Wavelet Sets and the Harmonic Analysis of a Discrete Affine Group

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The Discrete Affine Group

Let $\mathbb{D} := \{m2^n \in \mathbb{Q} \mid m, n \in \mathbb{Z}\}$ and $\vartheta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{D})$ be defined by

$$\vartheta(m)\beta = 2^{-m}\beta$$

for $\beta \in \mathbb{D}$, $m \in \mathbb{Z}$.

Semi-direct product $\mathbb{D} \rtimes_\vartheta \mathbb{Z}$ is a discrete subgroup of the one-dimensional affine group (a.k.a. $ax + b$ group).

Dilation and Translation Operators

For $\beta \in \mathbb{D}$, let $D, T_\beta : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$Df(t) := \sqrt{2} f(2t)$$
$$T_\beta f(t) := f(t-\beta)$$

for $f \in L^2(\mathbb{R})$.

Clearly, $D, T_\beta \in \mathcal{U}(L^2(\mathbb{R}))$.

Usual Notation: $T = T_1$.

Wavelet

Definition (Franklin-Strömberg). An (orthonormal) *wavelet* is a unit vector $\psi \in L^2(\mathbb{R})$ such that

$$\{D^n T^m \psi \mid n, m \in \mathbb{Z}\}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.

Note

$$D^n T^m \psi(t) = 2^{n/2} \psi(2^n t - m).$$

The Wavelet Group

It may be interesting to look at

$$\text{Group}(D, T) = \text{group generated by } D, T \text{ in } \mathcal{U}(L^2(\mathbb{R}))$$
$$= \{T_\beta D^n \mid \beta \in \mathbb{D}, n \in \mathbb{Z}\}.$$

Easy to see that

$$\text{Group}(D, T) \cong \mathbb{D} \rtimes_\vartheta \mathbb{Z}.$$

Wavelet Representation of $\mathbb{D} \rtimes_{\vartheta} \mathbb{Z}$

Look at the natural representation

$$\pi : \mathbb{D} \rtimes_{\vartheta} \mathbb{Z} \to \mathcal{U}(L^2(\mathbb{R}))$$
$$(\beta, n) \mapsto T_\beta U^n.$$

- $\pi(\mathbb{D} \rtimes_{\vartheta} \mathbb{Z}) = \text{Group}(D, T)$
- π faithful
- π cyclic (e.g. any wavelet is a cyclic vector)

(Recall $\pi : G \to \mathcal{U}(\mathcal{H})$ cyclic means $\overline{\text{span}}_{\mathbb{C}}\{\pi(x)\psi \mid x \in G\} = \mathcal{H}$)

Harmonic Analysis of $\mathbb{D} \rtimes_\vartheta \mathbb{Z}$

Objective: To decompose the representation $\pi : \mathbb{D} \rtimes_\vartheta \mathbb{Z} \to \mathcal{U}(L^2(\mathbb{R}))$, $(\beta, n) \mapsto T_\beta D^n$.

Result: π is unitarily equivalent to a direct integral of irreducible monomial representations indexed by a *wavelet set*.

Wavelet Sets

Definition (Dai and Larson). A measurable set $E \subseteq \mathbb{R}$ is called a *wavelet set* if

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right)$$

is a wavelet.

Example. Littlewood-Paley wavelet ψ_{LP} given by

$$\psi_{LP}(t) = \frac{\sin 2\pi t - \sin \pi t}{\pi t}$$

satisfies

$$\widehat{\psi}_{LP} = \frac{1}{\sqrt{2\pi}}\chi_{[-2\pi,-\pi)\cup[\pi,2\pi)}.$$

Example. Journé wavelet ψ_J satisfies

$$\widehat{\psi}_J = \frac{1}{\sqrt{2\pi}} \chi_{[-32\pi/7,-4\pi)\cup[-\pi,-4\pi/7)\cup(4\pi/7,\pi]\cup(4\pi,32\pi/7]}.$$

Theorem (Dai and Larson). $E \subseteq \mathbb{R}$ a measurable set. E is a wavelet set iff

i. $2^n E \cap 2^m E = \varnothing$ and $(E + 2n\pi) \cap (E + 2m\pi) = \varnothing$ whenever $n \neq m$;

ii. $\mathbb{R} \setminus \cup_{n\in\mathbb{Z}} 2^n E$ and $\mathbb{R} \setminus \cup_{n\in\mathbb{Z}} (E + 2n\pi)$ are both null sets.

Back to Our Result

Let $E \subseteq \mathbb{R}$ be a (any) wavelet set, say $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$.
For $t \in E$, define character

$$\begin{aligned} \chi^t : \mathbb{D} &\to \mathbb{T} \\ \beta &\mapsto e^{-i\beta t}. \end{aligned}$$

Then

$$\pi \cong \int_E^{\oplus} \mathrm{Ind}_{\mathbb{D}}^{\mathbb{D} \rtimes_\vartheta \mathbb{Z}} (\chi^t) \, d\mu_E(t).$$

Sketch of Proof

Let $\omega^t := \mathrm{Ind}_{\mathbb{D}}^{\mathbb{D}\rtimes_\vartheta\mathbb{Z}}(\chi^t) : \mathbb{D}\rtimes_\vartheta\mathbb{Z} \to \mathcal{U}(l^2(\mathbb{Z}))$. Usual construction of induced representation for semi-direct product groups* gives

$$[\omega^t(\beta,n)f](m) = e^{-i2^{-m}\beta t} f(m+n)$$

for $(\beta,n) \in \mathbb{D}\rtimes_\vartheta\mathbb{Z}$, $f \in l^2(\mathbb{Z})$.

(* see for example: A.A. Kirillov, *Elements of the Theory of Representations*, Grundlehren der mathematischen Wissenschaften, **220**, Springer-Verlag, Berlin Heidelberg, 1976)

Note that $\bigcup_{n\in\mathbb{Z}} 2^n E = \mathbb{R} - \{0\}$.

$E \times \mathbb{Z} \to \mathbb{R}, (t,n) \mapsto 2^{-m}t$ has inverse that is defined everywhere except 0 and so induces $\Phi : L^2(\mathbb{R}) \to L^2(E \times \mathbb{Z})$ where

$$(\Phi f)(t,n) = 2^{-m/2} f(2^{-m}t).$$

$$
\begin{array}{ccccc}
\mathcal{U}(L^2(\mathbb{R})) & \longrightarrow & \mathcal{U}(L^2(\mathbb{R})) & \longrightarrow & \mathcal{U}(L^2(E \times \mathbb{Z})) \\
A & \longmapsto & \widehat{A} = \mathcal{F} A \mathcal{F}^{-1} & \longmapsto & \tilde{A} = \Phi \widehat{A} \Phi^{-1}.
\end{array}
$$

For $f \in L^2(E \times \mathbb{Z})$,

$$
\begin{aligned}
\tilde{D}^n f(t,m) &= f(t, m+n) \\
\tilde{T}_\beta f(t,m) &= e^{-i2^{-m}\beta t} f(t,m).
\end{aligned}
$$

Now look at $\tilde{\pi} : \mathbb{D} \rtimes_\vartheta \mathbb{Z} \to \mathcal{U}(L^2(E \times \mathbb{Z}))$, $(\beta, n) \mapsto \tilde{T}_\beta \tilde{D}^n$. Clearly $\tilde{\pi} \cong \pi$.

$$
[\tilde{\pi}(\beta,n) f]\,(t,m) = \tilde{T}_\beta \tilde{D}^n f(t,m) = e^{-i2^{-m}\beta t} f(t, m+n).
$$

It remains to make the following identification

$$L^2(E \times \mathbb{Z}) \cong \int_E^{\oplus} (l^2(\mathbb{Z}))_t \, d\mu_E(t)$$

(roughly, given any $f \in L^2(E \times \mathbb{Z})$, $f(t, \cdot) \in l^2(\mathbb{Z})$ for each $t \in E$).

So

$$\tilde{\pi} \cong \int_E^{\oplus} \omega^t \, d\mu_E(t),$$

and

$$\pi \cong \int_E^{\oplus} \mathrm{Ind}_{\mathbb{D}}^{\mathbb{D} \rtimes_\vartheta \mathbb{Z}}(\chi^t) \, d\mu_E(t).$$

Generalization to Higher Dimensions

L., J. Packer and K. Taylor, "Direct integral decomposition of the wavelet representation," to appear in *PAMS*, preprint available from `http://xxx.lanl.gov/ps/math.FA/0003067`.

A a dilation matrix, ie. $A \in M(n, \mathbb{Z}) \cap \mathrm{GL}(n, \mathbb{Q})$ and all eigenvalues of A have absolute value > 1. $v \in \mathbb{Z}^n$.

- The affine group is $\mathbb{Q}_A \rtimes_\vartheta \mathbb{Z}$ where

  $$\mathbb{Q}_A = \bigcup_{j=0}^{\infty} \{A^{-j}v \mid v \in \mathbb{Z}^n\} \subseteq \mathbb{Q}^n$$

  and

  $$\vartheta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Q}_A), \vartheta(m)\beta = A^{-m}\beta.$$

- Dilation and translation operators are

$$D_A f(t) = |\mathrm{det}\, A|^{1/2} f(At),$$
$$T_v f(t) = f(t - v)$$

for $f \in L^2(\mathbb{R}^n)$.

- $\psi \in L^2(\mathbb{R}^n)$ is a wavelet iff $\{D_A^m T_v \psi \mid m \in \mathbb{Z}, v \in \mathbb{Z}^n\}$ is an orthonormal basis.

- A measurable $E \subseteq \mathbb{R}^n$ is a wavelet set iff

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\mu(E)}}\chi_E\right)$$

is a wavelet.

Similar Results for Higher Dimensions

Theorem (Dai, Larson and Speegle). Wavelet set exists for any dilation matrix A.

Our Result: Let $\pi : \mathbb{Q}_A \rtimes_\vartheta \mathbb{Z} \to \mathcal{U}(L^2(\mathbb{R}^n))$, $(\beta, m) \mapsto T_\beta D_A^m$. Then

$$\pi \cong \int_E^{\oplus} \mathrm{Ind}_{\mathbb{Q}_A}^{\mathbb{Q}_A \rtimes_\vartheta \mathbb{Z}}(\chi_t)\, d\mu_E(t).$$

where $\chi^t : \mathbb{Q}_A \to \mathbb{T}$, $\beta \mapsto e^{-i\langle t,\beta\rangle}$.