicitly when we invoke the Yudin–Nemirovski theorem.

Given any $x \neq 0$, observe that $x \in K$ if and only if $x/(b^{\mathsf{T}} x) \in P_b$. Thus the membership problem for K is equivalent to the membership problem for P_b. We show in the following that this extends, in an appropriate sense, to weak membership as well.

LEMMA 14. *Let $x \in \mathbb{Q}^n$ with $1/2 < \|x\| < 1$ and $b \in \mathbb{Q}^n$ with $b^{\mathsf{T}} x > 0$. Then the following problems are polynomial-time interreducible:*

(i) *Decide weak membership of x in K.*
(ii) *Decide weak membership of $y := x/(b^{\mathsf{T}}x)$ in P_b relative to H_b.*

Proof. Suppose that $0 < \delta < b^{\mathsf{T}}x/(2\|b\|)$. Let $z \in \mathbb{R}^n$ and $\|z\| \le \delta$. Clearly,

$$b^{\mathsf{T}}(x+z) = b^{\mathsf{T}}x + b^{\mathsf{T}}z \ge b^{\mathsf{T}}x - \|b\|\|z\| \ge \frac{1}{2}b^{\mathsf{T}}x > 0.$$

In the following, we let $y := x/(b^{\mathsf{T}}x)$ and $u := (x+z)/\big(b^{\mathsf{T}}(x+z)\big) \in H_b$.

Suppose that we can solve (i), i.e., for any rational $\delta > 0$ and $x \in \mathbb{Q}^n$ with $1/2 < \|x\| < 1$ we can decide whether $x \in S(K,\delta)$ or $x \notin S(K,-\delta)$. Let $\varepsilon > 0$ be rational and choose δ rational so that

$$\frac{(b^{\mathsf{T}}x)^2}{8\|b\|}\varepsilon < \delta < \frac{(b^{\mathsf{T}}x)^2}{4\|b\|}\varepsilon.$$

Consider first the case $x \notin S(K,-\delta)$. There exists $z \in \mathbb{R}^n$, $\|z\| \le \delta$ such that $x + z \notin K$. So $u \notin P_b$. Since

$$\begin{aligned} y - u &= \frac{1}{(b^{\mathsf{T}}x)(b^{\mathsf{T}}(x+z))}[(b^{\mathsf{T}}(x+z))x - (b^{\mathsf{T}}x)(x+z)] \\ &= \frac{1}{(b^{\mathsf{T}}x)(b^{\mathsf{T}}(x+z))}[(b^{\mathsf{T}}z)x - (b^{\mathsf{T}}x)z], \end{aligned}$$

we obtain

$$\|y-u\| \le \frac{2}{(b^{\mathsf{T}}x)^2}(2\|b\|\|x\|\|z\|) \le \frac{4\|b\|\delta}{(b^{\mathsf{T}}x)^2} < \varepsilon.$$

Hence $y \notin S_{H_b}(P_b,-\varepsilon)$.

Consider now the case $x \in S(K,\delta)$. There exists $z \in \mathbb{R}^n$, $\|z\| \le \delta$ such that $x + z \in K$. The same line of argument as above yields that $y \in S_{H_b}(P_b,\varepsilon)$. Together the two cases show that if we can decide WMEM in K with inputs x, δ, then we can decide WMEM in P_b relative to H_b with inputs y, ε.

Suppose we can solve (ii), i.e., for any rational $\varepsilon > 0$ and $x \in \mathbb{Q}^n$ with $1/2 < \|x\| < 1$, $b^{\mathsf{T}}x > 0$, we can decide whether $y \in S_{H_b}(P_b,\varepsilon)$ or $y \notin S_{H_b}(P_b,-\varepsilon)$.

Let $x \in \mathbb{Q}^n$ with $1/2 < \|x\| < 1$. We start by excluding the trivial case when $b^{\mathsf{T}}x \le 0$. By Lemma 13, $x \notin K$ and thus $x \notin S(K,-\delta)$ for any $\delta > 0$. So we may assume henceforth that $b^{\mathsf{T}}x > 0$. Let $\delta > 0$ be rational and set $\varepsilon := \delta/(b^{\mathsf{T}}x)$.

Consider first the case $y \notin S_{H_b}(P_b,-\varepsilon)$. There exists $v \in H_b \setminus P_b$ such that $\|v - y\| \le \varepsilon$. Let $z = (b^{\mathsf{T}}x)(v-y)$. So

$$\|z\| \le (b^{\mathsf{T}}x)\varepsilon = \delta.$$

Hence $(b^{\mathsf{T}}x)v = x + z \notin K$ and so $x \notin S(K,-\delta)$.

Consider now the case $y \in S_{H_b}(P_b,\varepsilon)$. The same line of argument as above yields that $x \in S(K,\delta)$. Together the two cases show that if we can decide WMEM in P_b relative to H_b with inputs y, ε, then we can decide WMEM in K with inputs x, δ. ◻

Lemma 14 may be viewed as a compactification result: We transform a problem involving a noncompact object K to a problem involving a compact object P_b. The motivation is so that we may apply the Yudin–Nemirovski theorem later.

THEOREM 15. *Let $K \subset \mathbb{R}^n$ be a proper cone and K^* be its dual. Let $a \in \mathbb{Q}^n$ and $b \in \mathbb{Q}^n$ be interior points of K and K^*, respectively, that satisfy $b^{\mathsf{T}}a = 1$. Then the* WMEM *problem for K^* is polynomial-time reducible to the* WMEM *problem for K.*

Proof. Note that such a pair of a and b must exist for any proper cone. Let $a, b \in \mathbb{Q}^n$ be interior points contained in balls of radii ε_a, $\varepsilon_b > 0$ within K, K^*, respectively. So $b^\mathsf{T} a > 0$. If $b^\mathsf{T} a = 1$, we are done. Otherwise set $a' = a/(b^\mathsf{T} a) \in \mathbb{Q}^n$. Then $b^\mathsf{T} a' = 1$ and a' is contained in a ball of radius $\varepsilon_{a'} = \varepsilon_a/(b^\mathsf{T} a)$ within K^*.

By Lemma 14, we just need to show that the WMEM problem for P_a^* relative to H_a is polynomial-time reducible to the WMEM problem for P_b relative to H_b. Since $b^\mathsf{T} a = 1$, $H_b - a = b^\perp$, the orthogonal complement of b, and can be identified with $\mathbb{R}^{n-1}$ by an orthogonal change of coordinates. We set $K_b \coloneqq P_b - a$, a compact closed set in $\mathbb{R}^{n-1}$ containing the origin $0 \in \mathbb{R}^{n-1}$. Moreover $B(0, \varepsilon_a) \subset K_b$, where $B(0, \varepsilon_a)$ here is an $(n-1)$-dimensional ball in $\mathbb{R}^{n-1}$. It is enough to show that the WMEM problem for P_a^* relative to H_a is polynomial-time reducible to the WMEM problem[4] for K_b. We would also need to invoke the fact that the WVAL problem for K_b is polynomial-time reducible to the WMEM problem for K_b by the Yudin–Nemirovski theorem. The following sequence of polynomial-time reductions outlines the idea of our proof:

$$\begin{aligned} &\text{WMEM for } K \Rightarrow \text{WMEM for } P_b \text{ relative to } H_b \\ &\qquad\qquad \Rightarrow \text{WMEM for } P_a^* \text{ relative to } H_a \Rightarrow \text{WMEM for } K^*. \end{aligned}$$

Let $c \in \mathbb{Q}^n \cap H_a$. Given a rational $\delta > 0$ we need to decide whether $c \notin S_{H_a}(P_a^*, -\delta)$ or $c \in S_{H_a}(P_a^*, \delta)$. Let $\varepsilon > 0$ be rational with

$$\varepsilon < \min \left\{ \frac{1}{4(1 + \|c\|)}, \frac{\delta}{4(1 + \|c\|)(\|b - c\|)} \right\}, \tag{12}$$

where $\delta/0 \coloneqq \infty$ if $b = c$. It follows from (12) that

$$\tau \coloneqq (1 + \|c\|)\varepsilon \le \frac{1}{4}, \qquad \left\| c - \frac{c + \tau b}{1 + \tau} \right\| \le \delta, \qquad \left\| c - \frac{c - 2\tau b}{(1 - 2\tau)} \right\| \le \delta. \tag{13}$$

Observe that c defines a linear functional $b^\perp \to \mathbb{R}$, $x \mapsto c^\mathsf{T} x$. Consider the WVAL problem for K_b with $\gamma = -c^\mathsf{T} a$: Either $c^\mathsf{T} x \ge -c^\mathsf{T} a - \varepsilon$ for all $x \in S_{H_b}(K_b, -\varepsilon)$ or $c^\mathsf{T} x \le -c^\mathsf{T} a + \varepsilon$ for some $x \in S_{H_b}(K_b, -\varepsilon)$. We will show that in the first case $c \in S_{H_a}(P_a^*, \delta)$ and in the second case $c \notin S_{H_a}(P_a^*, -\delta)$ for a corresponding $\delta > 0$.

Consider first the case $c^\mathsf{T} x \ge -c^\mathsf{T} a - \varepsilon$ for all $x \in S_{H_b}(K_b, -\varepsilon)$ or, equivalently, $c^\mathsf{T} y \ge -\varepsilon$ for all $y = x + a \in S_{H_b}(P_b, -\varepsilon)$. We claim that $c^\mathsf{T} y \ge -(1 + \|c\|)\varepsilon$ for all $y \in P_b$. This holds for $y \in S_{H_b}(P_b, -\varepsilon)$ since $c^\mathsf{T} y \ge -\varepsilon \ge -(1 + \|c\|)\varepsilon$. For $y \in P_b \setminus S_{H_b}(P_b, -\varepsilon)$, there exists $x \in S_{H_b}(P_b, -\varepsilon)$ such that $\|y - x\| \le \varepsilon$. Thus $c^\mathsf{T} y = c^\mathsf{T} x + c^\mathsf{T}(y - x) \ge -\varepsilon - \|c\|\|y - x\| = -(1 + \|c\|)\varepsilon$. Then for any $y \in P_b$,

$$\frac{1}{1 + \tau}(c + \tau b)^\mathsf{T} y \ge 0 \quad \Rightarrow \quad \frac{1}{1 + \tau}(c + \tau b) \in P_a^*.$$

By the middle inequality in (13), we obtain $c \in S_{H_a}(P_a^*, \delta)$.

Consider now the case $c^\mathsf{T} x \le -c^\mathsf{T} a + \varepsilon$ for some $x \in S_{H_b}(K_b, \varepsilon)$ or, equivalently, $c^\mathsf{T} y \le \varepsilon$ for some $y = x + a \in S_{H_b}(P_b, \varepsilon)$. Hence there exists $z \in P_b$ such that $\|z - y\| \le \varepsilon$ and so $c^\mathsf{T} z = c^\mathsf{T} y + c^\mathsf{T}(z - y) \le (1 + \|c\|)\varepsilon = \tau < 1/4$ by the left inequality in (13). Then

$$\frac{1}{1 - 2\tau}(c - 2\tau b)^\mathsf{T} z \le -\tau \quad \Rightarrow \quad \frac{1}{1 - 2\tau}(c - 2\tau b) \notin P_a^*.$$

[4]When we refer to the WMEM or WVAL problem for K_b, we mean its WMEM or WVAL problem as a subset of $\mathbb{R}^{n-1}$.

By the right inequality in (13), we obtain $c \notin S_{H_a}(P_a^*, -\delta)$. ∎

6. Approximation of Fenchel duals. Let $C \subseteq \mathbb{R}^n$ and $f : C \to \mathbb{R}$. Since the epigraph of f, $\operatorname{epi}(f) = \{(x,t) \in C \times \mathbb{R} : f(x) \le t\}$, is in general noncompact, we introduce the following variant that preserves all essential features of the epigraph but has the added advantage of facilitating complexity theoretic discussions. For any $\alpha \in \mathbb{R}$, we let

$$\operatorname{epi}_\alpha(f) = \{(x,t) \in C \times (-\infty, \alpha] : f(x) \le t\}$$

and call this the *α-epigraph* of f. Clearly f is a convex function if and only if $\operatorname{epi}_\alpha(f)$ is a convex set for all $\alpha \in \mathbb{R}$.

DEFINITION 16. *Let $C \subseteq \mathbb{R}^n$ be a bounded set with nonempty interior. Let $f : C \to \mathbb{R}$ be a bounded function. We define the following* approximation problems *(*APPROX*):*

(i) *Approximation problem for f: Given any $x \in \mathbb{Q}^n \cap C$ and any rational $\varepsilon > 0$, find an $\omega(x)$ such that $f(x) - \varepsilon < \omega(x) < f(x) + \varepsilon$.*

(ii) *Approximation problem for $\mu := \inf_{x \in C} f(x)$: Given any rational $\varepsilon > 0$, find $\mu(\varepsilon) \in \mathbb{Q}$ such $\mu - \varepsilon < \mu(\varepsilon) < \mu + \varepsilon$.*

Problem (i) is of course a generalization of Definition 9 from norms to a more general function. We will show that (i) and (ii) are polynomial-time interreducible. For this purpose, we will need a useful corollary [6, Corollary 4.3.12] of the Yudin–Nemirovski theorem (cf. theorem 4) with the WOPT problem in place of the WVAL problem.

COROLLARY 17 (Yudin–Nemirovski). *Let $C \subseteq \mathbb{R}^n$ be a compact convex set with nonempty interior for which we have knowledge of $a \in \mathbb{Q}^n$ and $0 < r \le R \in \mathbb{Q}$ such that $B(a,r) \subseteq C \subseteq B(a,R)$. Then the* WOPT *problem for C is polynomial-time reducible to the* WMEM *problem for C.*

We will rely on this to show that for a convex function $f : C \to \mathbb{R}$, the approximation problem for $\inf_{x \in C} f(x)$ is polynomial-time reducible to the approximation problem for f.

LEMMA 18. *Let $C \subseteq \mathbb{R}^n$ be a compact convex set with nonempty interior where* MEM *in C can be checked in polynomial time. Let $f : C \to \mathbb{R}$ be a continuous convex function with $|f(x)| \le \alpha$ for some rational $\alpha > 0$. Suppose that there exists a rational $\delta > 0$ such that*

$$\mu := \min_{x \in C} f(x) = \min_{x \in S(C,-\delta)} f(x). \tag{14}$$

Then the approximation problem for μ is polynomial-time reducible to the approximation problem for f.

Note that we require knowledge of the values of both α and δ, not just of their existence. We need the condition (14) to ensure that no minimizer of f lies on the boundary of C and that any minimizer is at least distance δ away from the boundary.

Proof of Lemma 18. We will show that WOPT in $\operatorname{epi}_{2\alpha}(f)$ yields a solution to APPROX for μ. The result then follows from two polynomial-time reductions: WOPT in $\operatorname{epi}_{2\alpha}(f)$ can be reduced to WMEM in $\operatorname{epi}_{2\alpha}(f)$, and WMEM in $\operatorname{epi}_{2\alpha}(f)$ can be reduced to APPROX for f.

As f is a continuous convex function and C is compact with nonempty interior, $C' := \operatorname{epi}_{2\alpha}(f)$ is a compact convex set with interior in $\mathbb{R}^{n+1}$. We claim that the

WMEM in C' is polynomial-time reducible to the approximation problem for f. Let $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < \alpha$ and $(x,t) \in \mathbb{Q}^{n+1}$. If $x \notin C$ or $t > 2\alpha$, then $(x,t) \notin C'$ and so $(x,t) \notin S(C', -\varepsilon)$. Now suppose $x \in C$ and $t \le 2\alpha$. An oracle call to the approximation problem for f gives us $\omega(x)$ with $\omega(x) - \varepsilon < f(x) < \omega(x) + \varepsilon$. If $t \ge \omega(x)$, then as $(x,t) + (0,\varepsilon) \in C'$, it follows that $(x,t) \in S(C', \varepsilon)$. If $t < \omega(x)$, then as $(x,t) - (0,\varepsilon) \notin C'$, it follows that $(x,t) \notin S(C', -\varepsilon)$.

By Corollary 17, WOPT in C' is polynomial-time reducible to WMEM in C'. Therefore given $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < \min(\alpha, \delta)$ and $\gamma = (0, \dots, 0, -1) \in \mathbb{Z}^{n+1}$, by an oracle call to WMEM in C', we may find $(y,s) \in S(C', \varepsilon)$ such that

$$\gamma^{\mathsf{T}}(x,t) = -t \le \gamma^{\mathsf{T}}(y,s) + \varepsilon = -s + \varepsilon$$

for all $(x,t) \in S(C', -\varepsilon)$. We claim that $s = \mu(\varepsilon)$, the required approximation to μ. Since $\varepsilon < \delta$, it follows that $S(C', -\varepsilon) \supseteq S(C', -\delta)$. The assumption (14) ensures that $(x^\star, \mu) \in S(C, -\delta)$, where $f(x^\star) = \mu$. Hence we deduce that $s \le \mu + \varepsilon$, i.e., $\mu \ge s - \varepsilon$. As $(y,s) \in S(C', \varepsilon)$, it follow that there exists $(x', t') \in C'$ such that $t' \ge f(x')$ and $|t' - s| \le \varepsilon$. So $s \ge t' - \varepsilon \ge \mu - \varepsilon$. Thus $\mu - \varepsilon \le s \le \mu + \varepsilon$, but starting with 2ε in place of ε allows us to replace "$\le$" by "$<$" as required by Definition 16(ii). ∎

We now turn to the computational complexity of the *Fenchel dual* [1, 12]. Our results here require that f be defined on all of $\mathbb{R}^n$. Recall that for a function $f : \mathbb{R}^n \to \mathbb{R}$, its Fenchel dual is defined to be the function $f^* : \mathbb{R}^n \to (-\infty, \infty]$,

$$f^*(y) \coloneqq \sup_{x \in \mathbb{R}^n} \bigl(y^{\mathsf{T}} x - f(x)\bigr).$$

The Fenchel dual is also known as the *Fenchel conjugate* and the map $f \mapsto f^*$ is sometimes called the *Legendre transform*. It is well-known that f^* is always a convex function, being the pointwise supremum of a family of affine functions $y \mapsto y^{\mathsf{T}} x - f(x)$. It is also well-known that f is a lower semicontinuous proper convex function if and only if $f^{**} = f$.

Suppose that given any inputs $x \in \mathbb{Q}^n$ and $0 < \varepsilon \in \mathbb{Q}$, we can compute $f(x)$ to within precision ε in polynomial-time. What can we say about the complexity of computing $f^*(y)$ for an input $y \in \mathbb{Q}^n$ to a certain precision? We will see that if f is not convex, then the computation of f^* can be NP-hard at least for some y. However, when f is convex and satisfies certain growth conditions, computing f^* is a problem that is polynomial-time reducible to computing f. Furthermore f^* would satisfy the same growth conditions so that computing f and computing f^* are in fact polynomial-time interreducible.

Let $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x,y,z) \mapsto \sum_{i,j,k=1}^n a_{ijk} x_i y_j z_k$ be a multilinear function. Let $D = \{(x,y,z) \in \mathbb{R}^{3n} : \|x\| \le 1,\ \|y\| \le 1,\ \|z\| \le 1\}$. We define a nonconvex function f as follows: For $(x,y,z) \in D$, $f(x,y,z) \coloneqq -g(x,y,z)$. For $(x,y,z) \notin D$, let $t = 1/\max(\|x\|, \|y\|, \|z\|)$ and set $f(x,y,z) \coloneqq -g(tx, ty, tz)$. It is trivial to compute f for any $(x,y,z) \in \mathbb{R}^{3n}$ but $f^*(0) = \max_{(x,y,z) \in D} g(x,y,z)$ is NP-hard to approximate in general [9, Theorem 10.2].

In what follows let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous convex function. We will assume that f satisfies the growth condition

$$k_f \|x\|^s \le f(x) \le K_f \|x\|^t \quad \text{whenever} \quad \|x\| \ge r \tag{15}$$

for some constants $0 < k_f \le K_f$, $1 < s \le t$, and $r > 0$ depending on f. We now show that f^* must satisfy similar growth conditions

$$k_{f^*} \|y\|^{s'} \le f^*(y) \le K_{f^*} \|y\|^{t'} \quad \text{whenever} \quad \|y\| \ge r', \tag{16}$$

but with possibly different constants.

LEMMA 19. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $f^* : \mathbb{R}^n \to (-\infty, \infty]$ be its Fenchel dual. Then f satisfies* (15) *if and only if f^* satisfies* (16).

Proof. For $\|x\| \geq r$, the lower bound in (15) and $y^\mathsf{T} x \leq \|y\|\|x\|$ give

$$y^\mathsf{T} x - f(x) \leq \|y\|\|x\| - k_f\|x\|^s = \|x\|(\|y\| - k_f\|x\|^{s-1}). \tag{17}$$

Observe that for $z \in [0, \infty)$, the maximum of $h(z) \coloneqq \|y\|z - k_f z^s$ is attained at

$$z^\star = \left(\frac{\|y\|}{k_f s}\right)^{1/(s-1)}$$

with maximum value

$$h(z^\star) = \frac{s-1}{s}\|y\|z^\star = \frac{s-1}{s(k_f s)^{1/(s-1)}}\|y\|^{s/(s-1)}.$$

Let $\mu \coloneqq \min_{\|x\|\leq r} f(x)$. Then

$$\max_{\|x\|\leq r} \left(y^\mathsf{T} x - f(x)\right) \leq \|y\|r - \mu.$$

Combine this with (17) and we obtain

$$f^*(y) \leq \max\left(\|y\|r - \mu, \frac{s-1}{s(k_f s)^{1/(s-1)}}\|y\|^{s/(s-1)}\right).$$

This last inequality yields the upper bound in (16) with

$$K_{f^*} = \frac{s-1}{s(k_f s)^{1/(s-1)}}, \qquad t' = \frac{s}{s-1}, \qquad r' \geq r_1,$$

for a corresponding r_1 that depends on k_f, s, r, μ. More precisely, either $r_1 = 0$ or r_1 is the unique positive solution of

$$r_1 r - \mu = \frac{s-1}{s(k_f s)^{1/(s-1)}} r_1^{s/(s-1)}.$$

To deduce the lower bound in (16), let y be such that

$$\|y\| \geq r^{t-1} K_f t.$$

Choose $x = cy$ such that

$$\|x\| = \left(\frac{\|y\|}{K_f t}\right)^{1/(t-1)}.$$

It follows that $\|x\| \geq r$ and so the upper bound in (15) yields $f^*(y) \geq \|y\|\|x\| - K_f\|x\|^t$. Hence we have the lower bound in (16) with

$$k_{f^*} = \frac{t-1}{t(K_f t)^{1/(t-1)}}, \qquad s' = \frac{t}{t-1}. \qquad \square$$

THEOREM 20. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying* (15). *Then the approximation problem for f^* is polynomial-time reducible to the approximation problem for f.*

Proof. We will compute an approximation of $f^*(y)$ with oracle calls to approximations of $f(x)$.

Suppose first that $y = 0$ and we need to compute an approximation of $f^*(0) = \sup_{x\in\mathbb{R}^n} -f(x)$. By the lower bound in (15), there is some $\rho_0 = \rho(r, k_f, s) \in \mathbb{Q}\cap(0,\infty)$ such that $-f(x) < -f(0)$ whenever $\|x\| \geq \rho_0$. Hence

$$f^*(0) = \max_{\|x\|\leq\rho_0} -f(x) = -\min_{\|x\|\leq\rho_0} f(x) = -\min_{\|x\|\leq\rho_0+1} f(x).$$

Let $C = B(0, \rho_0 + 1)$. Since MEM in a Euclidean ball $B(0, \rho)$ is clearly polynomial-time decidable, the conditions of Lemma 18 are satisfied. Hence APPROX for $f^*(0)$ is polynomial-time reducible to APPROX for f.

Suppose now that $y \neq 0$. Clearly $f^*(y) \geq -f(0)$. Let $\rho > r$, where r is as in (15). Let $f^*_\rho(y) := \max_{\|x\|=\rho} \bigl(y^\mathsf{T} x - f(x)\bigr)$. As $y^\mathsf{T} x \leq \|y\|\|x\|$, the lower bound in (15) gives

$$f^*_\rho(y) \leq \|x\|(\|y\| - k_f\|x\|^{s-1}) = \rho(\|y\| - k_f\rho^{s-1}).$$

Hence there exists $\rho_1 = \rho(\|y\|, k_f, s) \in \mathbb{Q}\cap(r,\infty)$ such that $-f(0) > f^*_\rho(y)$ for all $y \in \mathbb{R}^n$ whenever $\rho \geq \rho_1$. Therefore

$$f^*(y) = -\min_{\|x\|\leq\rho_1} \bigl(f(x) - y^\mathsf{T} x\bigr) = -\min_{\|x\|\leq\rho_1+1} \bigl(f(x) - y^\mathsf{T} x\bigr).$$

Let $0 \neq y \in \mathbb{Q}^n$ and $C = B(0, \rho_1 + 1)$. Then the conditions of Lemma 18 are satisfied. Hence APPROX for $f^*(y)$ is polynomial-time reducible to APPROX for f. ∎

Since $f^{**} = f$ for a convex function and by Lemma 15, f and f^* both satisfy the polynomial growth condition if either one does, we obtain the following.

COROLLARY 21. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying* (15). *The approximation problem for f^* is polynomial-time computable (resp., NP-hard) if and only if the approximation problem for f is polynomial-time computable (resp., NP-hard).*

7. Conclusion. In this article, we have focused on establishing equivalence in the computational complexity of dual objects for several common convex objects and common notions of duality. These results are expected to have immediate applications in many areas. We conclude our article with two such examples.

Drawing from our own work, we rely on the results in sections 3 and 4 to deduce that the nuclear norm for higher-order tensors is NP-hard to compute [5, Corollary 8.8] and likewise for the dual norm of an operator (p, q)-norm when $1 \leq q < p \leq \infty$ or when $p = q \notin \{1, 2, \infty\}$ [5, section 7].

Following the notation in [10], we let $\Sigma^2_{\nabla^2_{n,4}}$ denote the cone of SOS-convex quartic forms [8] and $\Sigma^2_{n,4} \cap \mathbb{S}^{n^4}_{\mathrm{cvx}}$ denote the cone of convex quartic forms that are SOS. Using the results in section 5 and [10, Proposition 5.1 and Theorem 5.4], we easily deduce that membership in the dual cone of $\Sigma^2_{\nabla^2_{n,4}}$ is polynomial-time whereas membership in the dual cone of $\Sigma^2_{n,4} \cap \mathbb{S}^{n^4}_{\mathrm{cvx}}$ is NP-hard—observations that are new to the best of our knowledge. Furthermore, if we assume that $P \neq NP$, then it follows that the containment of $\Sigma^2_{\nabla^2_{n,4}}$ in $\Sigma^2_{n,4} \cap \mathbb{S}^{n^4}_{\mathrm{cvx}}$ is strict, verifying [10, Theorem 4.1].

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