A Combinatorial Framework for Nonlinear Dynamics

Konstantin Mischaikow

Dept. of Mathematics/BioMaPS Institute
Rutgers
mischaik@math.rutgers.edu

I. WHY?

Assume: there exists a multiparameter deterministic model for the dynamics $f : X \times \Lambda \to X$ (X is compact)

Phase Space (X) Parameter Space (Λ)

$f_\lambda(\cdot) = f(\cdot, \lambda) : X \to X$ Iterations define the dynamics

Assume: there exists a multiparameter deterministic model for the dynamics $f : X \times \Lambda \to X$ (X is compact)

Phase Space (↑ X) Parameter Space (↑ Λ)

$f_\lambda(\cdot) = f(\cdot, \lambda) : X \to X$ Iterations define the dynamics

Objects of Interest: Invariant sets

Bounded subsets $S_\lambda \subset X$ such that $f_\lambda(S_\lambda) = S_\lambda$

Assume: there exists a multiparameter deterministic model for the dynamics $f: X \times \Lambda \to X$ (X is compact)

Phase Space (X) — Parameter Space (Λ)

$f_\lambda(\cdot) = f(\cdot, \lambda): X \to X$ Iterations define the dynamics

Objects of Interest: Invariant sets

Bounded subsets $S_\lambda \subset X$ such that $f_\lambda(S_\lambda) = S_\lambda$

Invariant sets are associated to asymptotic dynamics

Example: If $f(x) = \frac{1}{2}x$ then $S = \{0\}$

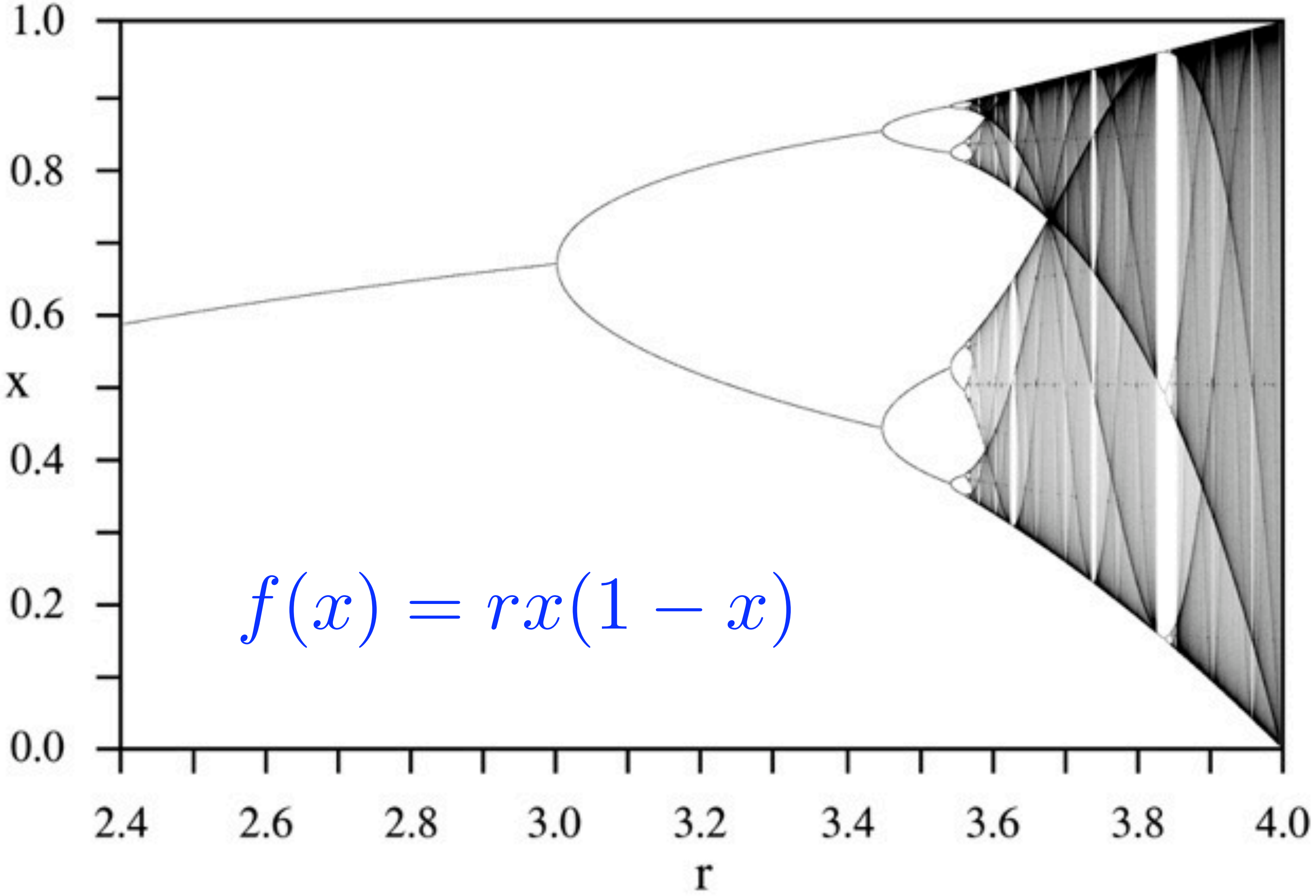
Three Problems associated with Invariant Sets.

Three Problems associated with Invariant Sets.

1. Time series data is transient.

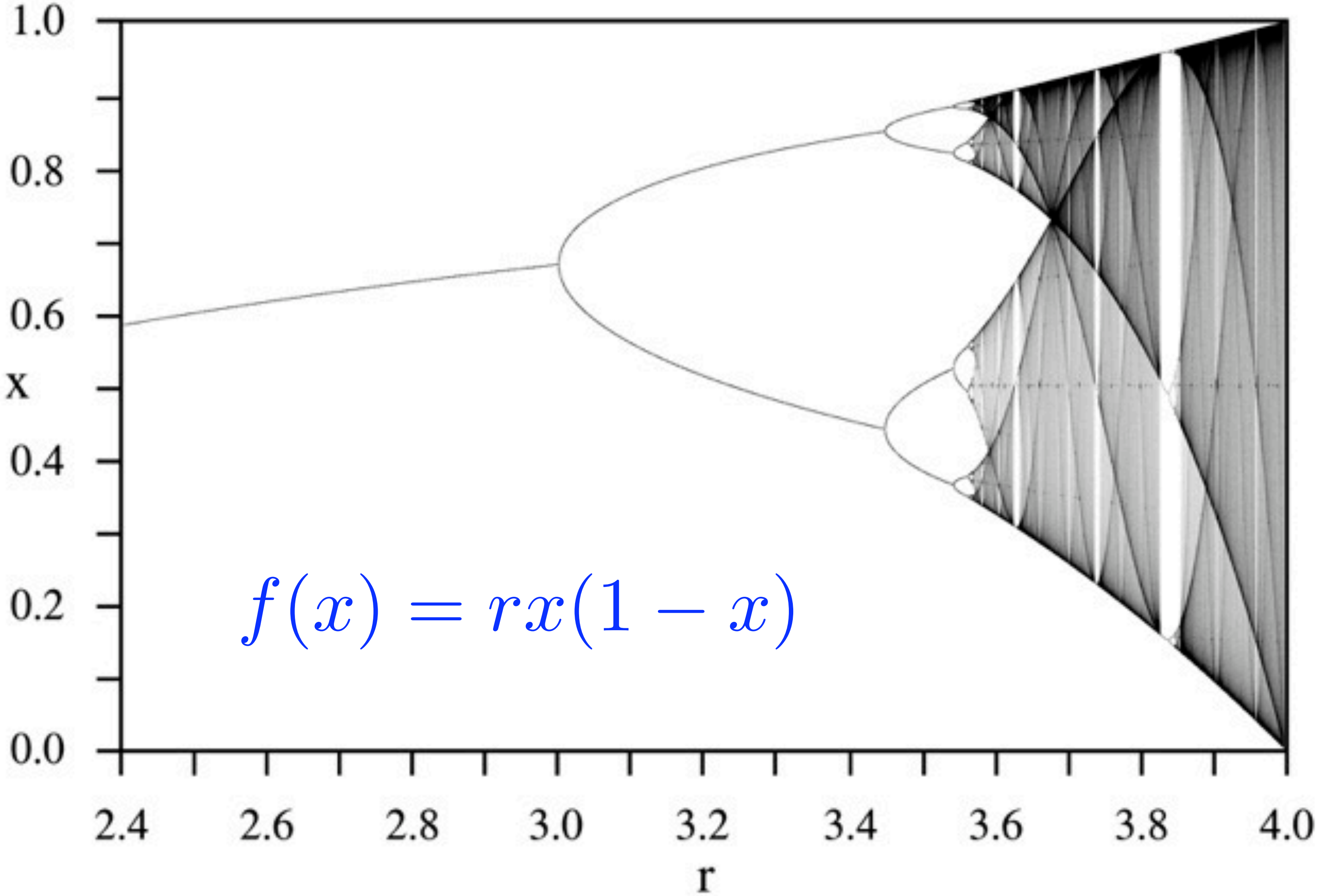
Three Problems associated with Invariant Sets.

1. Time series data is transient.



Three Problems associated with Invariant Sets.

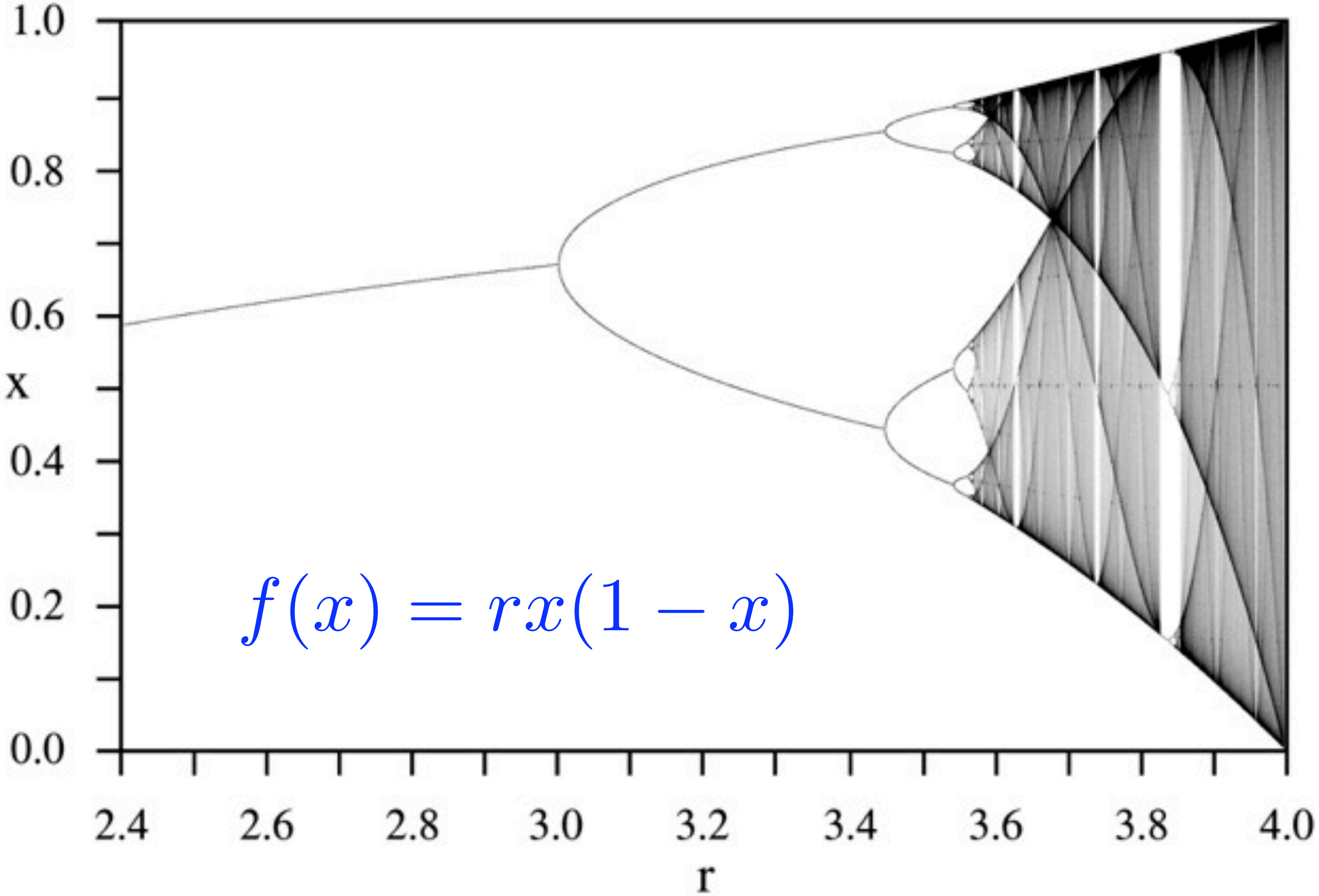
1. Time series data is transient.



2. Nonlinear systems exhibit chaos: for each parameter value there can be uncountably many topologically distinct orbits.

Three Problems associated with Invariant Sets.

1. Time series data is transient.



2. Nonlinear systems exhibit chaos: for each parameter value there can be uncountably many topologically distinct orbits.

3. Bifurcations can occur on Cantor sets of positive measure

II. Rigorous Computational Results for Multiparameter Systems

Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics

Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics

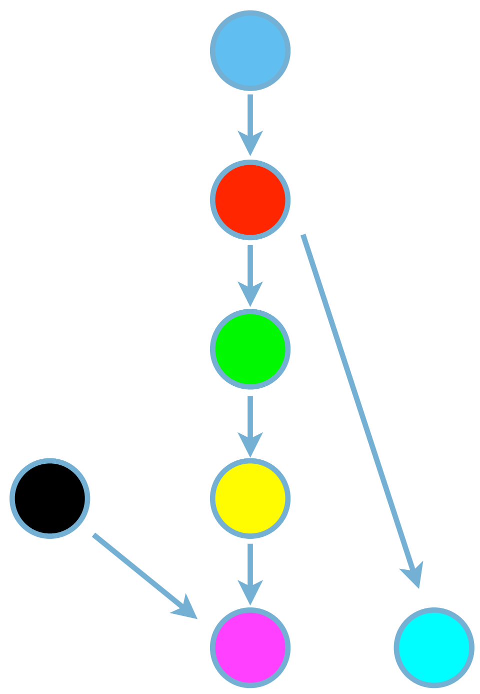
A Morse decomposition M of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of compact disjoint invariant sets of $M(p) \subset S$, called Morse sets, such that for every $x \notin \bigcup_{p \in \mathcal{P}} M(p)$ there are indices $q < p$ in P such that the forward orbit of x limits to $\mathsf{M}(q)$ and the backward orbit of x limits to $\mathsf{M}(p)$

Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics

A Morse decomposition M of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of compact disjoint invariant sets of $M(p) \subset S$, called Morse sets, such that for every $x \notin \bigcup_{p \in \mathcal{P}} M(p)$ there are indices $q < p$ in P such that the forward orbit of x limits to $\mathsf{M}(q)$ and the backward orbit of x limits to $\mathsf{M}(p)$

The labelling by P implies that a Morse decomposition can be represented as an acyclic directed graph $\mathcal{MG}$ called the Morse graph.

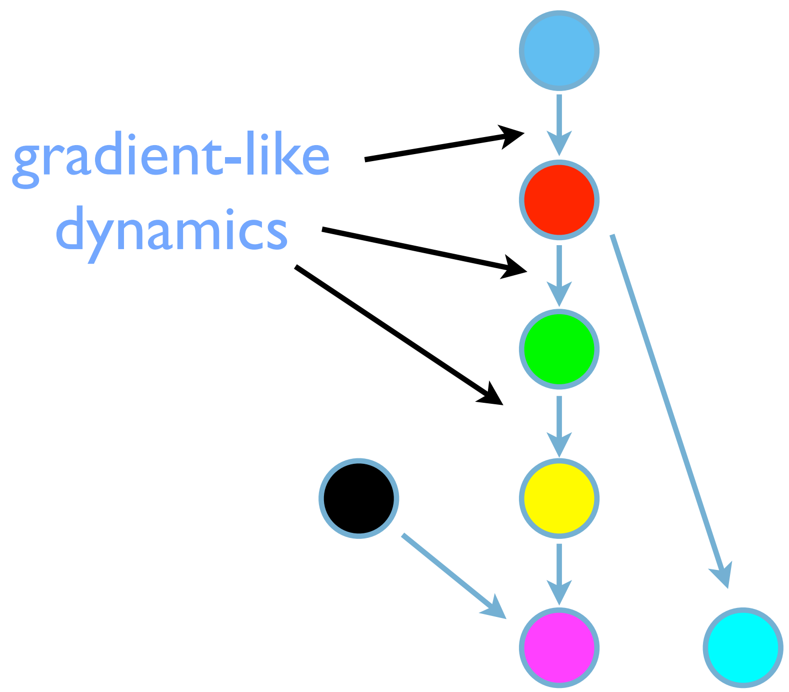


Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics

A Morse decomposition M of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of compact disjoint invariant sets of $M(p) \subset S$, called Morse sets, such that for every $x \notin \bigcup_{p \in \mathcal{P}} M(p)$ there are indices $q < p$ in P such that the forward orbit of x limits to $\mathsf{M}(q)$ and the backward orbit of x limits to $\mathsf{M}(p)$

The labelling by P implies that a Morse decomposition can be represented as an acyclic directed graph $\mathcal{MG}$ called the Morse graph.

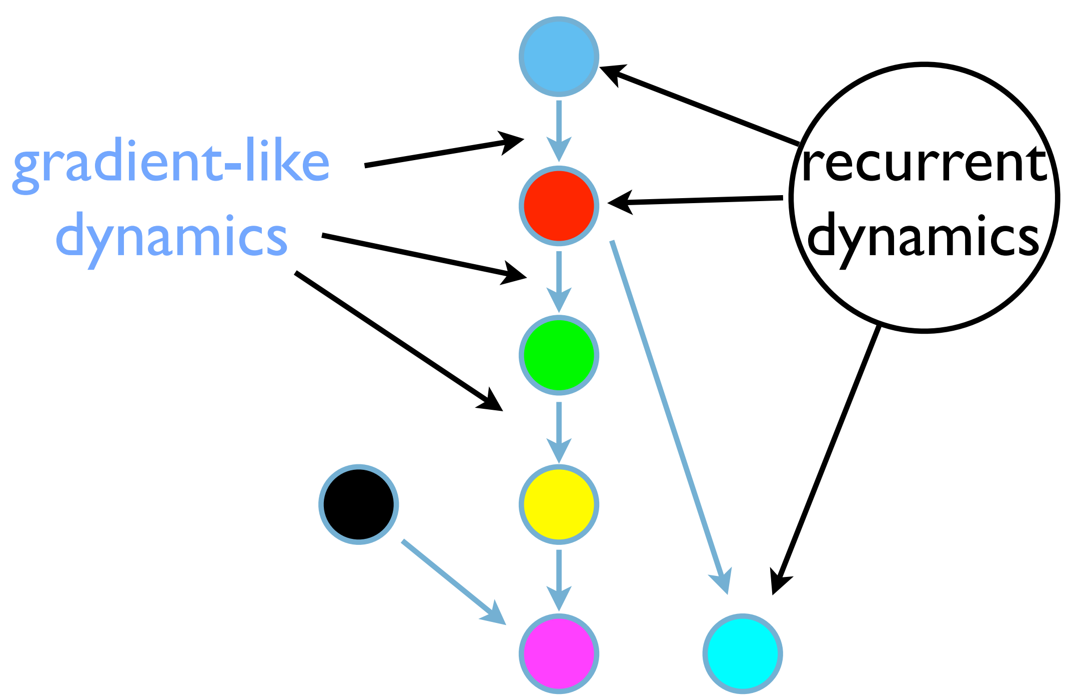


Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics

A Morse decomposition M of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of compact disjoint invariant sets of $M(p) \subset S$, called Morse sets, such that for every $x \notin \bigcup_{p \in \mathcal{P}} M(p)$ there are indices $q < p$ in P such that the forward orbit of x limits to $\mathsf{M}(q)$ and the backward orbit of x limits to $\mathsf{M}(p)$

The labelling by P implies that a Morse decomposition can be represented as an acyclic directed graph $\mathcal{MG}$ called the Morse graph.

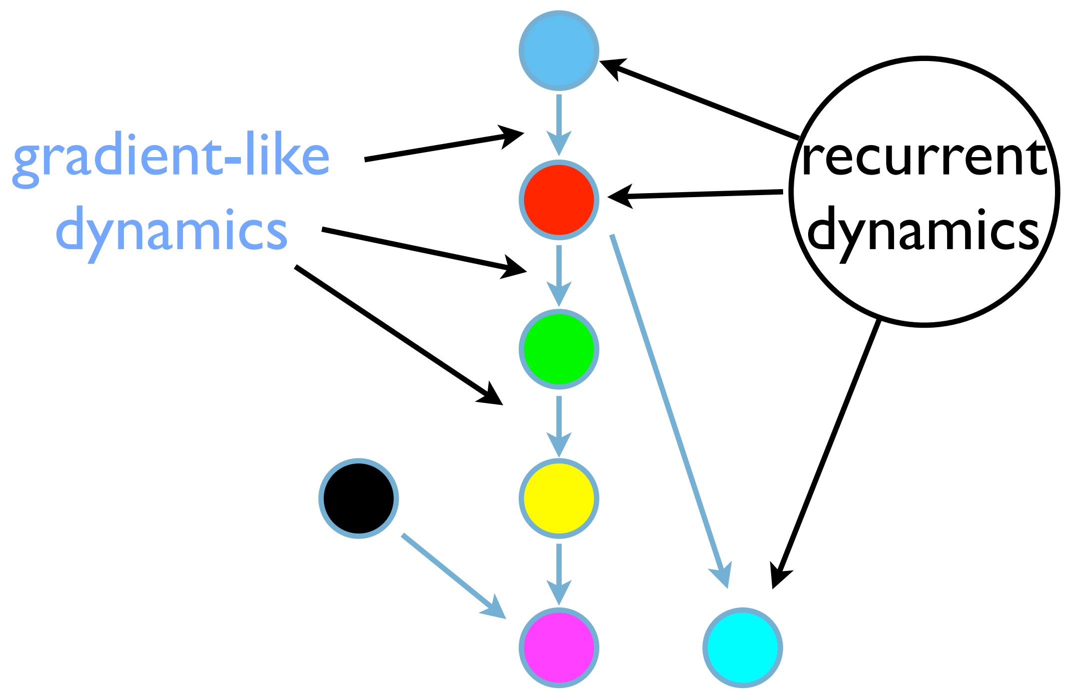


Fundamental Decomposition:

Recurrent Dynamics vs. Gradient-like Dynamics

A Morse decomposition M of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of compact disjoint invariant sets of $M(p) \subset S$, called Morse sets, such that for every $x \notin \bigcup_{p \in \mathcal{P}} M(p)$ there are indices $q < p$ in P such that the forward orbit of x limits to $\mathsf{M}(q)$ and the backward orbit of x limits to $\mathsf{M}(p)$

The labelling by P implies that a Morse decomposition can be represented as an acyclic directed graph $\mathcal{MG}$ called the Morse graph.



An Example

A density dependent Leslie model:

1st year pop.
2nd year pop.

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\ 0.7x \end{bmatrix}$$

$$\begin{array}{rcl} f : \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y; \theta_1, \theta_2) & & \end{array}$$

An Example

A density dependent Leslie model:

$$\begin{array}{l} \text{1st year pop.} \\ \text{2nd year pop.} \end{array} \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\ 0.7x \end{bmatrix} \quad \begin{array}{ccc} f: \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y; \theta_1, \theta_2) & & \end{array}$$

We can construct a mathematically rigorous, queryable database for the global dynamics on the phase space

$$[0, \infty) \times [0, \infty)$$

and for all parameters

$$\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]$$

An Example

A density dependent Leslie model:

$$\begin{matrix}\text{1st year pop.}\\ \text{2nd year pop.}\end{matrix}\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\ 0.7x \end{bmatrix} \quad \begin{matrix} f : \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x, y; \theta_1, \theta_2) & & \end{matrix}$$

We can construct a mathematically rigorous, queryable database for the global dynamics on the phase space

$$[0, \infty) \times [0, \infty)$$

and for all parameters

$$\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]$$

Input: Nonlinear map, Phase space, Parameter space
Resolution in phase space
Resolution in parameter space

An Example

A density dependent Leslie model:

1st year pop. / 2nd year pop.

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x + \theta_2 y)e^{-0.1(x+y)} \\ 0.7x \end{bmatrix} \quad f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y; \theta_1, \theta_2)$$

We can construct a mathematically rigorous, queryable database for the global dynamics on the phase space

$$[0, \infty) \times [0, \infty)$$

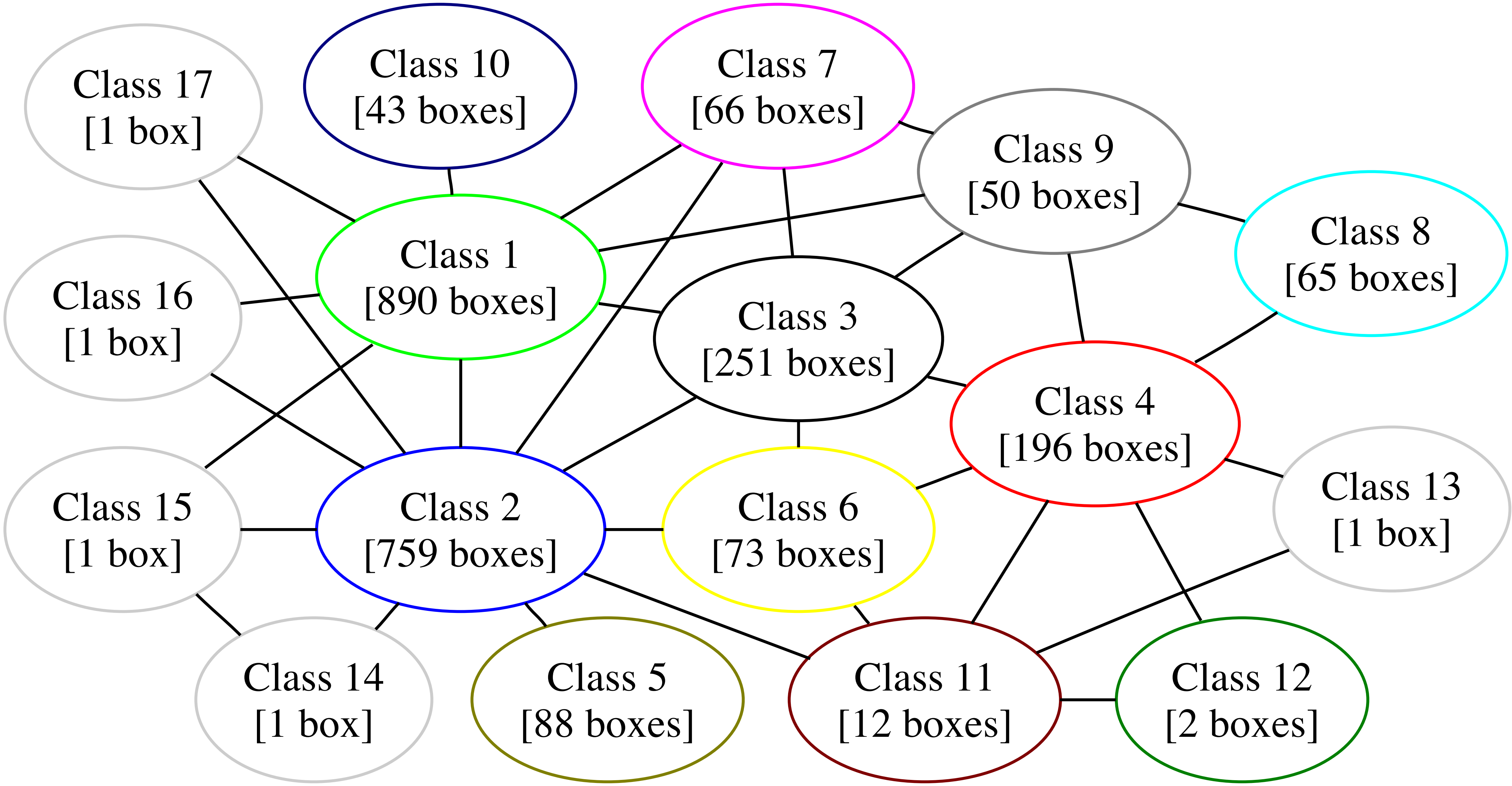
and for all parameters

$$\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]$$

Input: Nonlinear map, Phase space, Parameter space

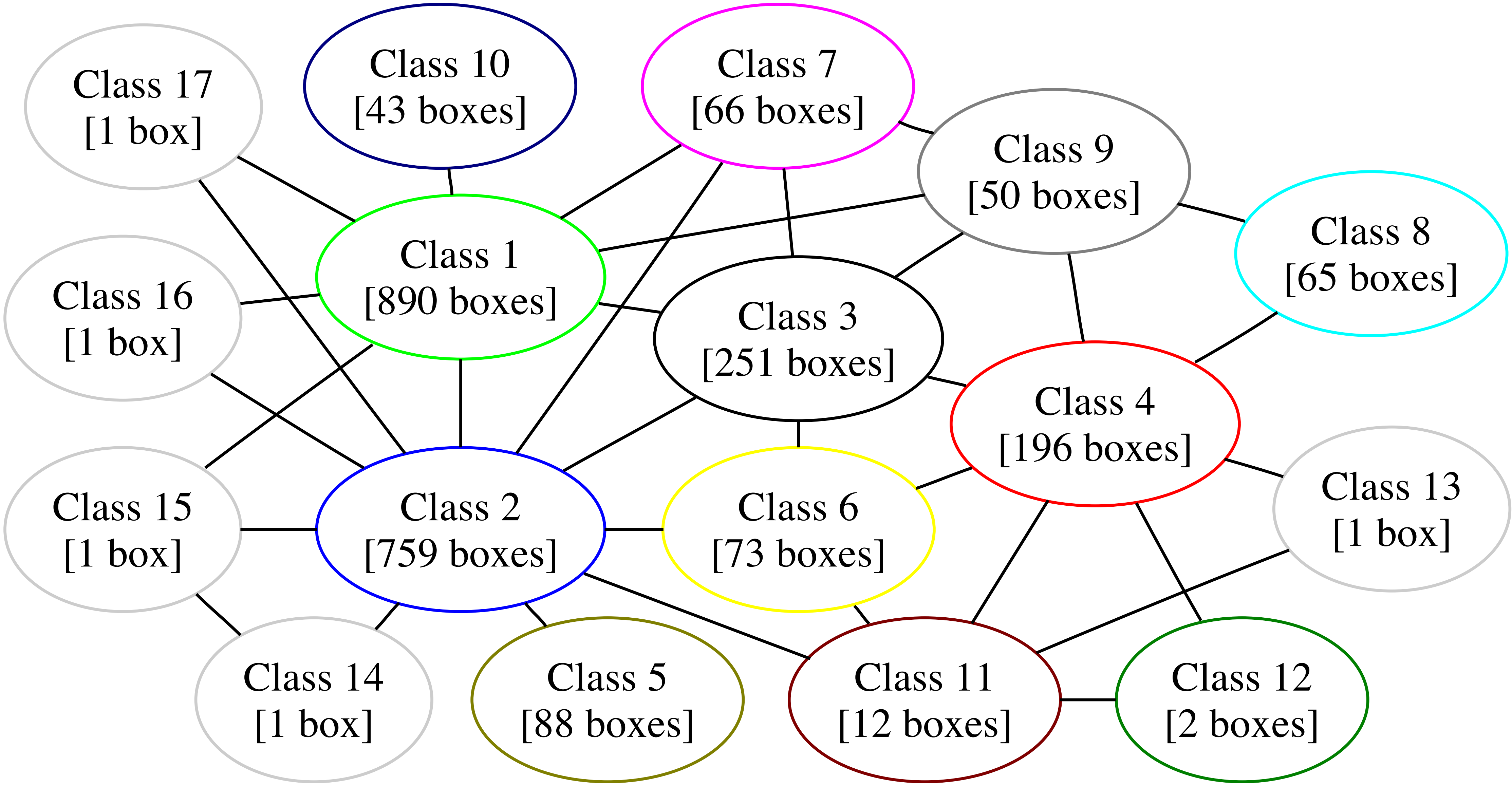
Resolution in phase space
Resolution in parameter space

The Data Base



The Continuation Graph

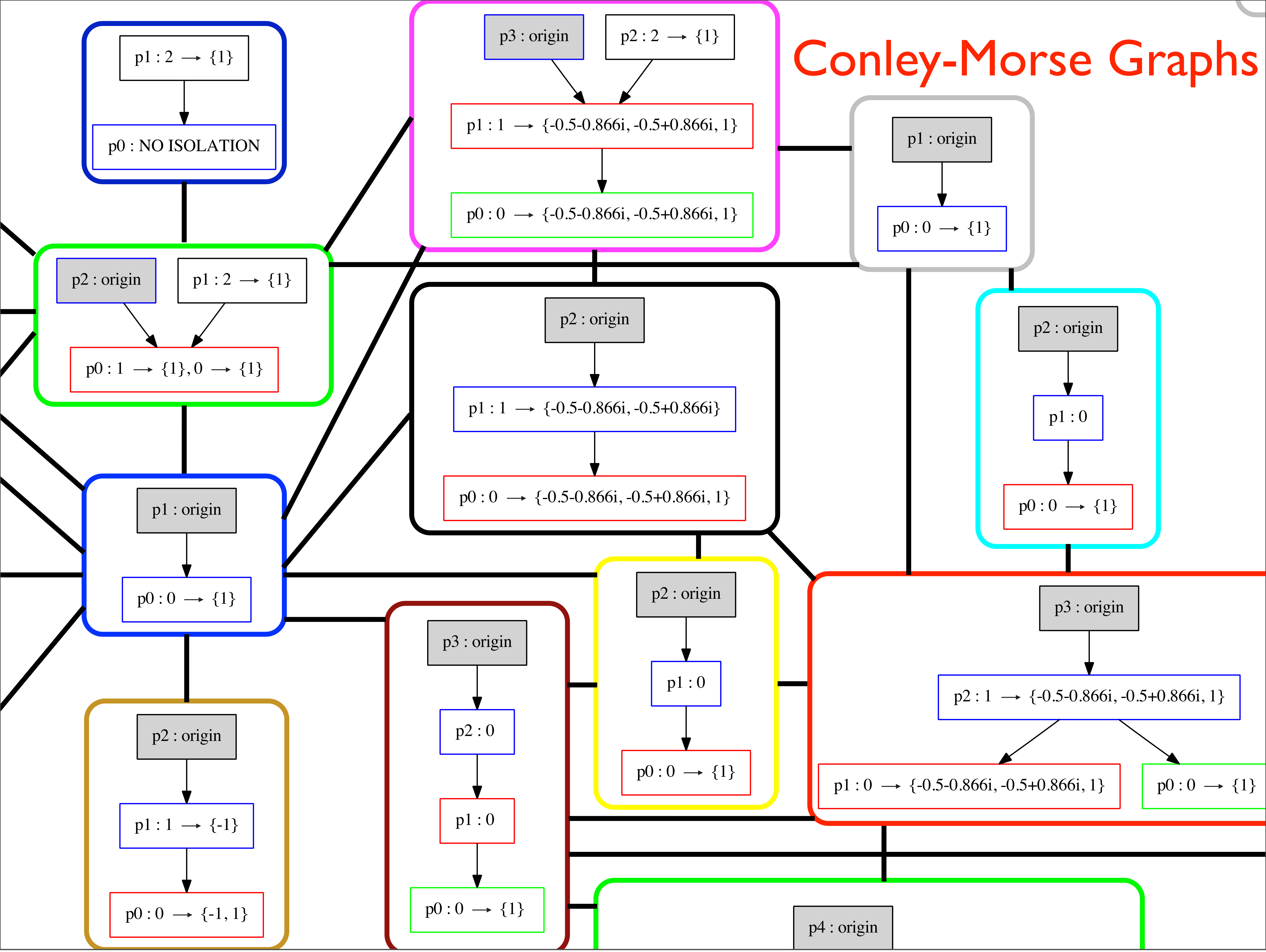
The Data Base



The Continuation Graph

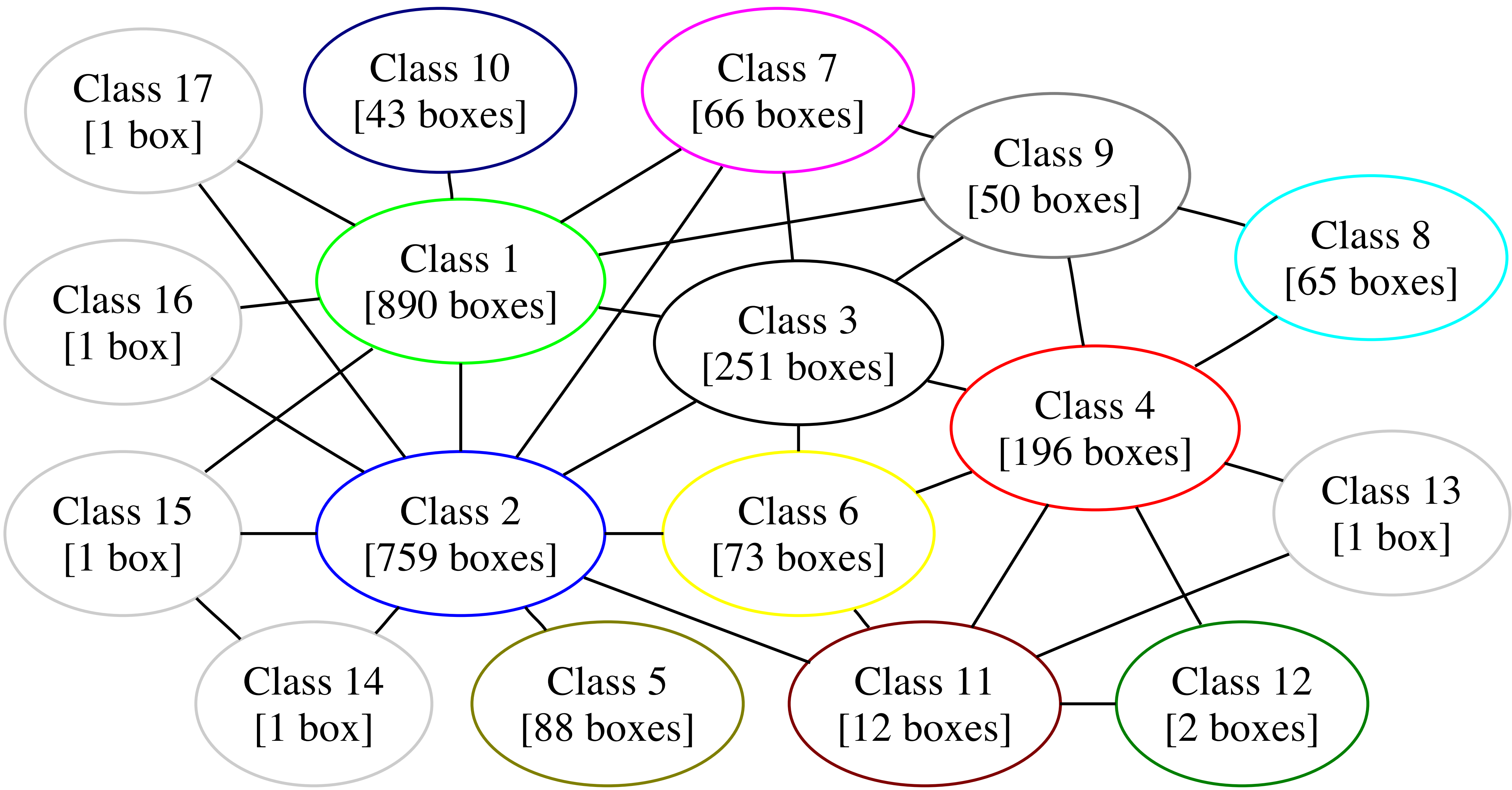
Nodes represent Conley-Morse Graphs

Conley-Morse Graphs



Saturday, July 3, 2010

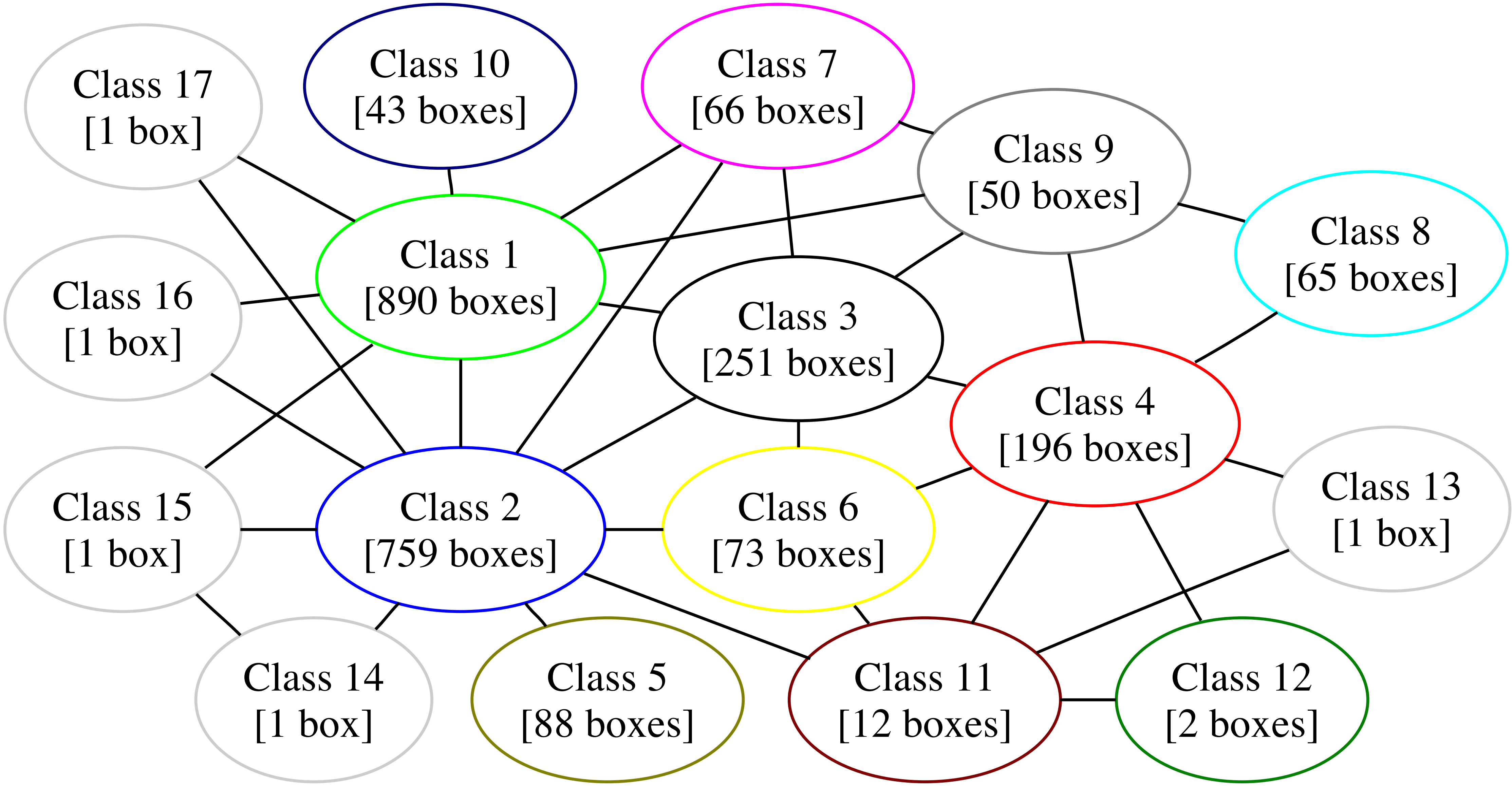
The Data Base



The Continuation Graph

Nodes represent Conley-Morse Graphs

The Data Base

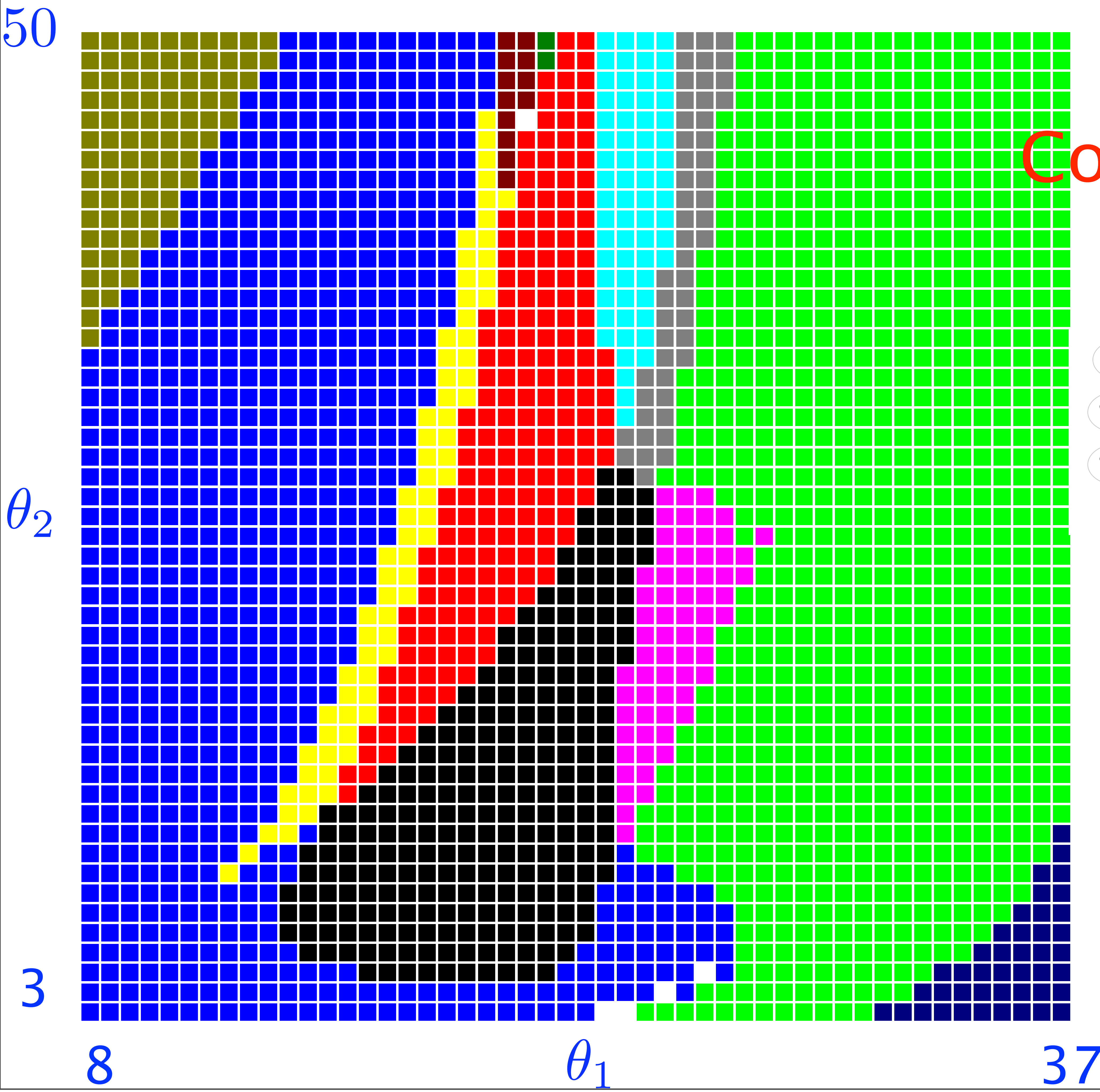


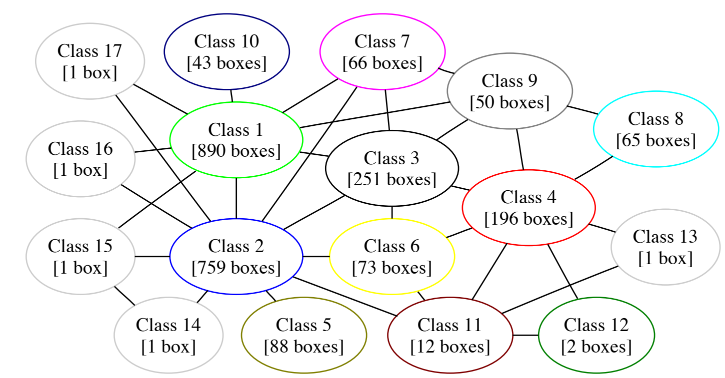
The Continuation Graph

Nodes represent Conley-Morse Graphs

Edges indicate connectivity in parameter space

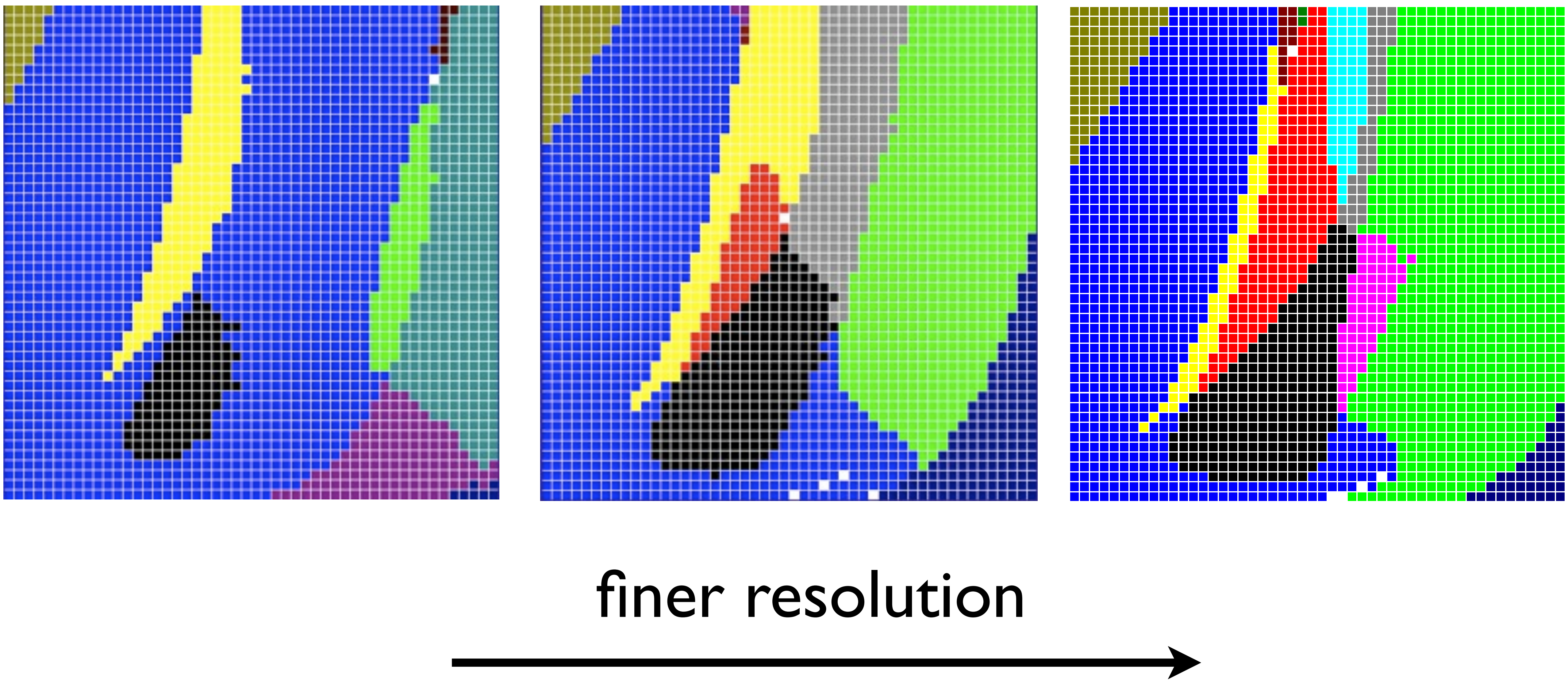
The Continuation Diagram





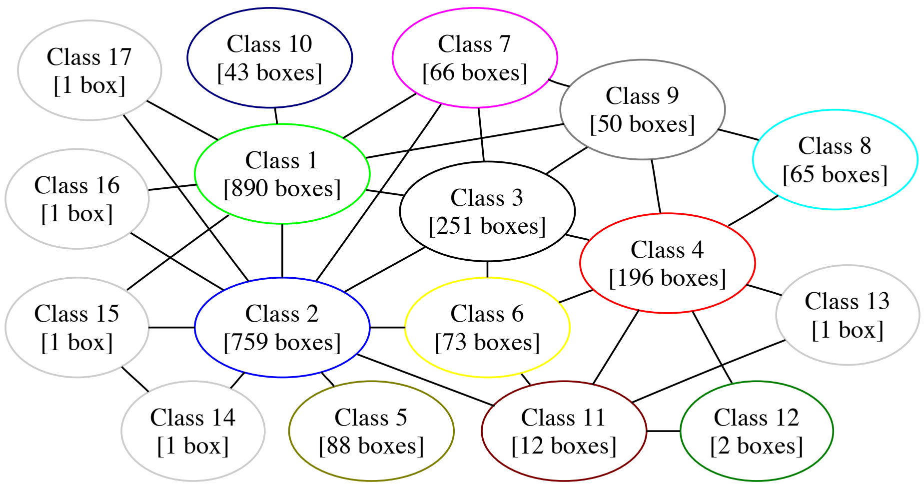
Different colors represent different continuation classes

Database results are never wrong, BUT they depend on the resolution!

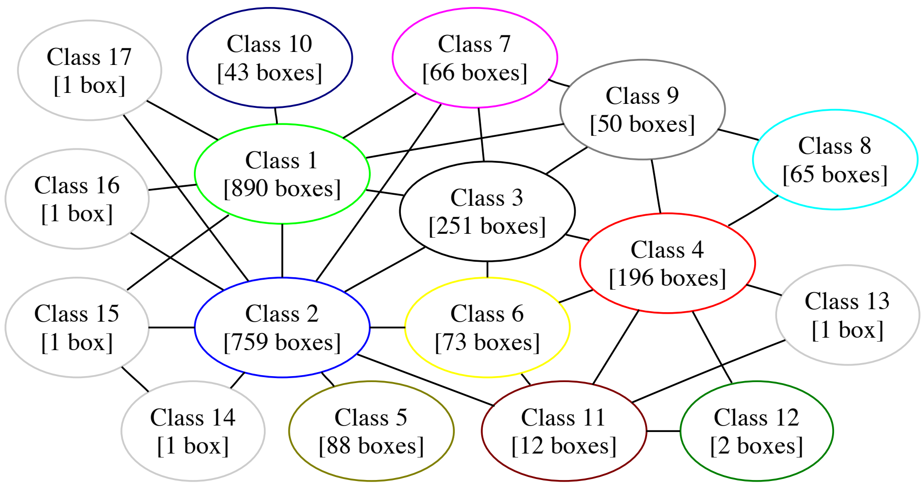


Appropriate resolution is problem dependent!

Querying the Database: *Are there multiple basins of attraction?*



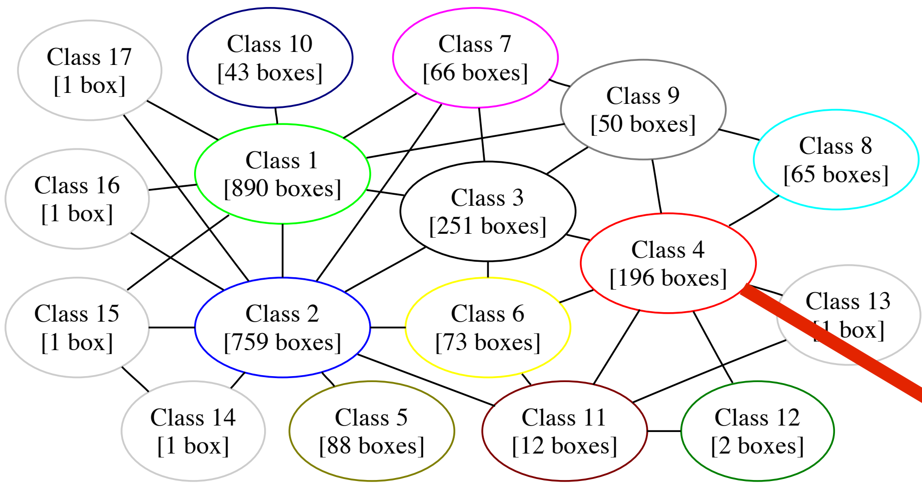
Querying the Database: *Are there multiple basins of attraction?*



Query the gradient-like structure:

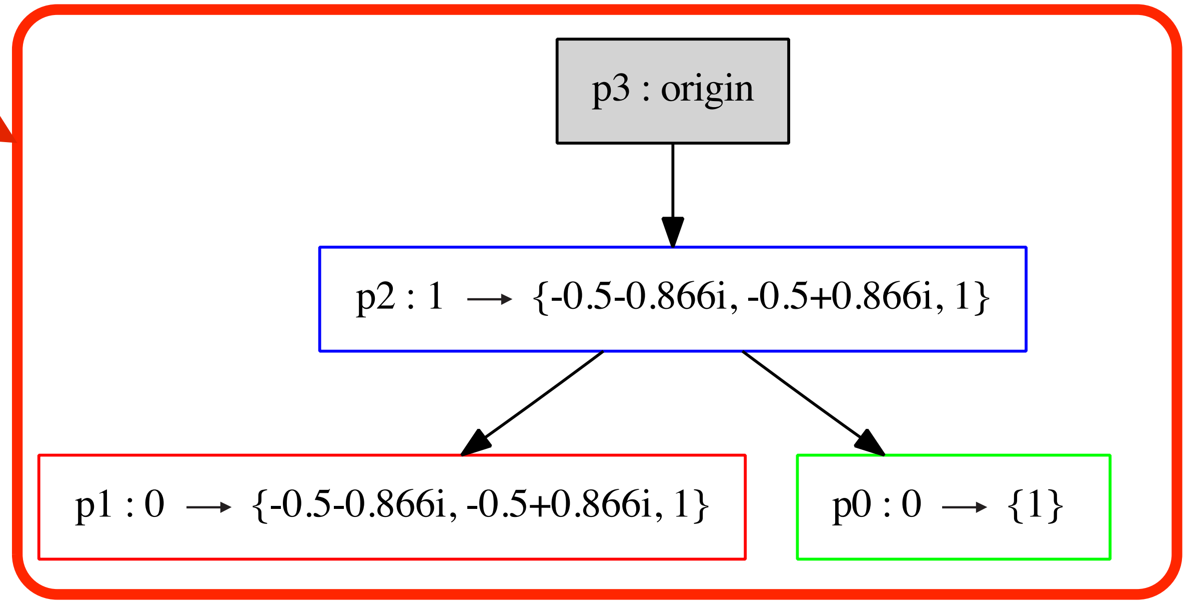
Is there a Morse graph with multiple minimal elements?

Querying the Database: *Are there multiple basins of attraction?*

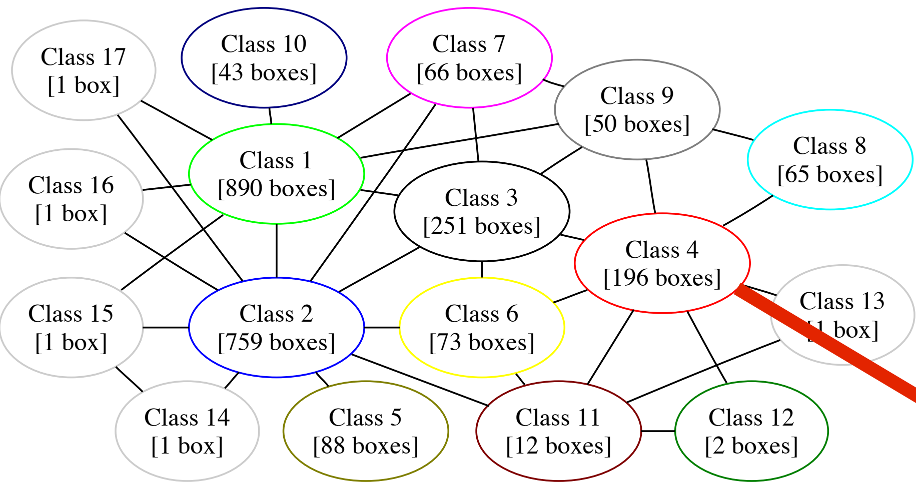


Query the gradient-like structure:

Is there a Morse graph with multiple minimal elements?

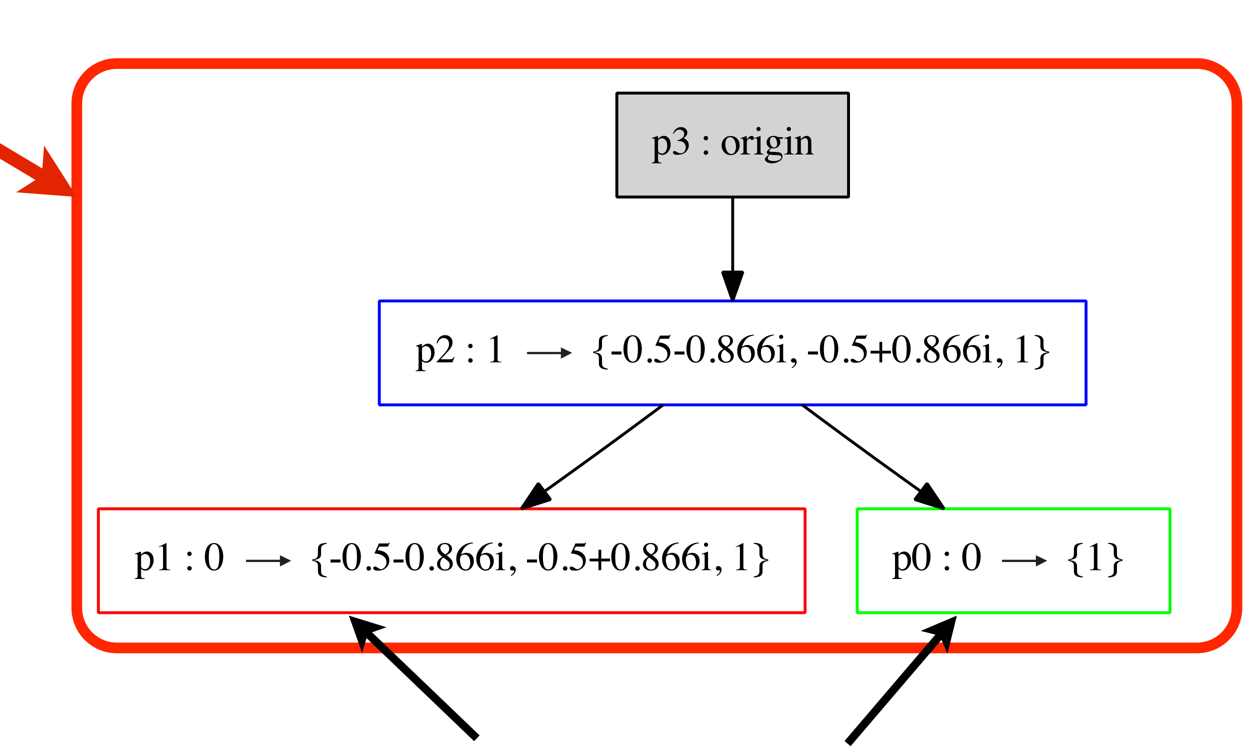


Querying the Database: *Are there multiple basins of attraction?*



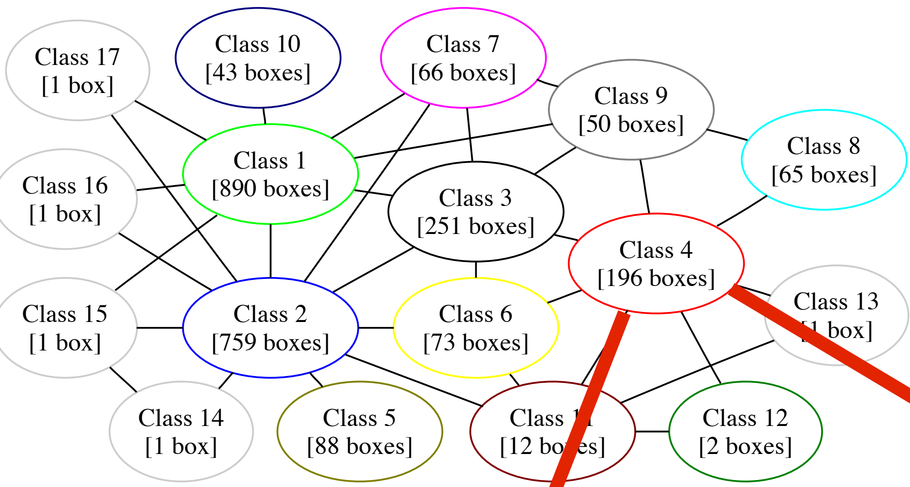
Query the gradient-like structure:

Is there a Morse graph with multiple minimal elements?



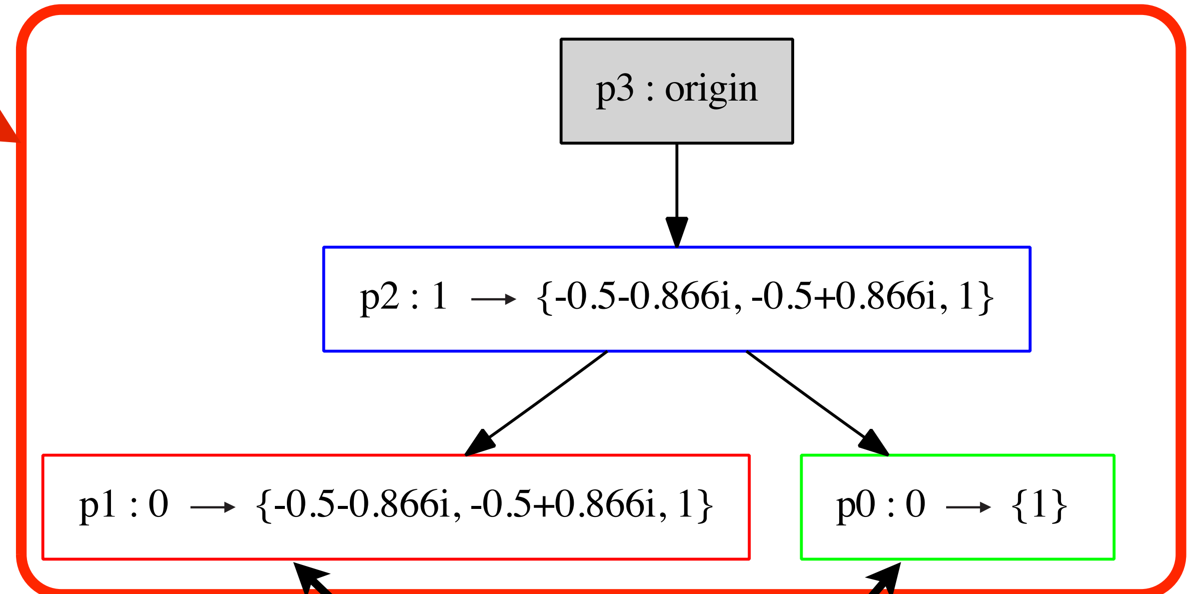
2 observable basins of attraction

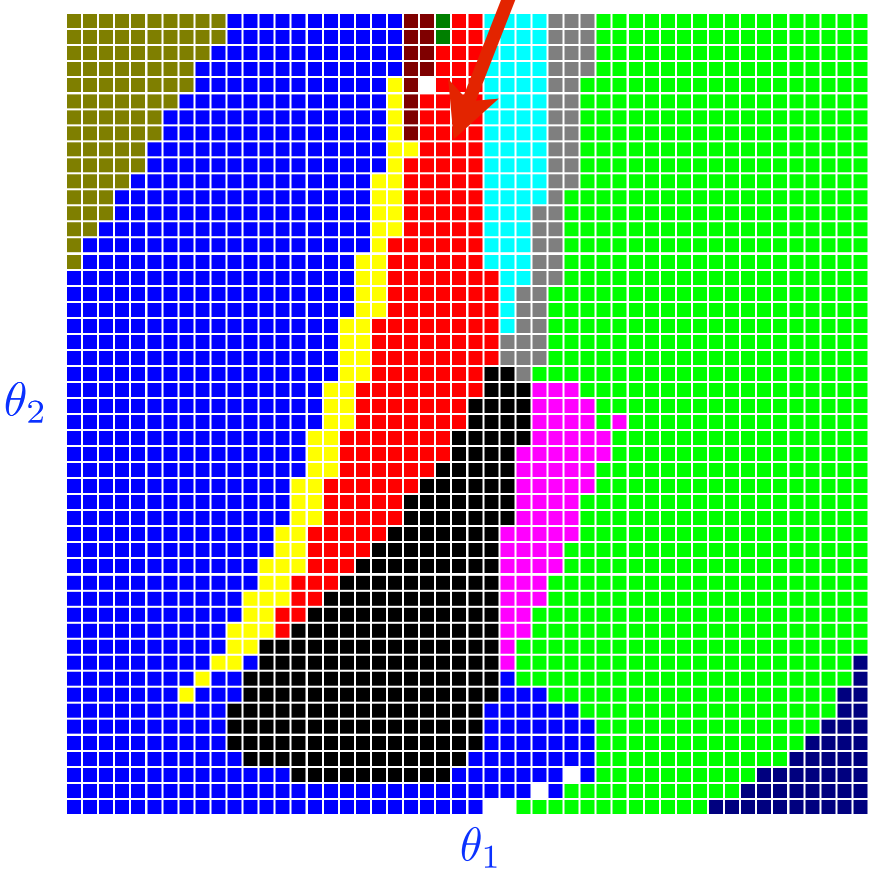
Querying the Database: *Are there multiple basins of attraction?*



Query the gradient-like structure:

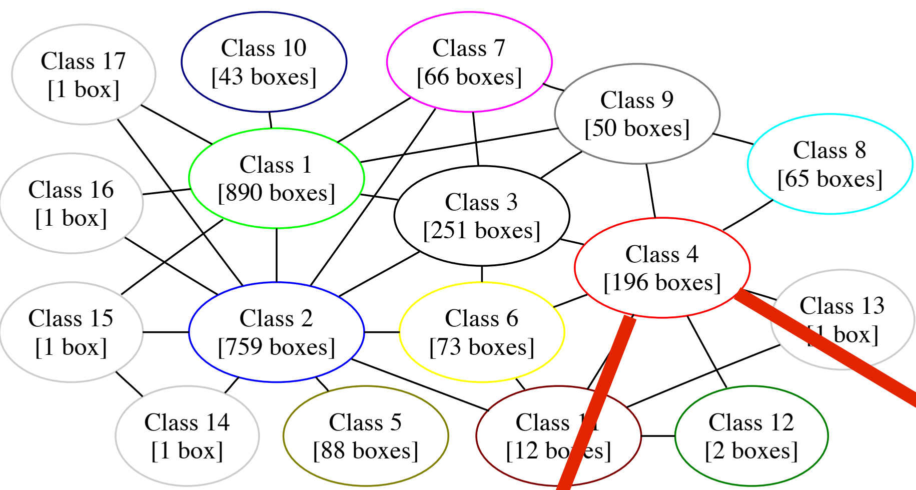
Is there a Morse graph with multiple minimal elements?





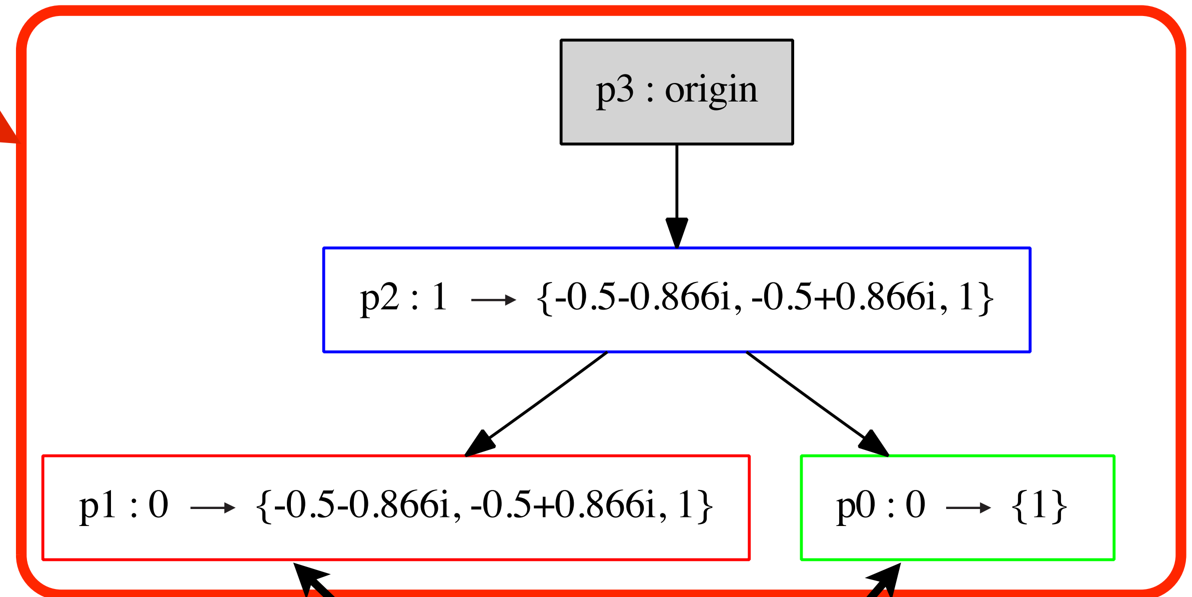
2 observable basins of attraction

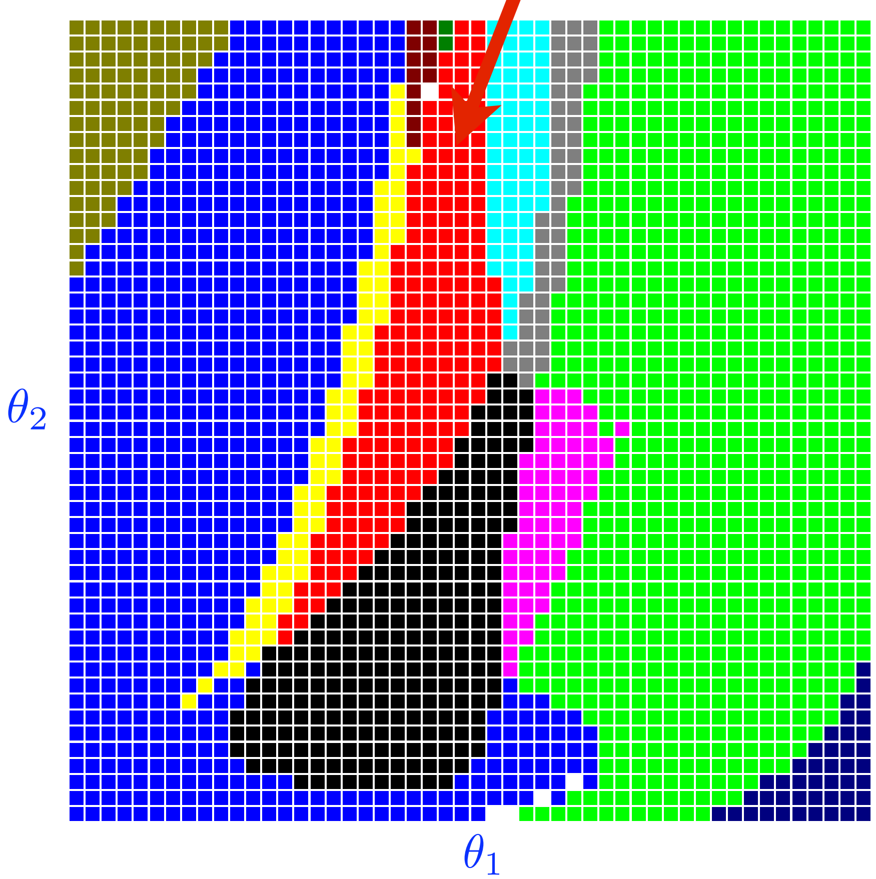
Querying the Database: *Are there multiple basins of attraction?*



Query the gradient-like structure:

Is there a Morse graph with multiple minimal elements?

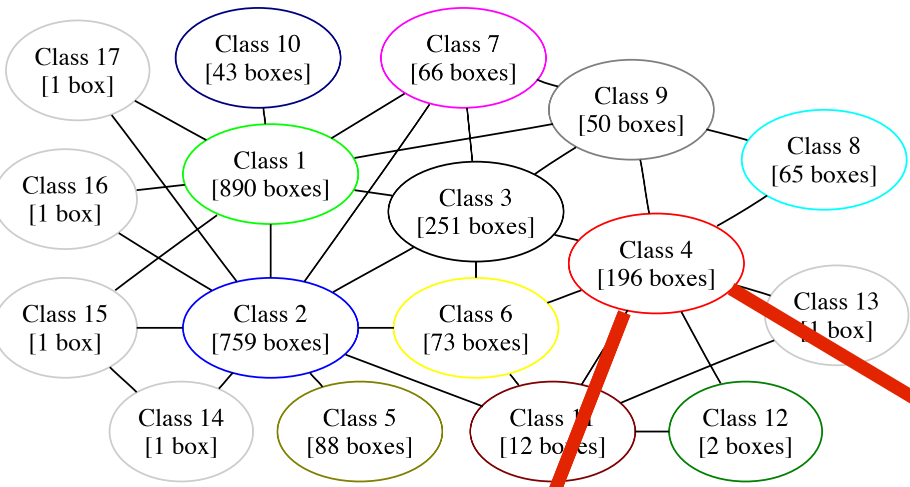


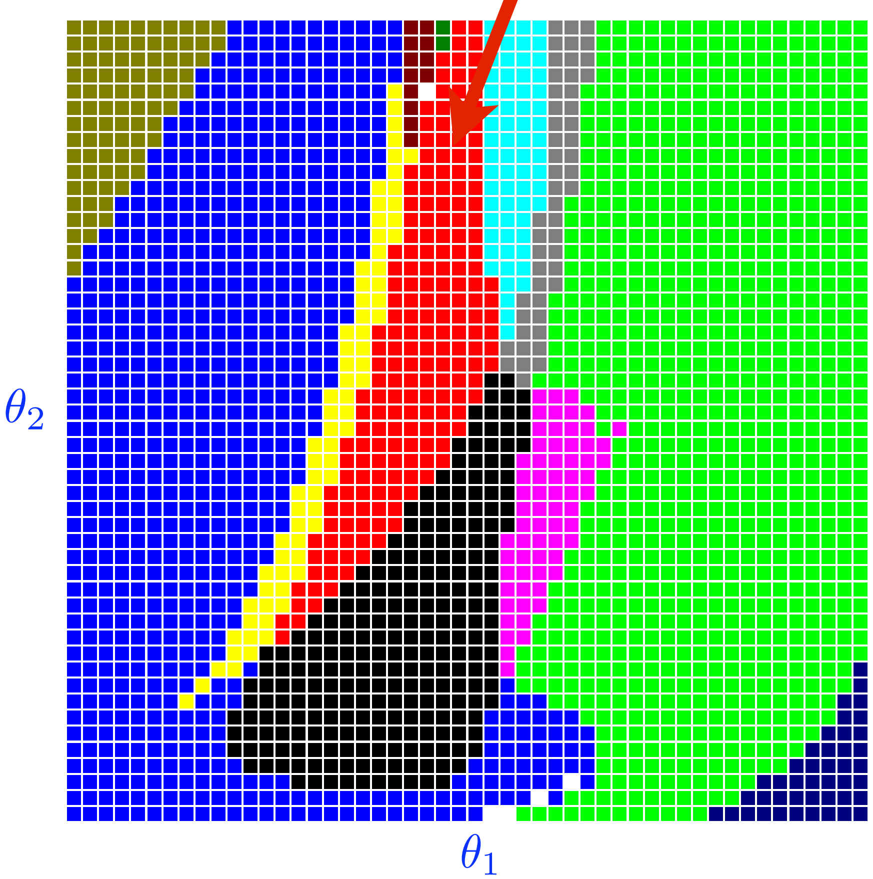


2 observable basins of attraction

Can we characterize the attracting dynamics?

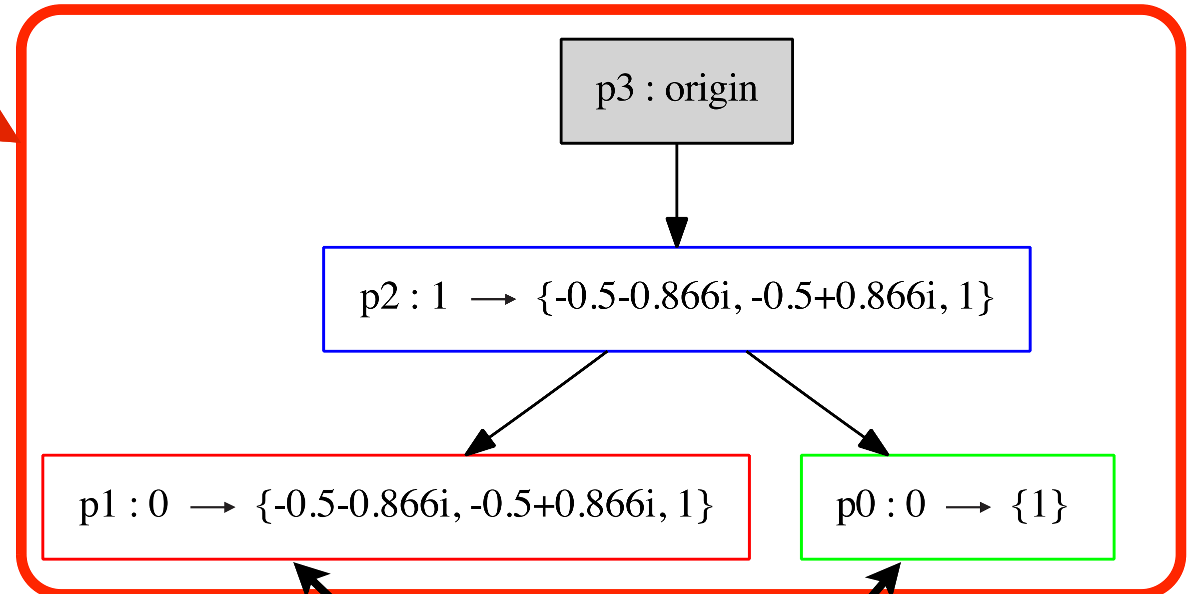
Querying the Database: *Are there multiple basins of attraction?*





Query the gradient-like structure:

Is there a Morse graph with multiple minimal elements?

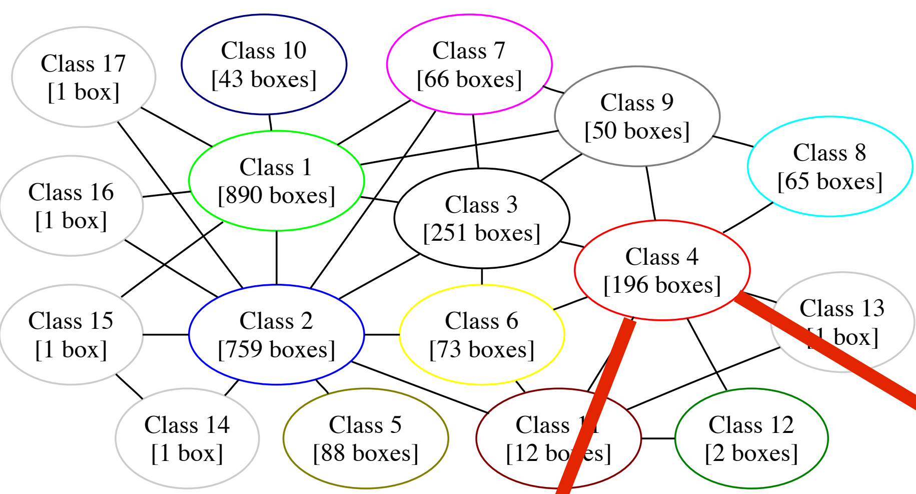


2 observable basins of attraction

Can we characterize the attracting dynamics?

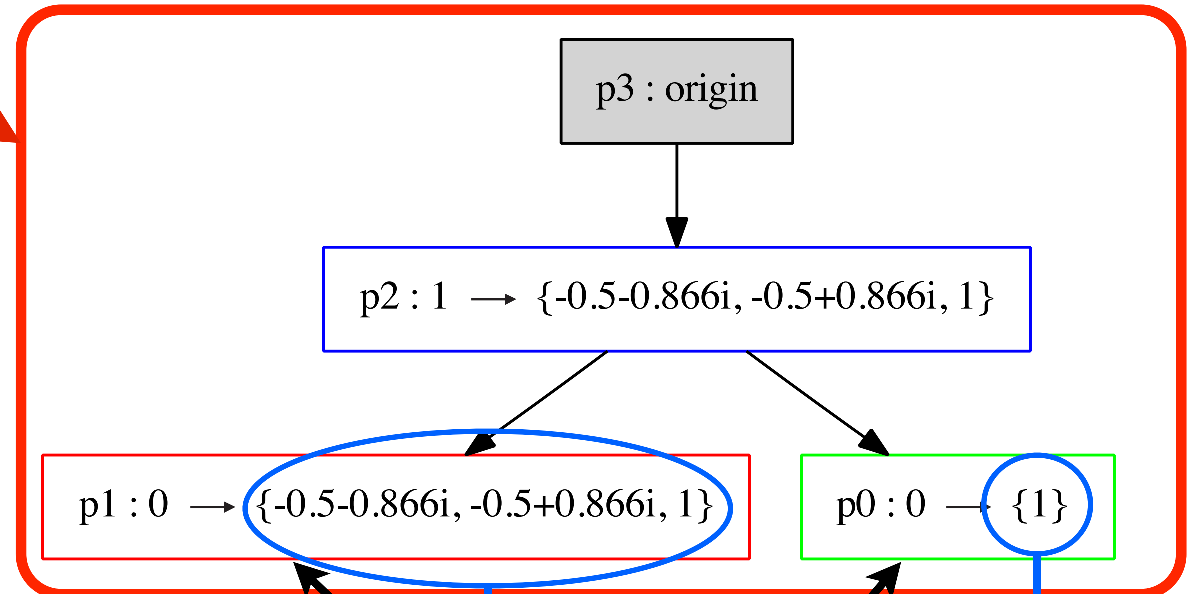
Query the Conley index:

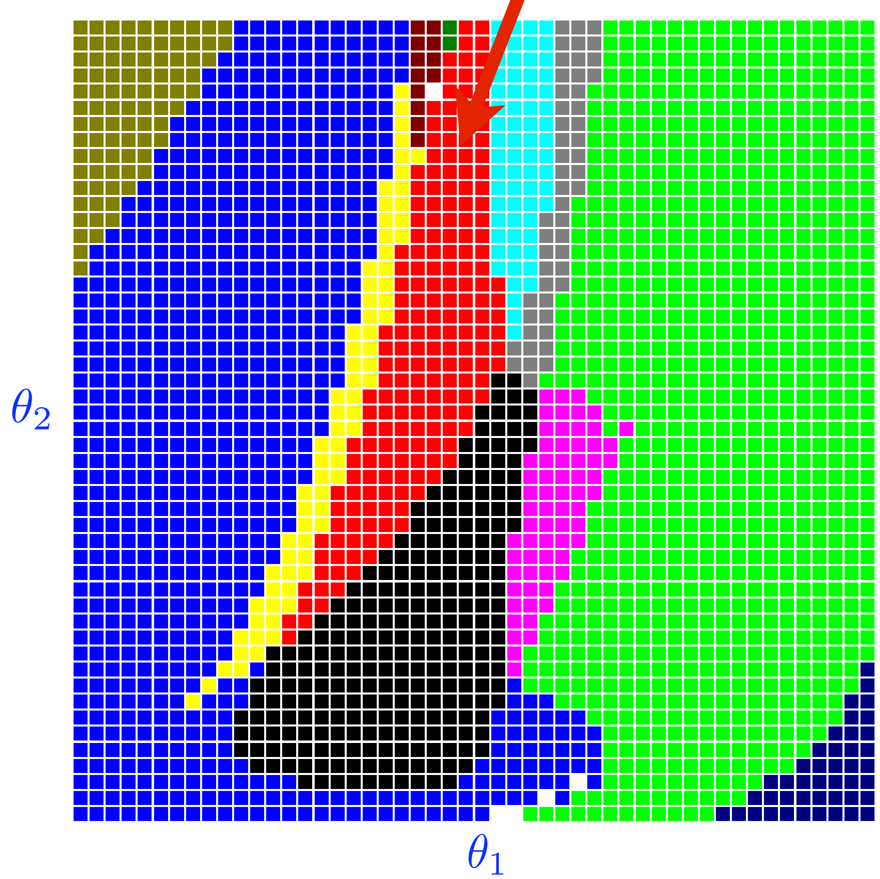
Querying the Database: *Are there multiple basins of attraction?*



Query the gradient-like structure:

Is there a Morse graph with multiple minimal elements?





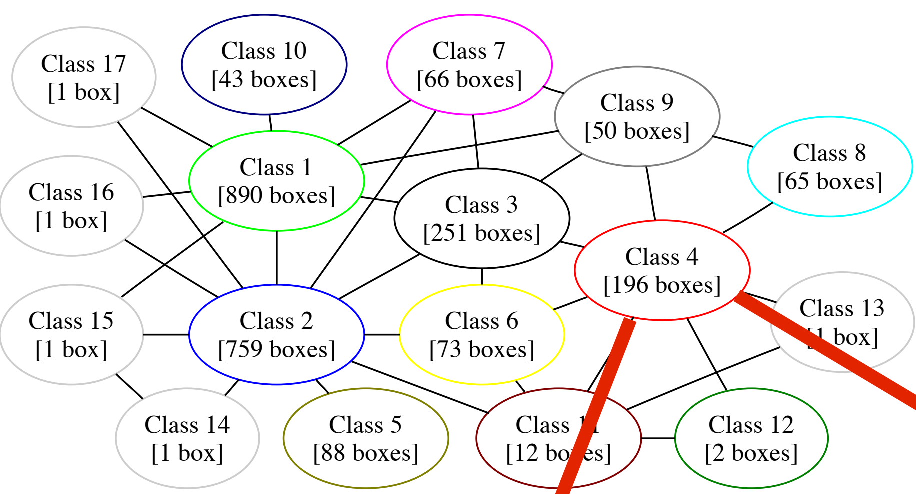
2 observable basins of attraction

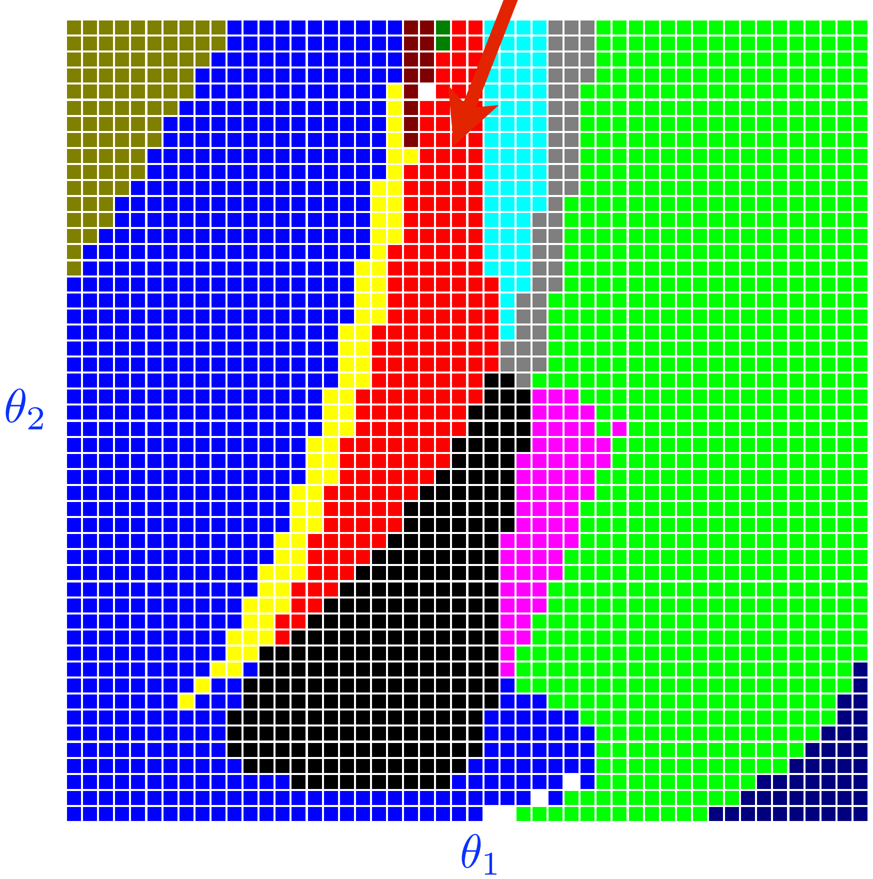
Can we characterize the attracting dynamics?

Query the Conley index:

"3 cycle" "1 cycle"

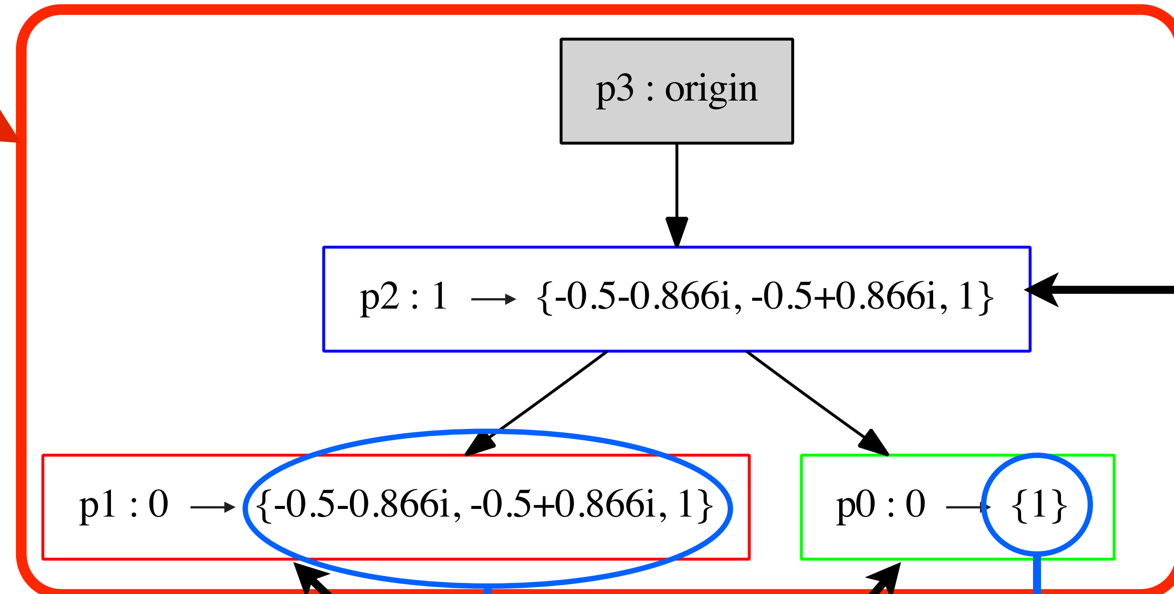
Querying the Database: *Are there multiple basins of attraction?*





Query the gradient-like structure:

Is there a Morse graph with multiple minimal elements?



"Critical transition Dynamics"

2 observable basins of attraction

Can we characterize the attracting dynamics?

Query the Conley index:

"3 cycle" "1 cycle"

III. Theoretical Framework

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

What is geometrically observable?

We assume existence (not knowledge) of a model $f\colon X \times \Lambda \to X$

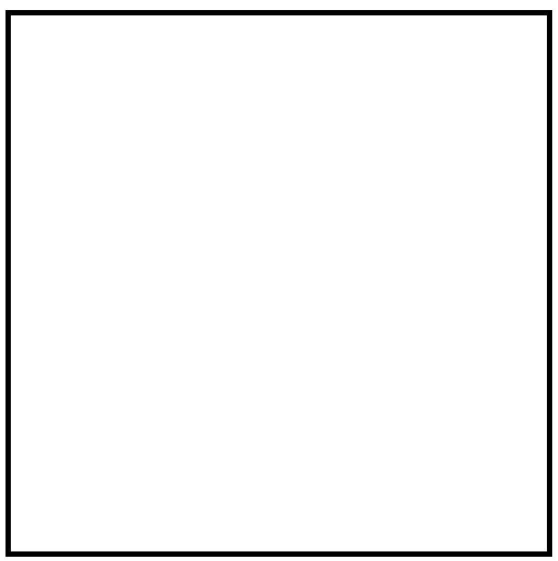
Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \operatorname{int}(N)$$

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that



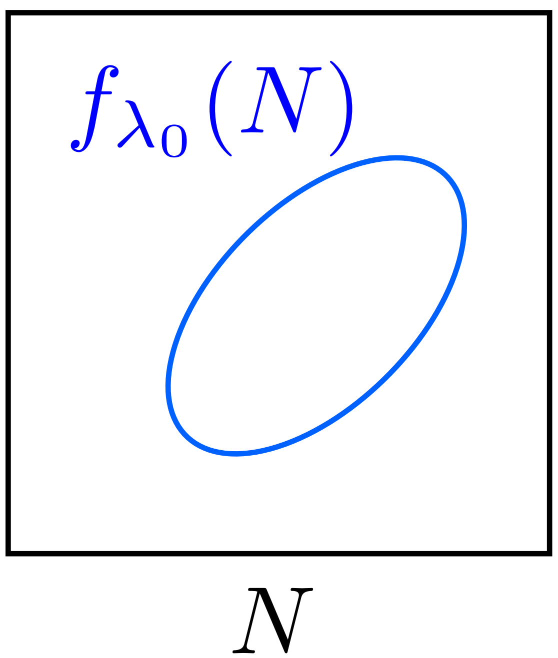
N

$$f_{\lambda_0}(N) \subset \operatorname{int}(N)$$

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that



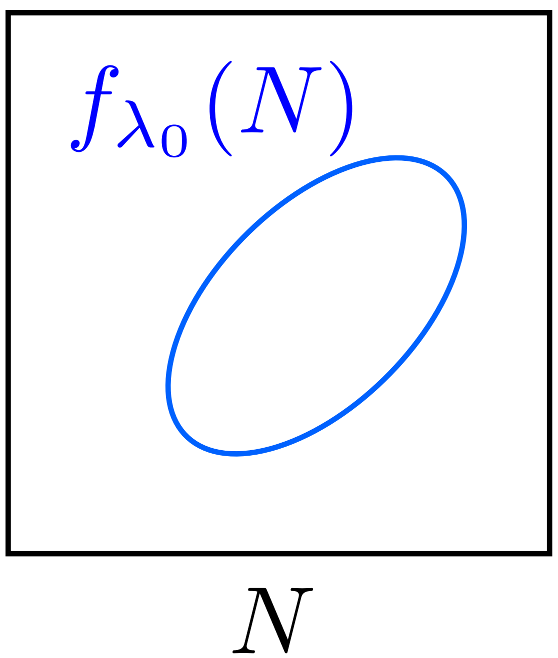
$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$



Robust with respect to:

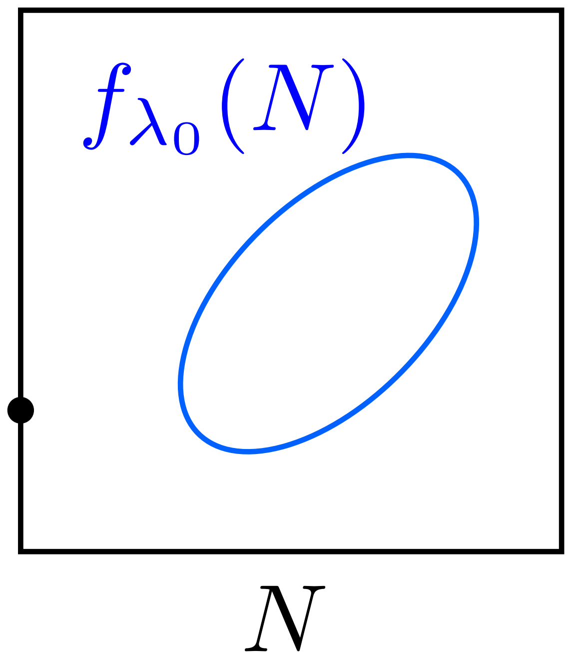
1. Measurement error

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \text{int}(N)$$



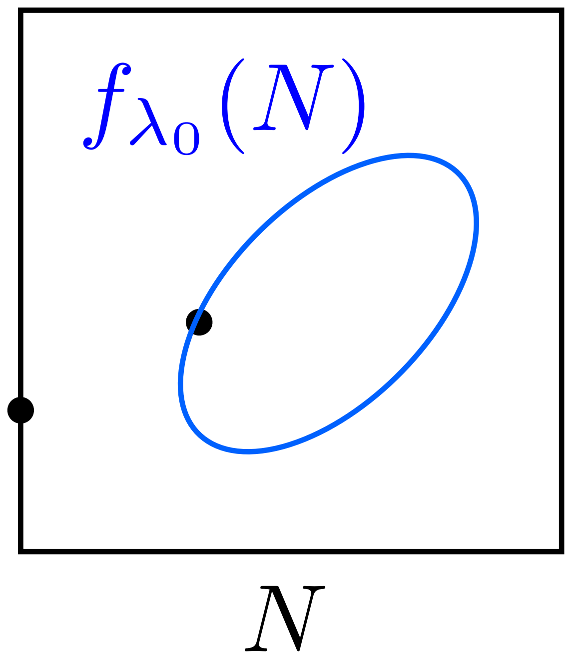
Robust with respect to:

1. Measurement error

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that



$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$

Robust with respect to:

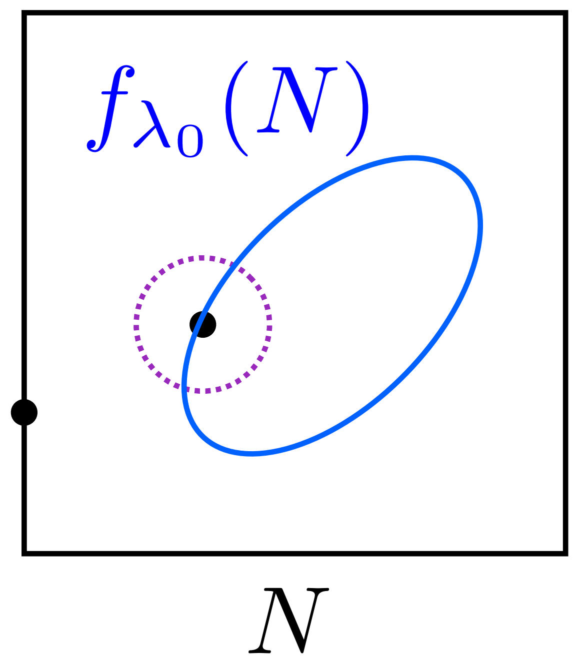
1. Measurement error

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$



Robust with respect to:

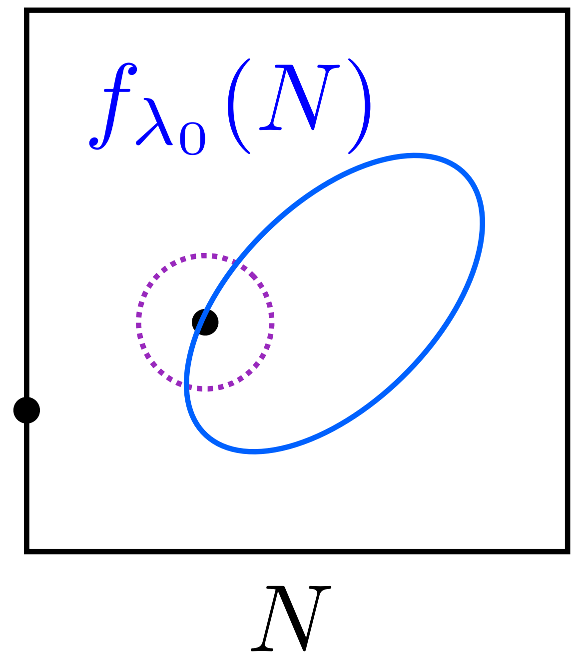
1. Measurement error

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$



Robust with respect to:

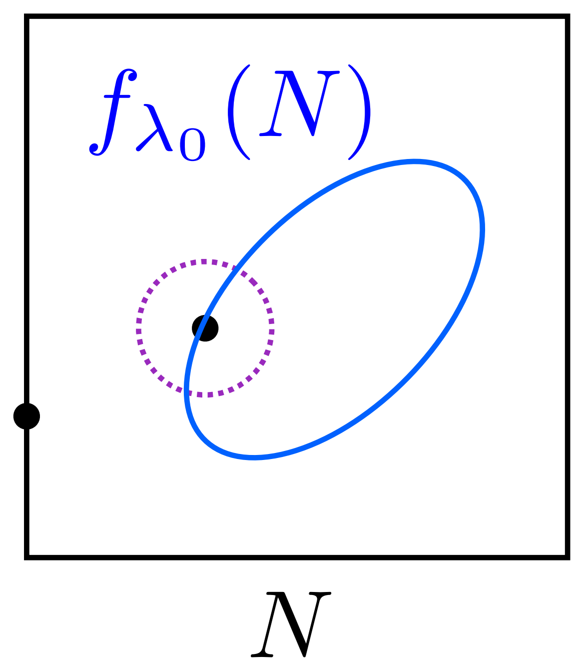
1. Measurement error
2. Model error

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \operatorname{int}(N)$$



Robust with respect to:

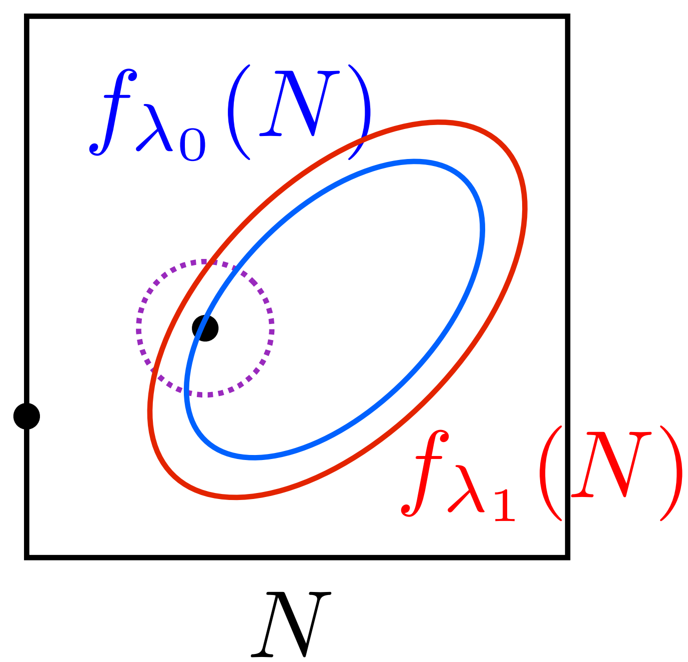
1. Measurement error
2. Model error $\lambda_1 \approx \lambda_0$

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$



Robust with respect to:

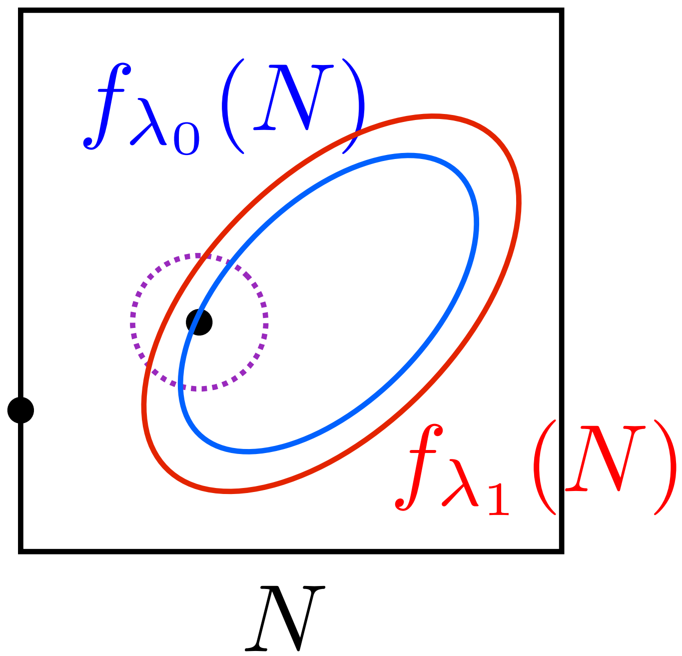
1. Measurement error
2. Model error $\lambda_1 \approx \lambda_0$

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that

$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$



Robust with respect to:

1. Measurement error

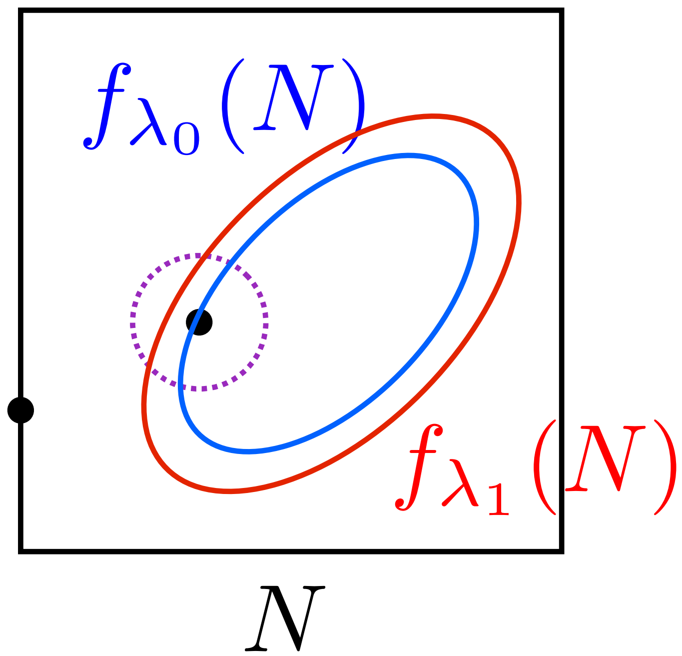
2. Model error $\lambda_1 \approx \lambda_0$

Remarks: 1. The set of attractor blocks defines a (large) lattice under $\cap$ and $\cup$.

What is geometrically observable?

We assume existence (not knowledge) of a model $f : X \times \Lambda \to X$

Attractor block: A compact subset $N \subset X$ such that



$$f_{\lambda_0}(N) \subset \mathrm{int}(N)$$

Robust with respect to:

1. Measurement error
2. Model error $\lambda_1 \approx \lambda_0$

Remarks: 1. The set of attractor blocks defines a (large) lattice under ∩ and ∪.

2. The separatrix dynamics is not explicit in the lattice of attractor blocks.

What about the dynamics?

What about the dynamics?

The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$
is a compact invariant set:

What about the dynamics?

The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \text{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$
is a compact invariant set:

Attractor

What about the dynamics?

The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$
is a compact invariant set:

⇕

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in N

What about the dynamics?

The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$
is a compact invariant set:

⇕

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

What about the dynamics?

The Omega limit set is a compact invariant set: $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

We can generalize this.

A compact set $N \subset X$ is an isolating neighborhood for f_{λ_0} if the maximal invariant set in N lies in the interior of N.

$$S = \mathrm{Inv}(N, f_{\lambda_0}) \subset \mathrm{int}(N)$$

What about the dynamics?

The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$

is a compact invariant set:

↕

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

We can generalize this.

A compact set $N \subset X$ is an isolating neighborhood for f_{λ_0} if the maximal invariant set in N lies in the interior of N.

N

$$S = \mathrm{Inv}(N, f_{\lambda_0}) \subset \mathrm{int}(N)$$

What about the dynamics?

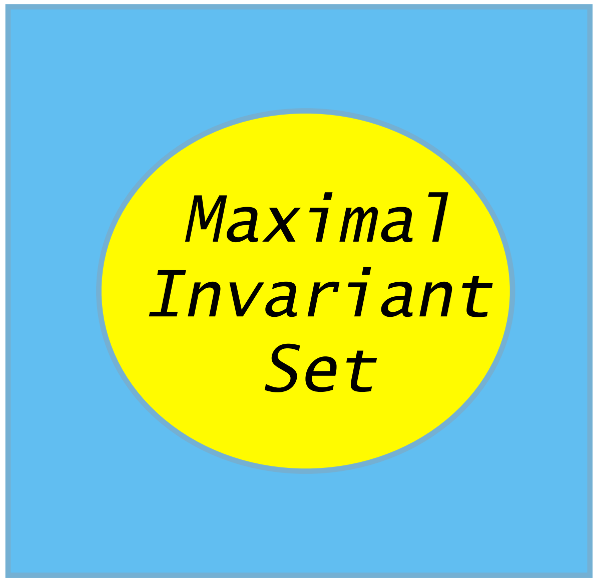
The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$
is a compact invariant set:

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

We can generalize this.

A compact set $N \subset X$ is an isolating neighborhood for f_{λ_0} if the maximal invariant set in N lies in the interior of N.



N

$$S = \mathrm{Inv}(N, f_{\lambda_0}) \subset \mathrm{int}(N)$$

What about the dynamics?

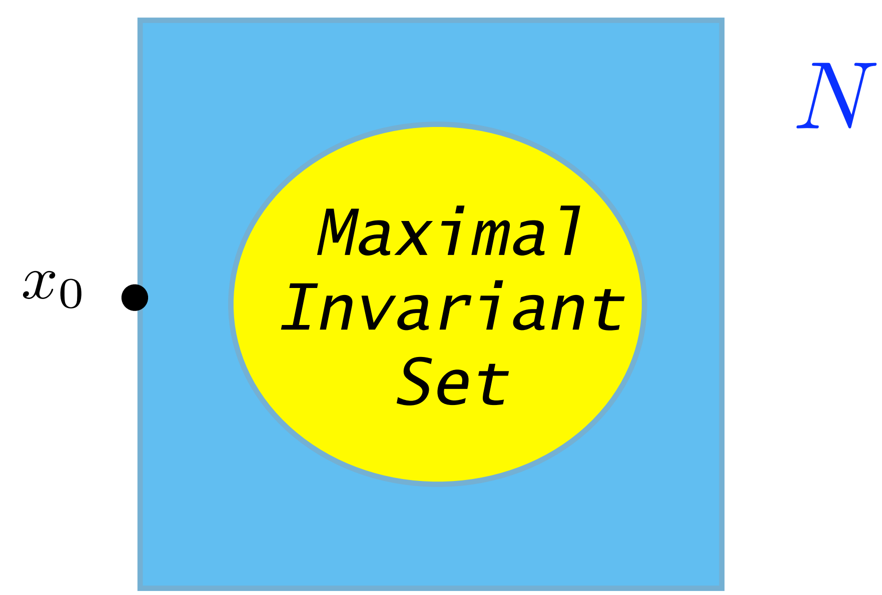
The Omega limit set is a compact invariant set: $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N) \right)$

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

We can generalize this.

A compact set $N \subset X$ is an isolating neighborhood for f_{λ_0} if the maximal invariant set in N lies in the interior of N.



$$S = \mathrm{Inv}(N, f_{\lambda_0}) \subset \mathrm{int}(N)$$

What about the dynamics?

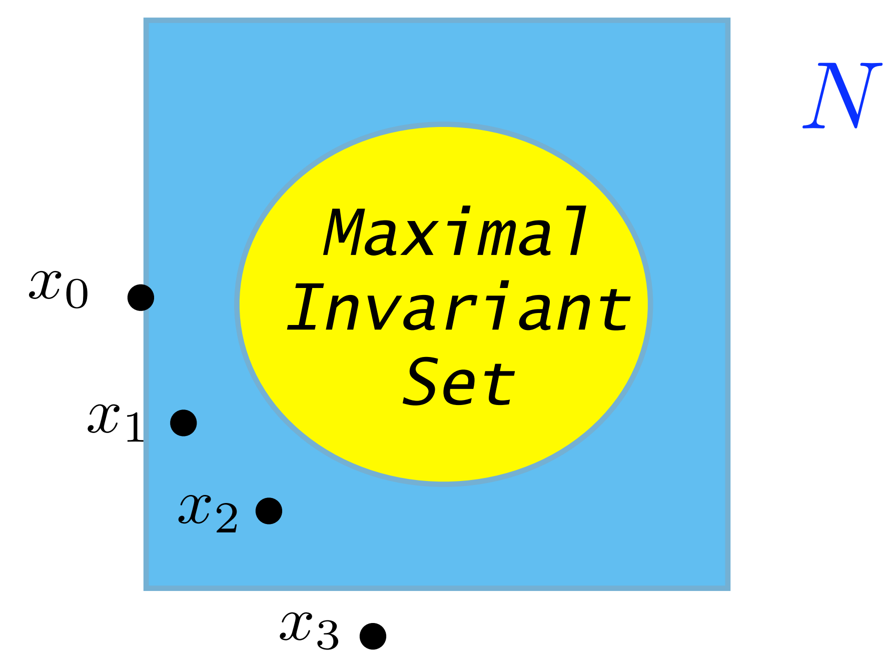
The Omega limit set is a compact invariant set: $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$

Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

We can generalize this.

A compact set $N \subset X$ is an isolating neighborhood for f_{λ_0} if the maximal invariant set in N lies in the interior of N.



$$S = \mathrm{Inv}(N, f_{\lambda_0}) \subset \mathrm{int}(N)$$

What about the dynamics?

The Omega limit set $\omega(N, f_{\lambda_0}) := \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{k=n}^{\infty} f_{\lambda_0}(N)\right)$
is a compact invariant set:

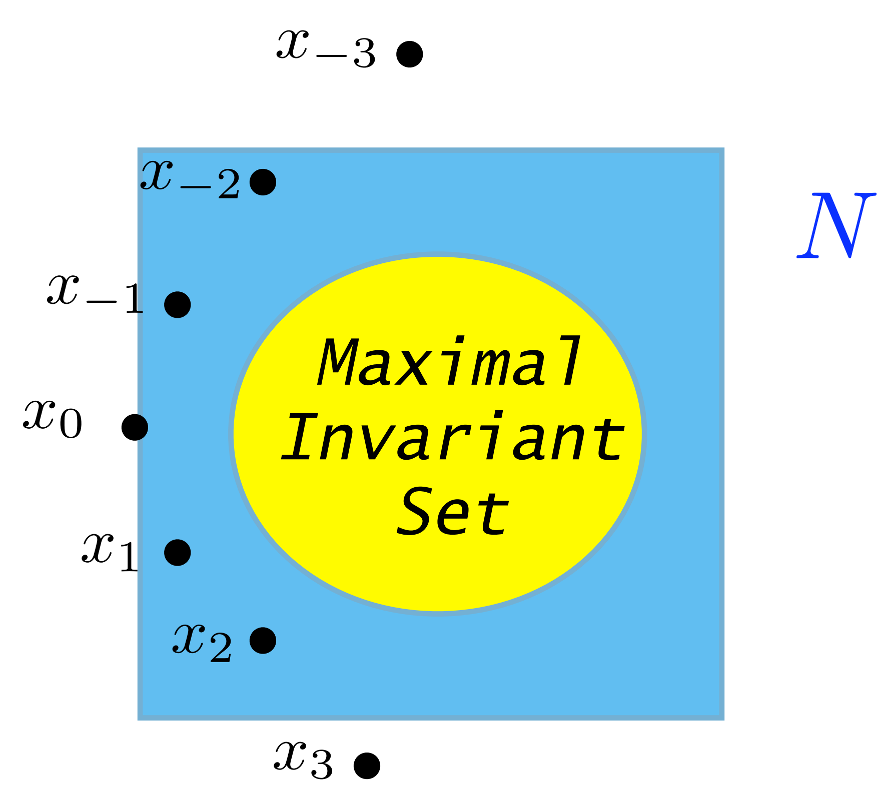
Attractor

The maximal invariant set $\mathrm{Inv}(N, F_{\Lambda_0})$ in ~~N~~ $\mathrm{int}(N)$

We can generalize this.

A compact set $N \subset X$ is an isolating neighborhood for f_{λ_0} if the maximal invariant set in N lies in the interior of N.

$$S = \mathrm{Inv}(N, f_{\lambda_0}) \subset \mathrm{int}(N)$$



A Morse covering of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of disjoint non-empty isolating neighborhoods $\mathsf{B} = \{B(p) \mid p \in (\mathsf{P}, \leq)\}$ with the property that given an orbit $\gamma := \{x_n \in X \mid n \in \mathbb{Z}, x_{n+1} = f(x_n)\}$ either

- there exists $p \in \mathsf{P}$ such that $\gamma \subset B(p)$, or
- there exists $q, p \in \mathsf{P}$ and $t_q, t_p \in \mathbb{Z}$ such that $q < p$ and $t_q > t_p$ for which

$$\begin{aligned} \{x_n \mid n \leq t_p\} &\subset B(p) \\ \{x_n \mid n \geq t_q\} &\subset B(q) \\ \{x_n \mid t_p < n < t_q\} &\cap (B(p) \cup B(q)) = \emptyset \end{aligned}$$

A Morse covering of X consists of a finite poset $(\mathsf{P}, \leq)$ that labels a collection of disjoint non-empty isolating neighborhoods $\mathsf{B} = \{B(p) \mid p \in (\mathsf{P}, \leq)\}$ with the property that given an orbit $\gamma := \{x_n \in X \mid n \in \mathbb{Z}, x_{n+1} = f(x_n)\}$ either

- there exists $p \in \mathsf{P}$ such that $\gamma \subset B(p)$, or
- there exists $q, p \in \mathsf{P}$ and $t_q, t_p \in \mathbb{Z}$ such that $q < p$ and $t_q > t_p$ for which

$$\begin{aligned} \{x_n \mid n \leq t_p\} &\subset B(p) \\ \{x_n \mid n \geq t_q\} &\subset B(q) \\ \{x_n \mid t_p < n < t_q\} &\cap (B(p) \cup B(q)) = \emptyset \end{aligned}$$

Prop: $\mathsf{M} := \{(p, M(p)) \mid p \in (\mathsf{P}, \leq), M(p) = \mathrm{Inv}(B(p))\}$ is a Morse decomposition

A Discrete Representation of the Dynamics

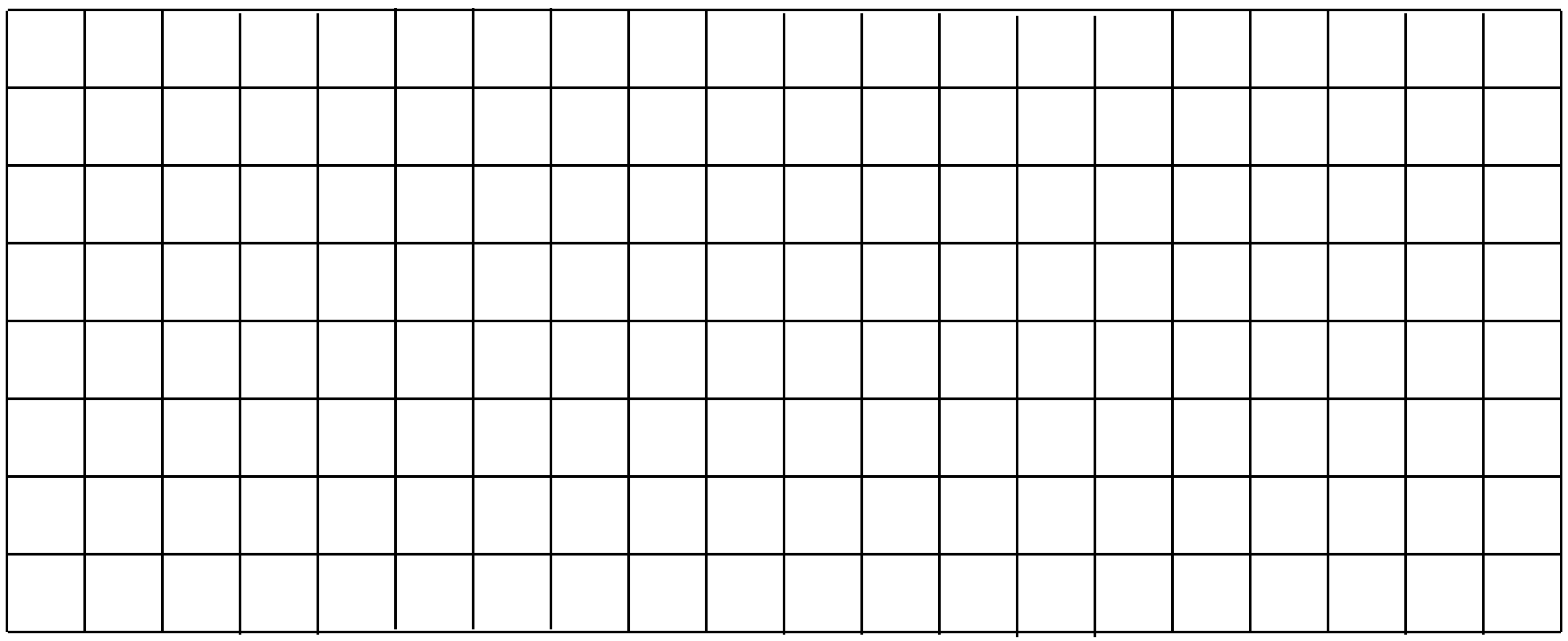
A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

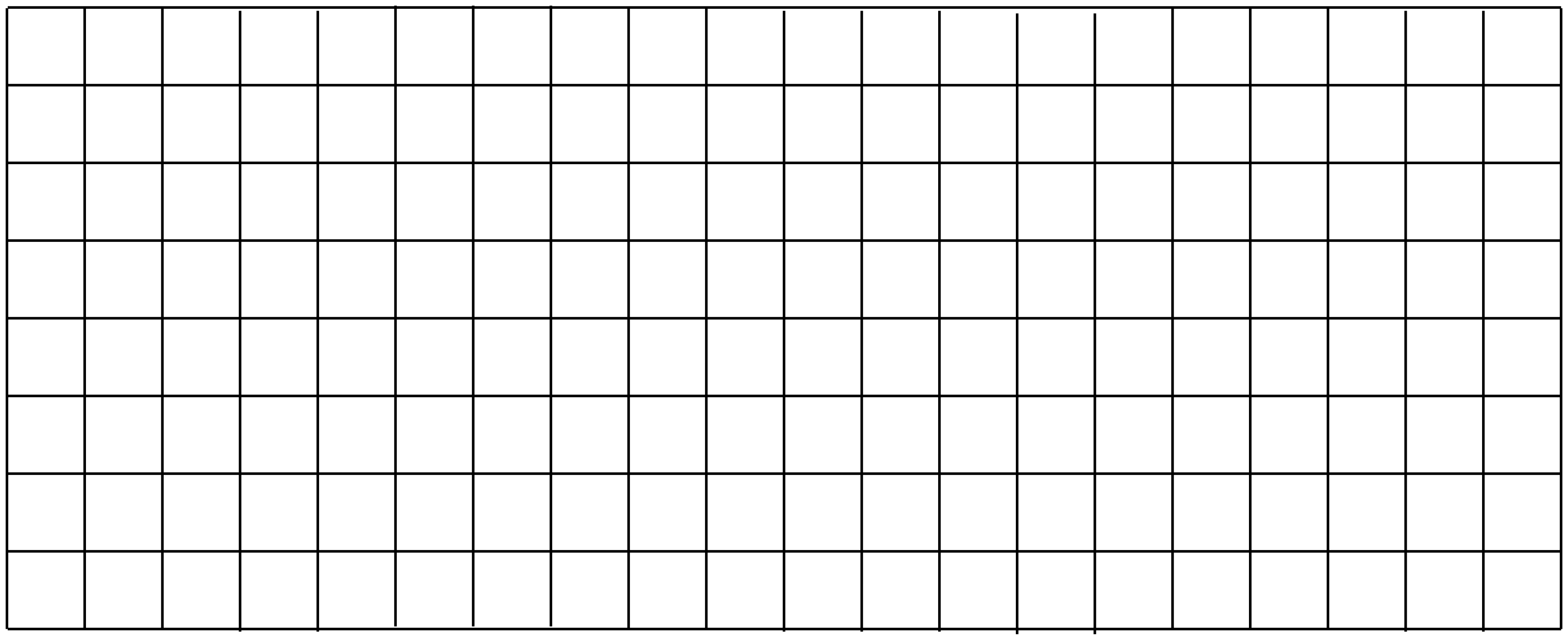
Choose a (cubical) grid $\mathcal{X}$ that covers X



A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X

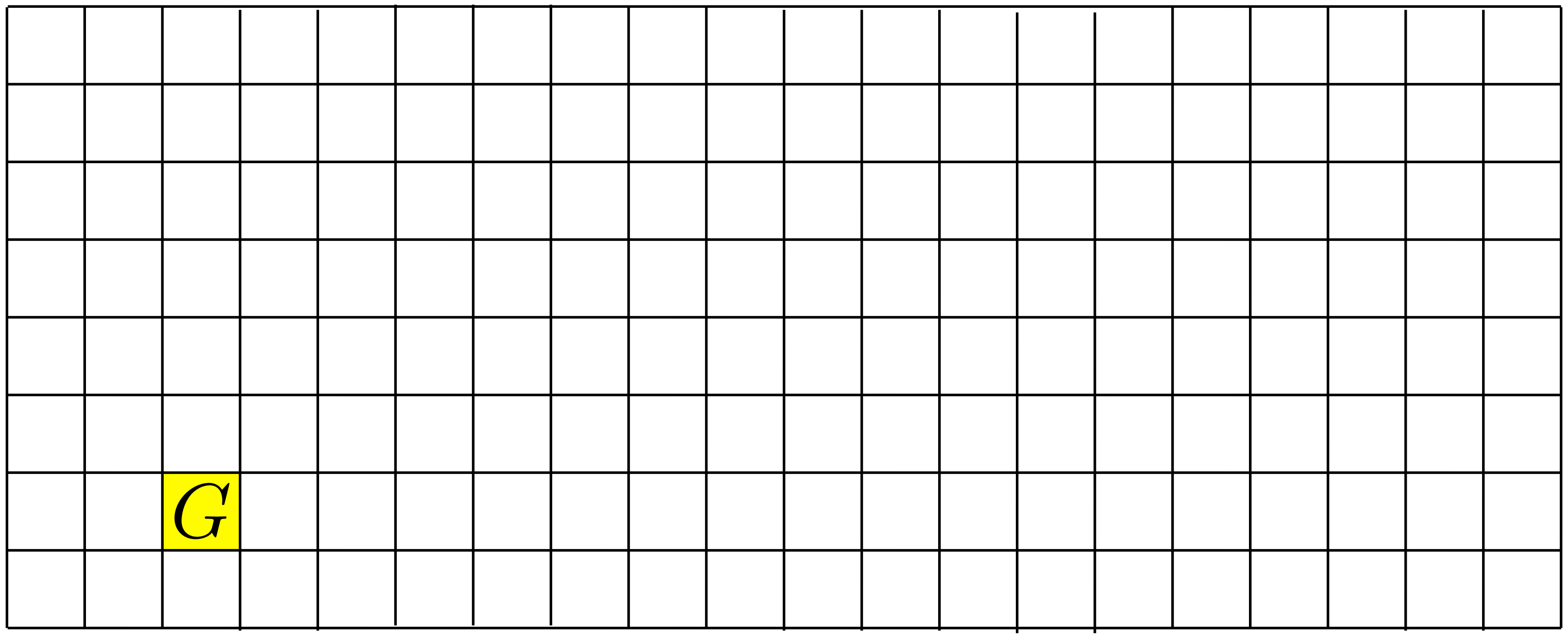


Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X

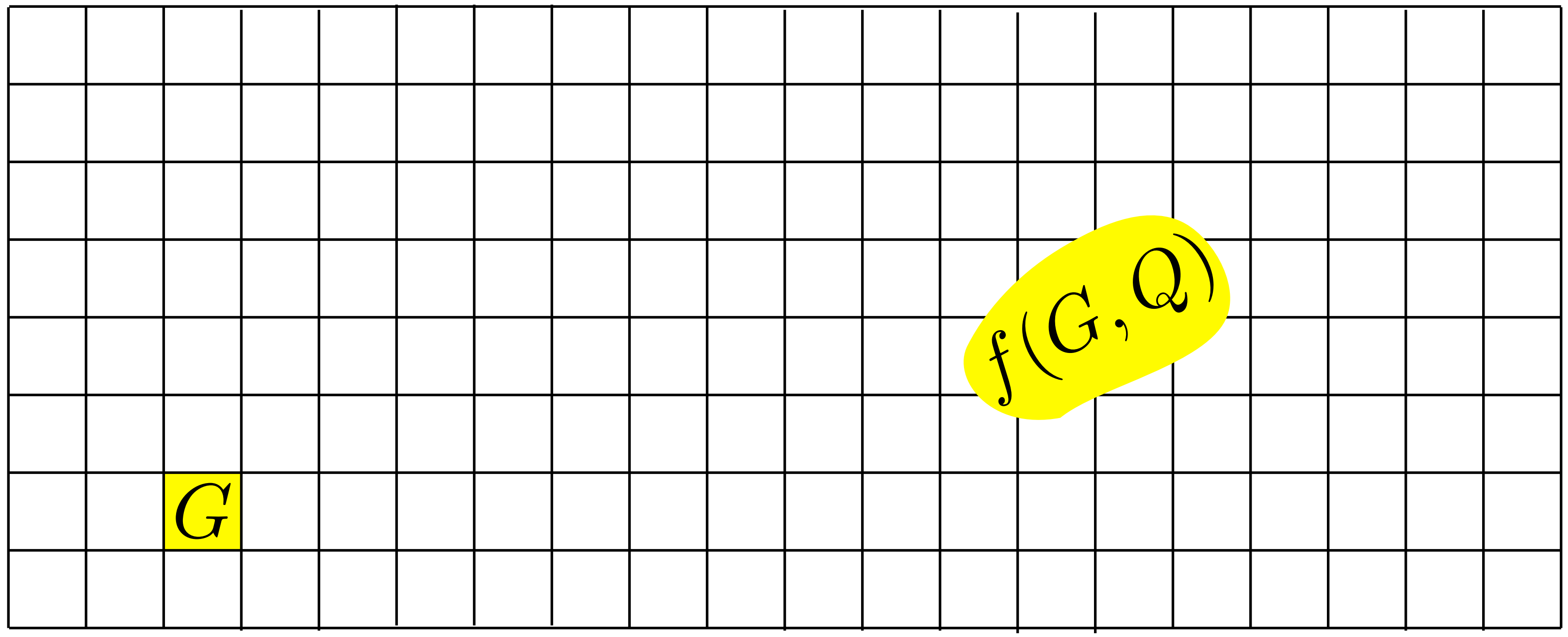


Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X

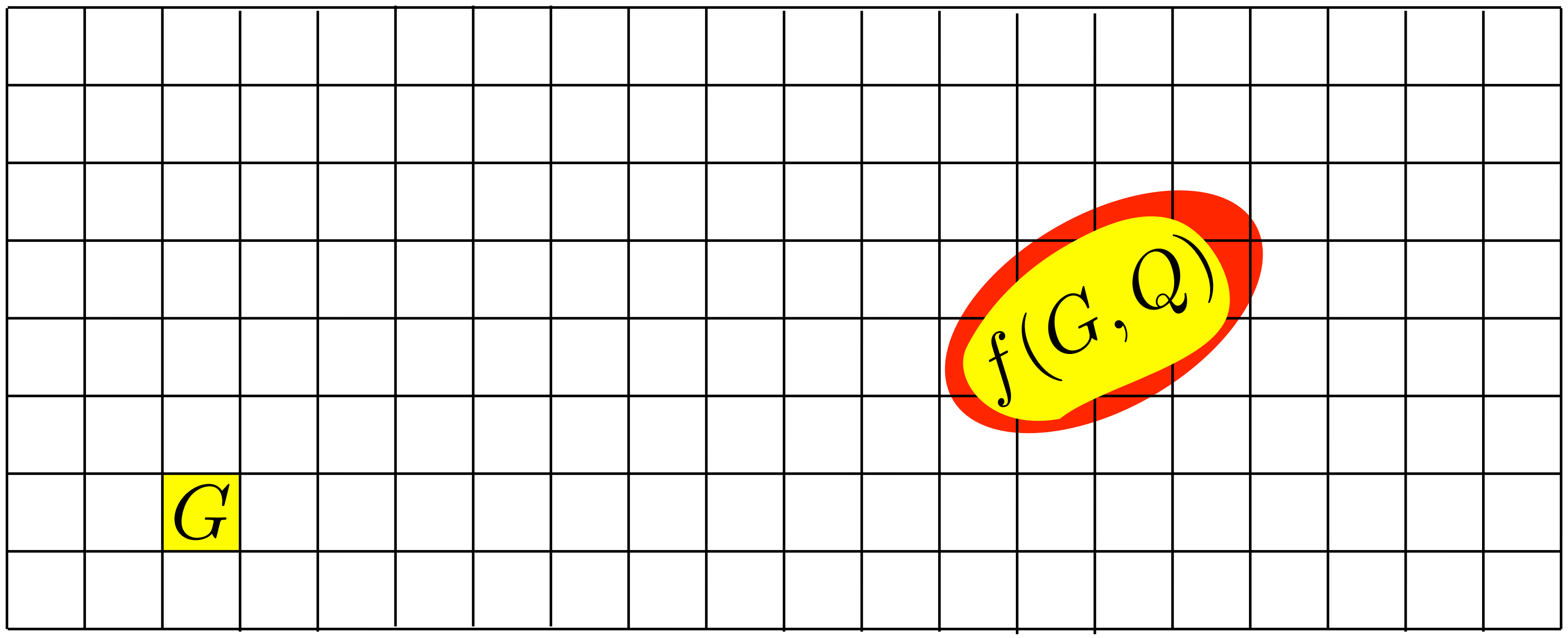


Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X



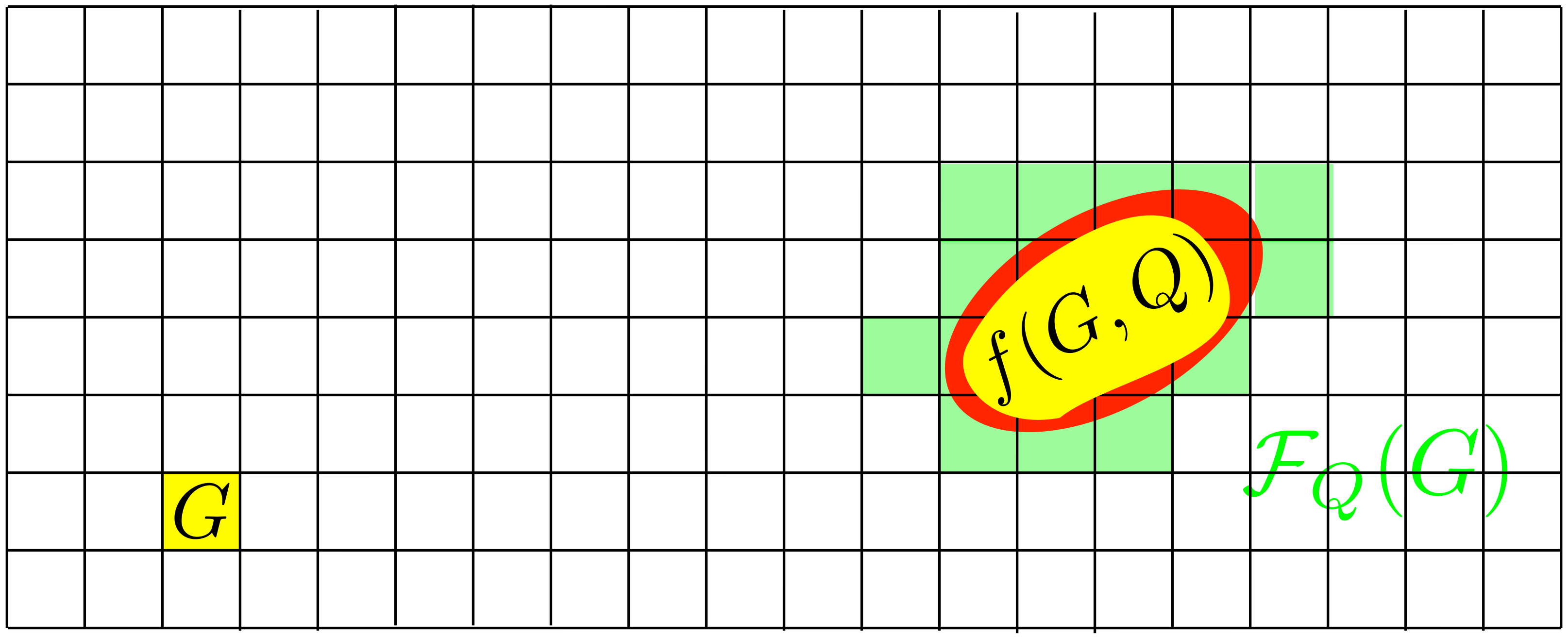
Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

Numerical/Experimental Error

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X



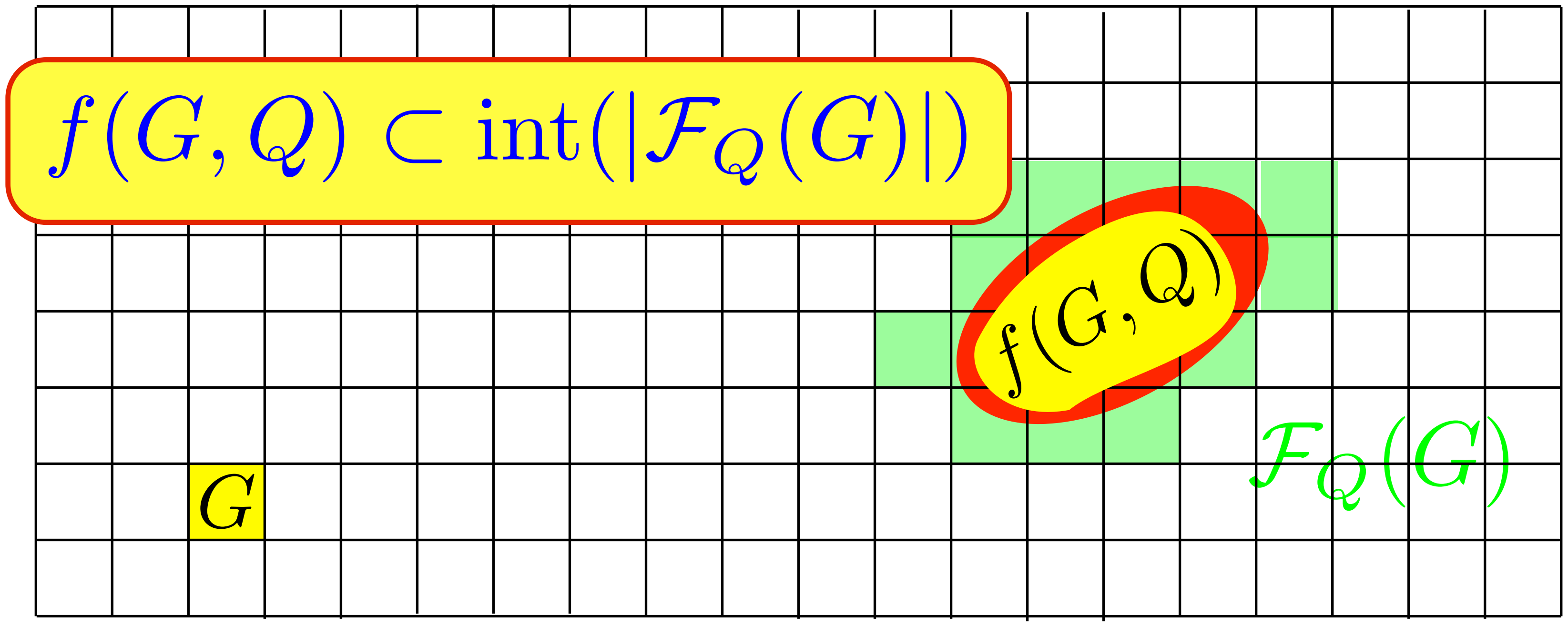
Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

Numerical/Experimental Error

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X



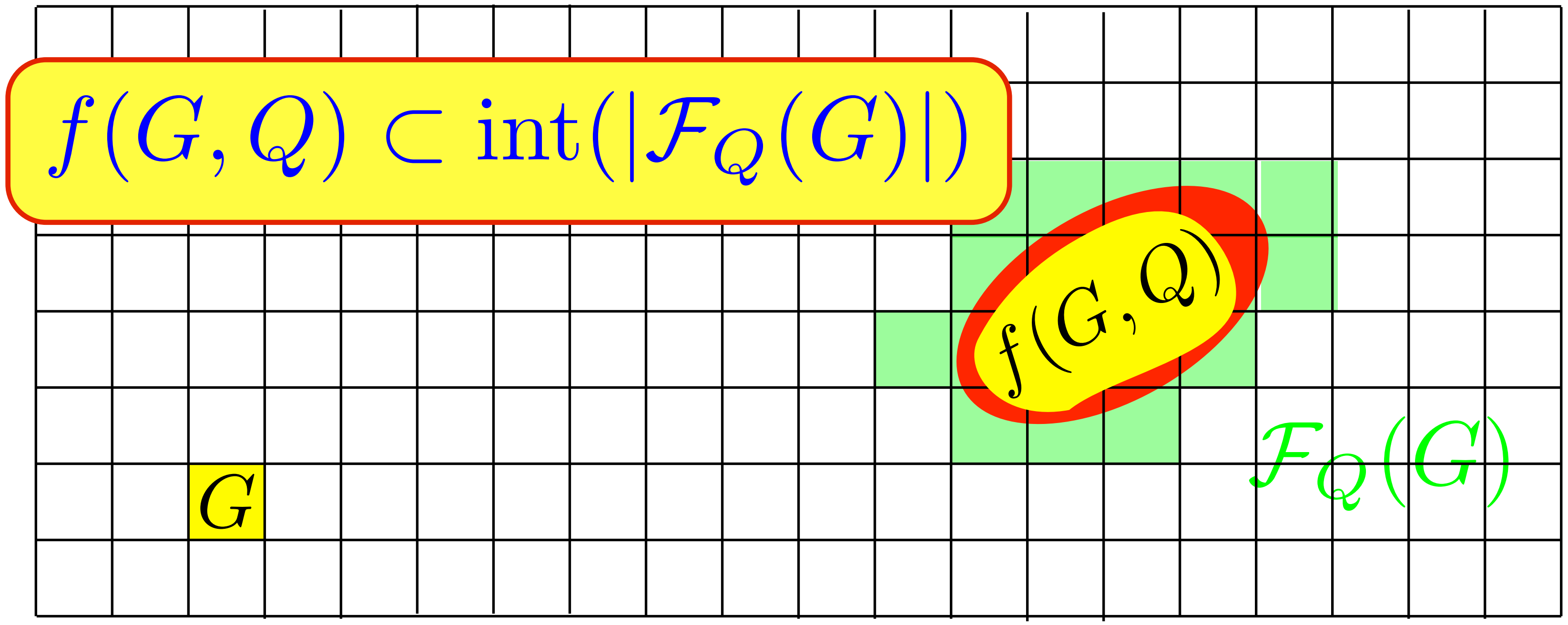
Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

Numerical/Experimental Error

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X



Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

Numerical/Experimental Error

$\mathcal{F}_Q$ is a directed graph:

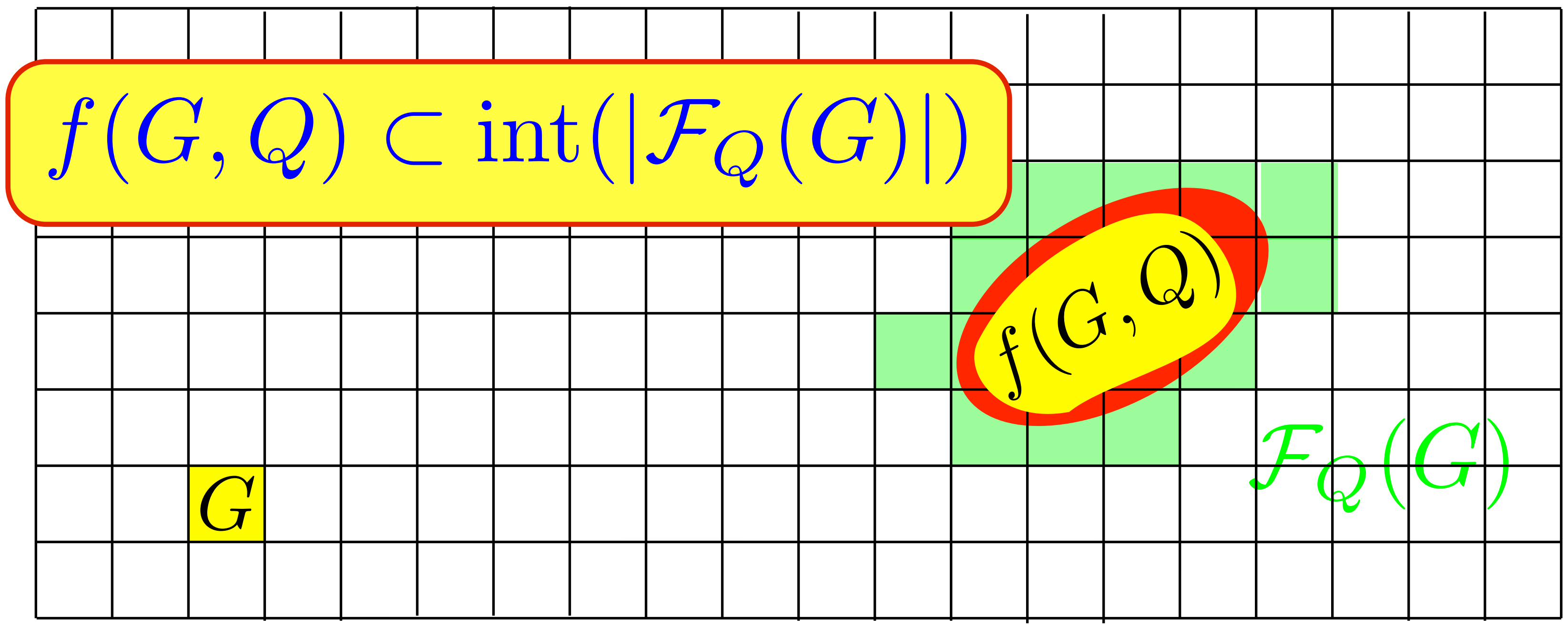
Vertices $G \in \mathcal{X}$

Edges $H \in \mathcal{F}_Q(G) \ \Rightarrow \ G \to H$

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X



Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

Numerical/Experimental Error

$\mathcal{F}_Q$ is a directed graph:

Vertices $G \in \mathcal{X}$

Edges $H \in \mathcal{F}_Q(G) \;\Rightarrow\; G \to H$

Recurrence in a Directed Graph

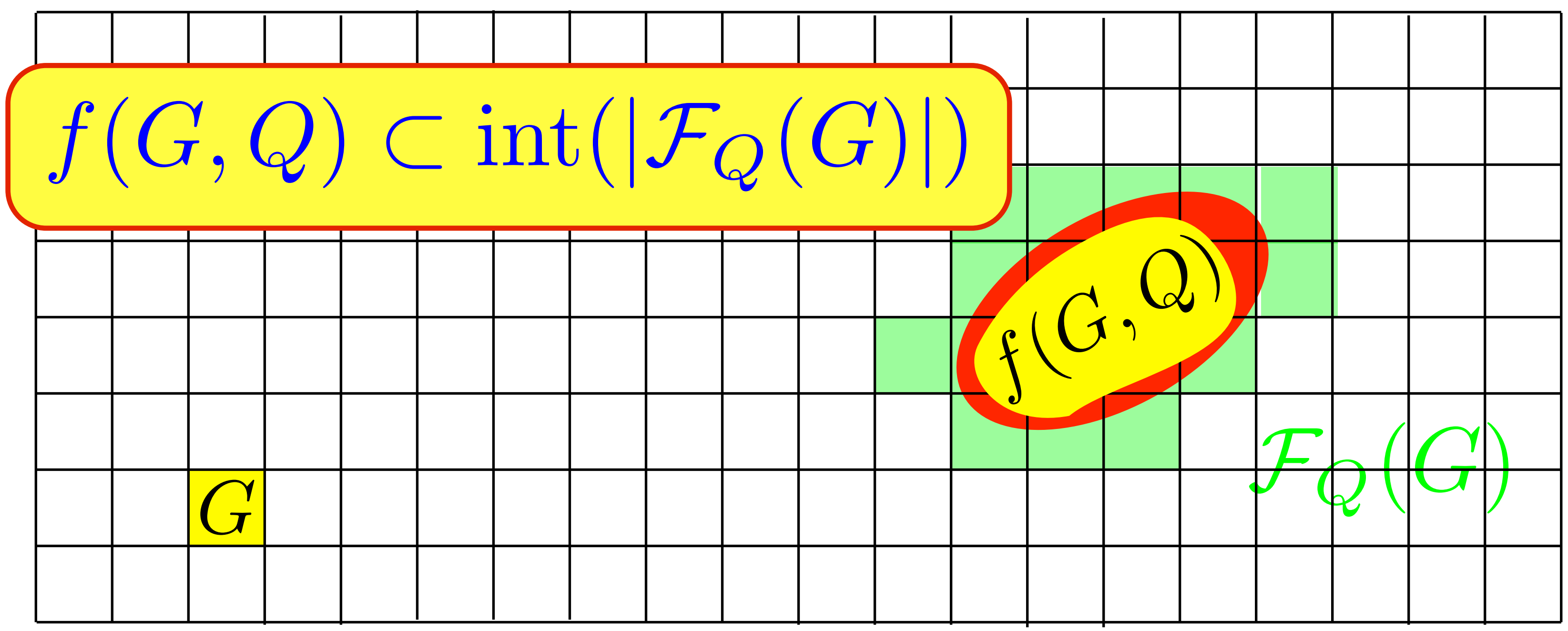
$\Updownarrow$

Strongly Connected Path Components

A Discrete Representation of the Dynamics

Choose a compact region in parameter space: $Q \subset \Lambda$

Choose a (cubical) grid $\mathcal{X}$ that covers X



Define a multivalued map: $\mathcal{F}_Q : \mathcal{X} \rightrightarrows \mathcal{X}$

Numerical/Experimental Error

$\mathcal{F}_Q$ is a directed graph:

Vertices $G \in \mathcal{X}$

Edges $H \in \mathcal{F}_Q(G) \;\Rightarrow\; G \to H$

Recurrence in a Directed Graph

$\Updownarrow$

Strongly Connected Path Components

1. Can be computed in linear time
2. Define a Morse Cover

Birkhoff’s Representation Theorem

$$P$$

$$\Big\downarrow \; O$$

$$O(P)$$

$$\Big\downarrow \; J^{\vee}$$

$$J^{\vee}\left(O(P)\right)$$

$$\Big\downarrow \; \cong$$

$$P$$

Birkhoff's Representation Theorem

Finite Poset

$$P \xrightarrow{\;O\;} O(P) \xrightarrow{\;J^{\vee}\;} J^{\vee}(O(P)) \xrightarrow{\;\cong\;} P$$

Birkhoff’s Representation Theorem

Category

Finite Poset | Posets

$$P \xrightarrow{O} O(P) \xrightarrow{J^{\vee}} J^{\vee}(O(P)) \xrightarrow{\cong} P$$

Birkhoff's Representation Theorem

Category

Finite Poset

Posets

construct the collection of lower sets

$$P \xrightarrow{O} O(P) \xrightarrow{J^{\vee}} J^{\vee}(O(P)) \xrightarrow{\cong} P$$

Birkhoff's Representation Theorem

Category

Finite Poset — P — Posets

construct the collection of lower sets — O ↓

Finite Distributive Lattice ($\cup$,$\cap$) — $O(P)$

$J^{\vee}$ ↓

$J^{\vee}(O(P))$

$\cong$ ↓

P

Birkhoff's Representation Theorem

Category

Finite Poset — P — Posets

construct the collection of lower sets — O ↓ — contravariant functor ↓

Finite Distributive Lattice $(\cup,\cap)$ — $O(P)$ — Lattices

$J^{\vee}$ ↓

$J^{\vee}(O(P))$

$\cong$ ↓

P

Birkhoff's Representation Theorem

Category

Finite Poset — P — Posets

construct the collection of lower sets — O ↓ — contravariant functor

Finite Distributive Lattice $(\cup,\cap)$ — $O(P)$ — Lattices

choose the join irreducible elements — $J^{\vee}$ ↓

$J^{\vee}(O(P))$

$\cong$ ↓

P

Birkhoff's Representation Theorem

Category

Finite Poset — P — Posets

construct the collection of lower sets — O ↓ — contravariant functor

Finite Distributive Lattice $(\cup,\cap)$ — $O(P)$ — Lattices

choose the join irreducible elements — $J^{\vee}$ ↓

Finite Poset — $J^{\vee}(O(P))$

$\cong$ ↓

P

Birkhoff's Representation Theorem

Category

Finite Poset		P	Posets	
	construct the collection of lower sets	O ↓	↓	contravariant functor
Finite Distributive Lattice (∪,∩)		O(P)	Lattices	
	choose the join irreducible elements	$J^{\vee}$ ↓	↓	contravariant functor
Finite Poset		$J^{\vee}$ (O(P))	Posets	
		≅ ↓		
		P		

Birkhoff's Representation Theorem

			Category	
Finite Poset		P	Posets	
	construct the collection of lower sets	O ↓	↓	contravariant functor
Finite Distributive Lattice $(\cup,\cap)$		$O(P)$	Lattices	
	choose the join irreducible elements	$J^{\vee}$ ↓	↓	contravariant functor
Finite Poset		$J^{\vee}(O(P))$	Posets	
	Birkhoff proved the existence of a poset isomorphism	$\cong$ ↓	↓	poset isomorphism
		P	Posets	

Birkhoff's Representation Theorem

				Category	
Finite Poset		P	M	Posets	
	construct the collection of lower sets	$\downarrow O$	$\downarrow O$	$\downarrow$	contravariant functor
Finite Distributive Lattice $(\cup,\cap)$		$O(P)$	$O(M)$	Lattices	
	choose the join irreducible elements	$\downarrow J^{\vee}$	$\downarrow J^{\vee}$	$\downarrow$	contravariant functor
Finite Poset		$J^{\vee}(O(P))$	$J^{\vee}(O(M))$	Posets	
	Birkhoff proved the existence of a poset isomorphism	$\downarrow \cong$	$\downarrow \cong$	$\downarrow$	poset isomorphism
		P	M	Posets	

Morse Decomposition

Combinatorial Theory

Morse Decomposition

$$M \xrightarrow{\;\mathcal{O}\;} \mathcal{O}(M) \xrightarrow{\;J^{\vee}\;} J^{\vee}(\mathcal{O}(M)) \xrightarrow{\;\text{Birkhoff}\;} M$$

In the Computer

Combinatorial Theory

Structures of Nonlinear Dynamics

Morse Decomposition

M

$\downarrow$ O

$\mathsf{O}(\mathsf{M})$

$\downarrow$ $\mathsf{J}^{\vee}$

$\mathsf{J}^{\vee}(\mathsf{O}(\mathsf{M}))$

$\downarrow$ Birkhoff

M

In the Computer

Combinatorial Theory

Structures of Nonlinear Dynamics

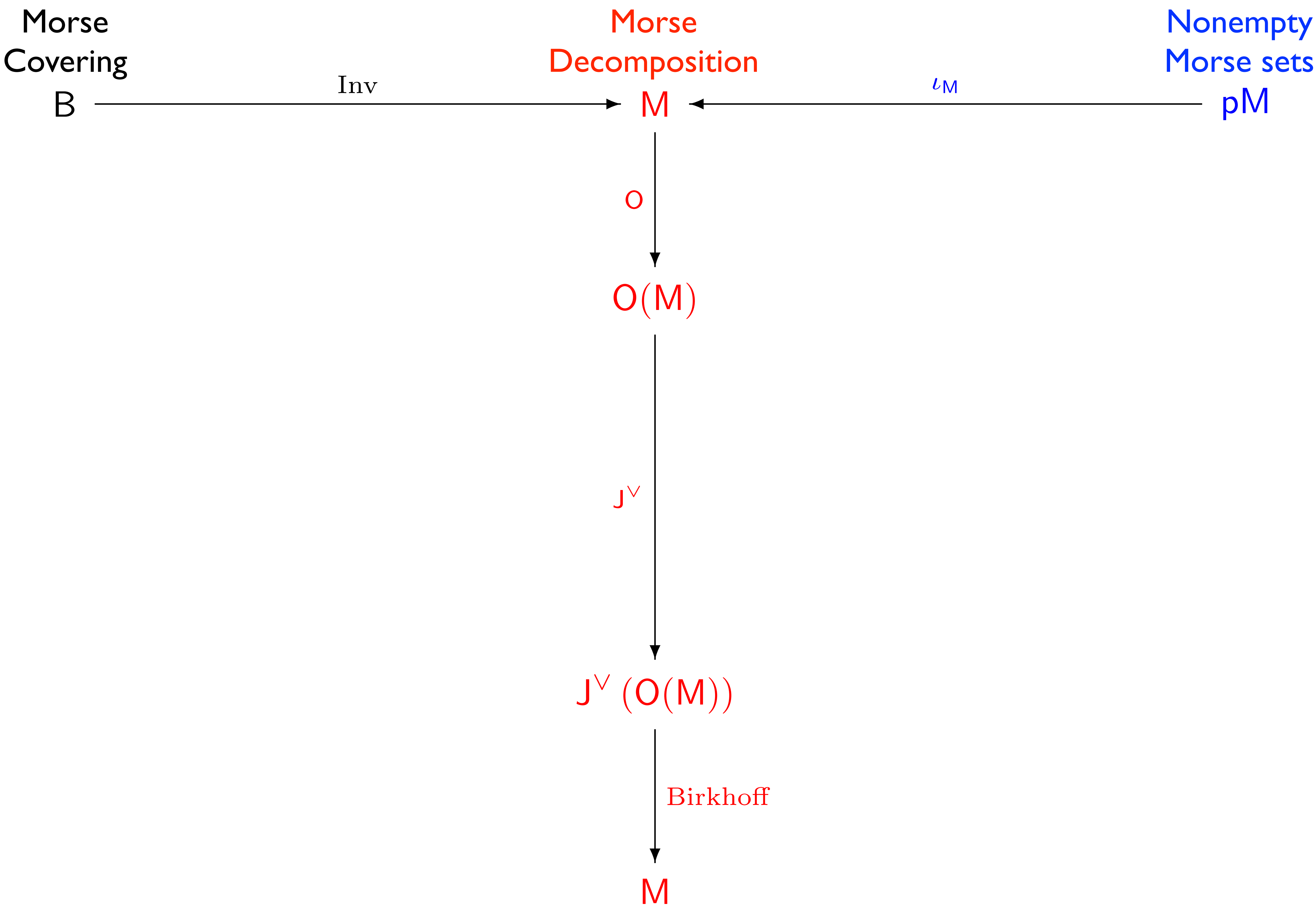
Morse Covering

Morse Decomposition

$$\mathsf{B} \xrightarrow{\mathrm{Inv}} \mathsf{M}$$

$$\mathsf{M} \xrightarrow{\mathsf{O}} \mathsf{O}(\mathsf{M}) \xrightarrow{\mathsf{J}^{\vee}} \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{M})) \xrightarrow{\mathrm{Birkhoff}} \mathsf{M}$$

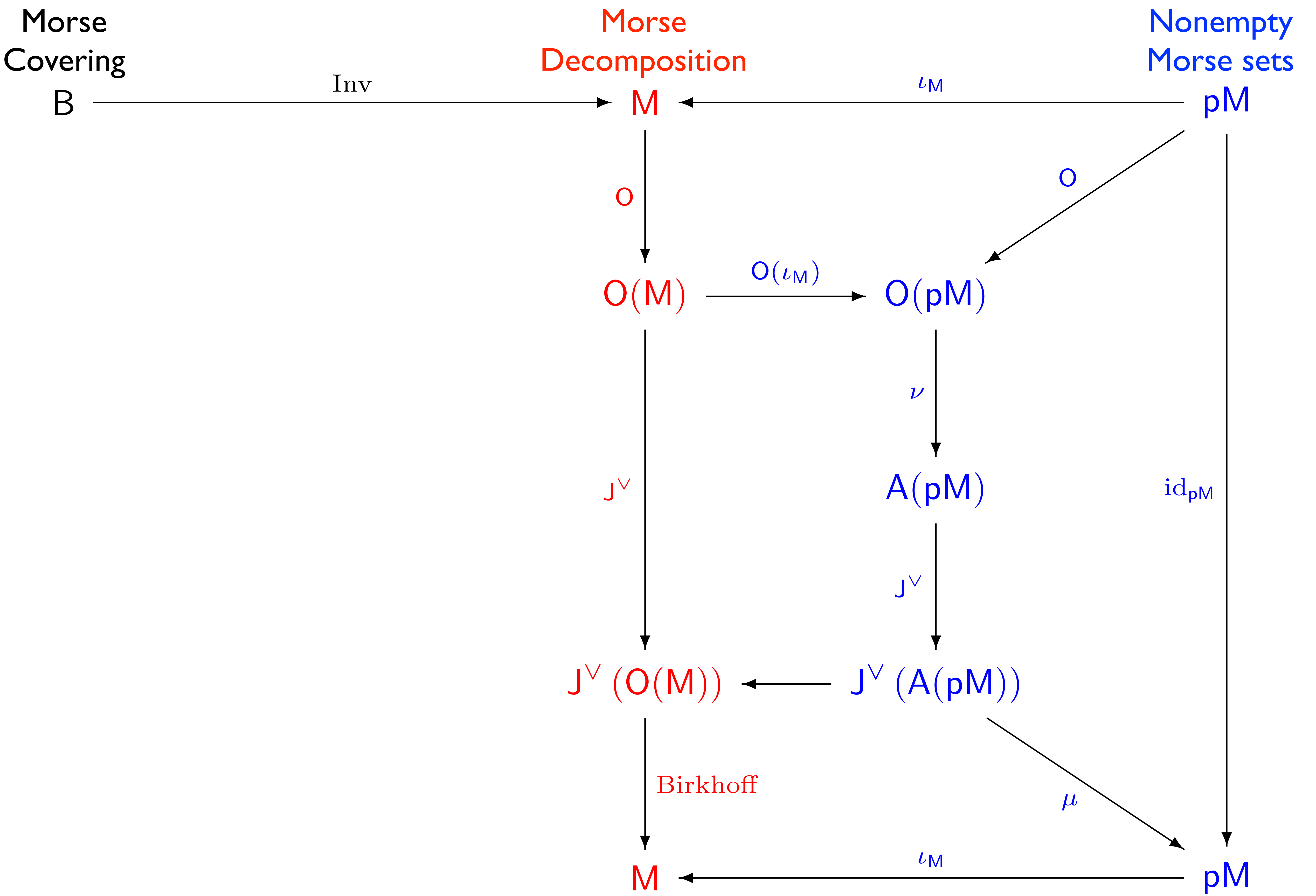
In the Computer | Combinatorial Theory | Structures of Nonlinear Dynamics



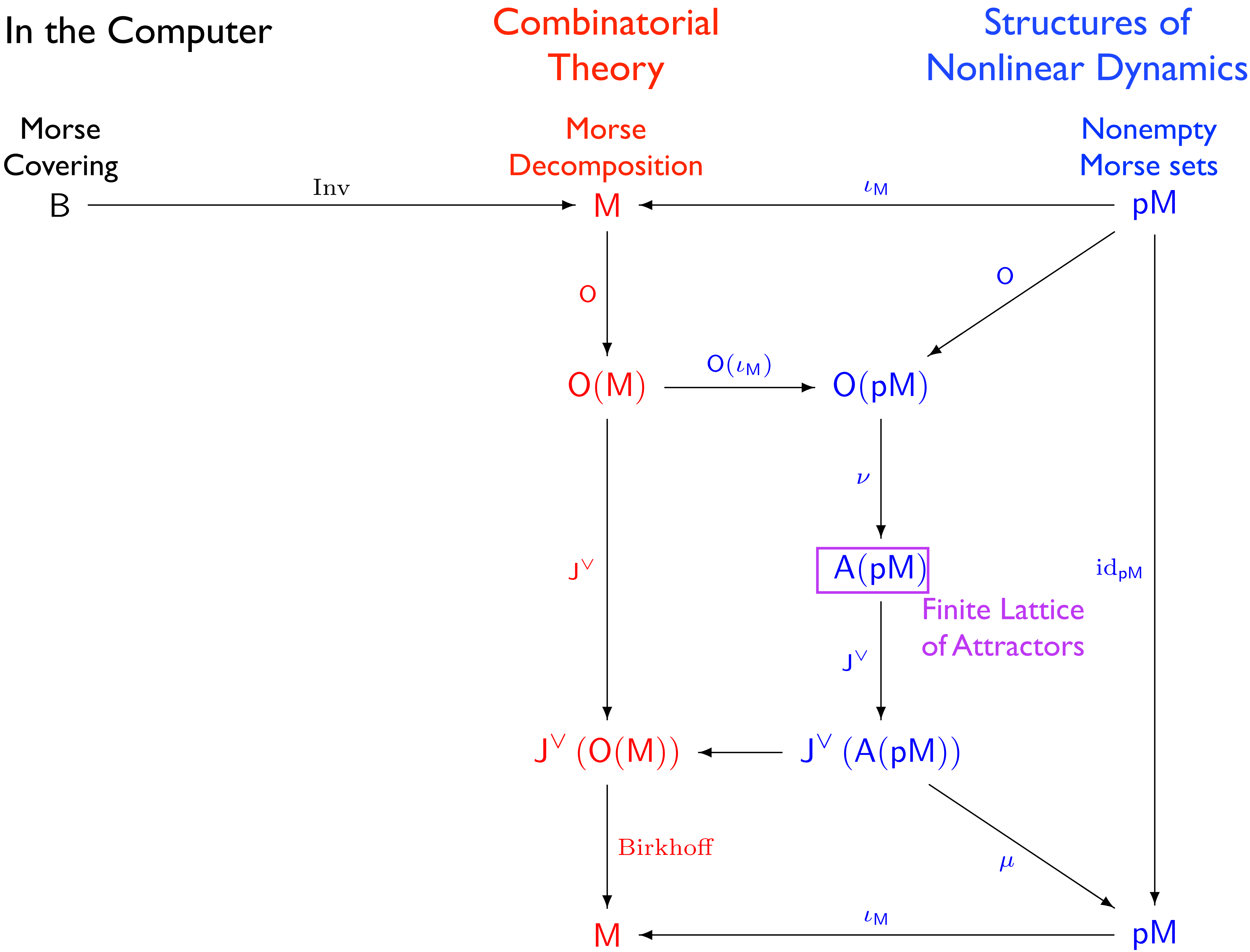
In the Computer

Combinatorial Theory

Structures of Nonlinear Dynamics



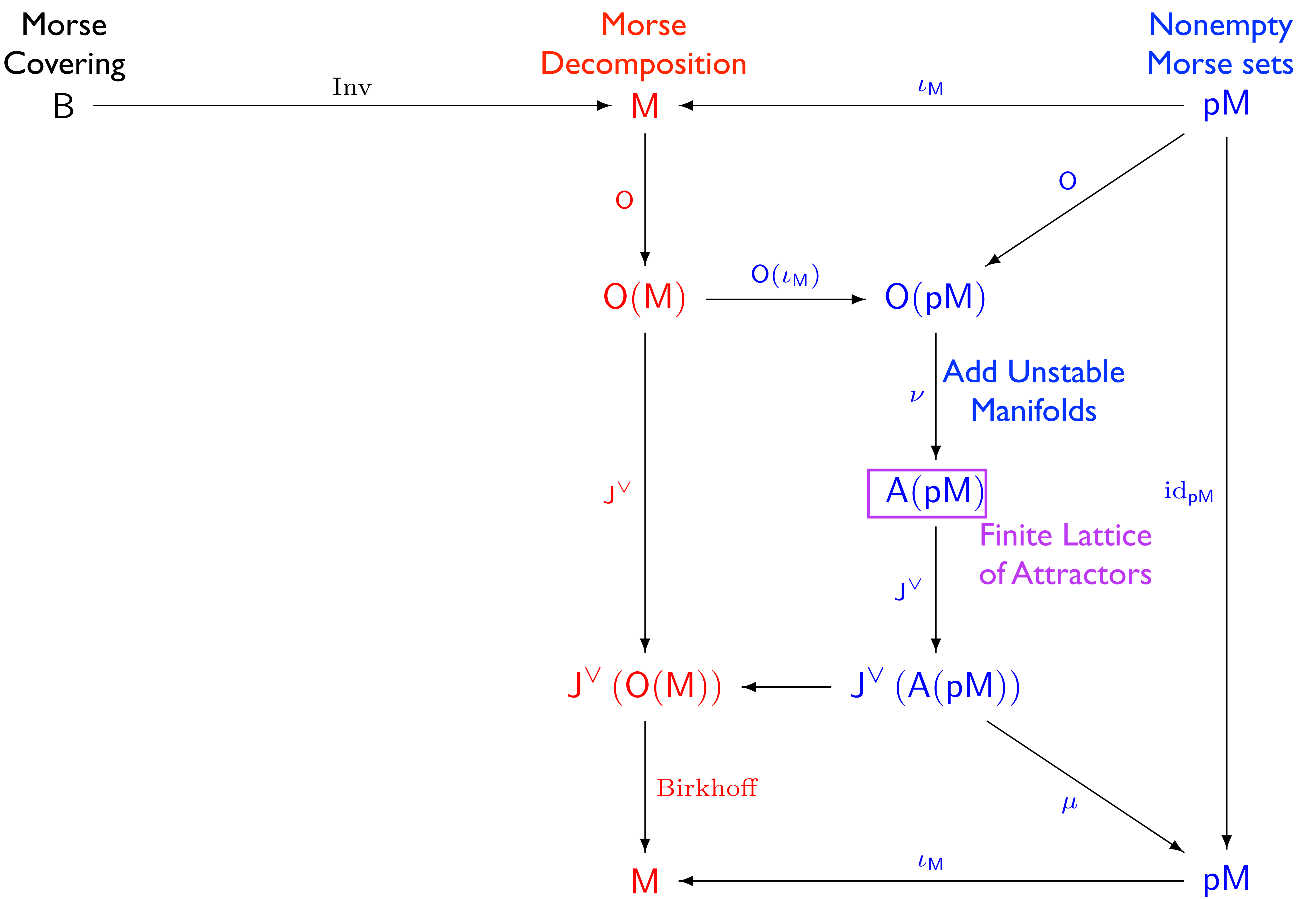
In the Computer



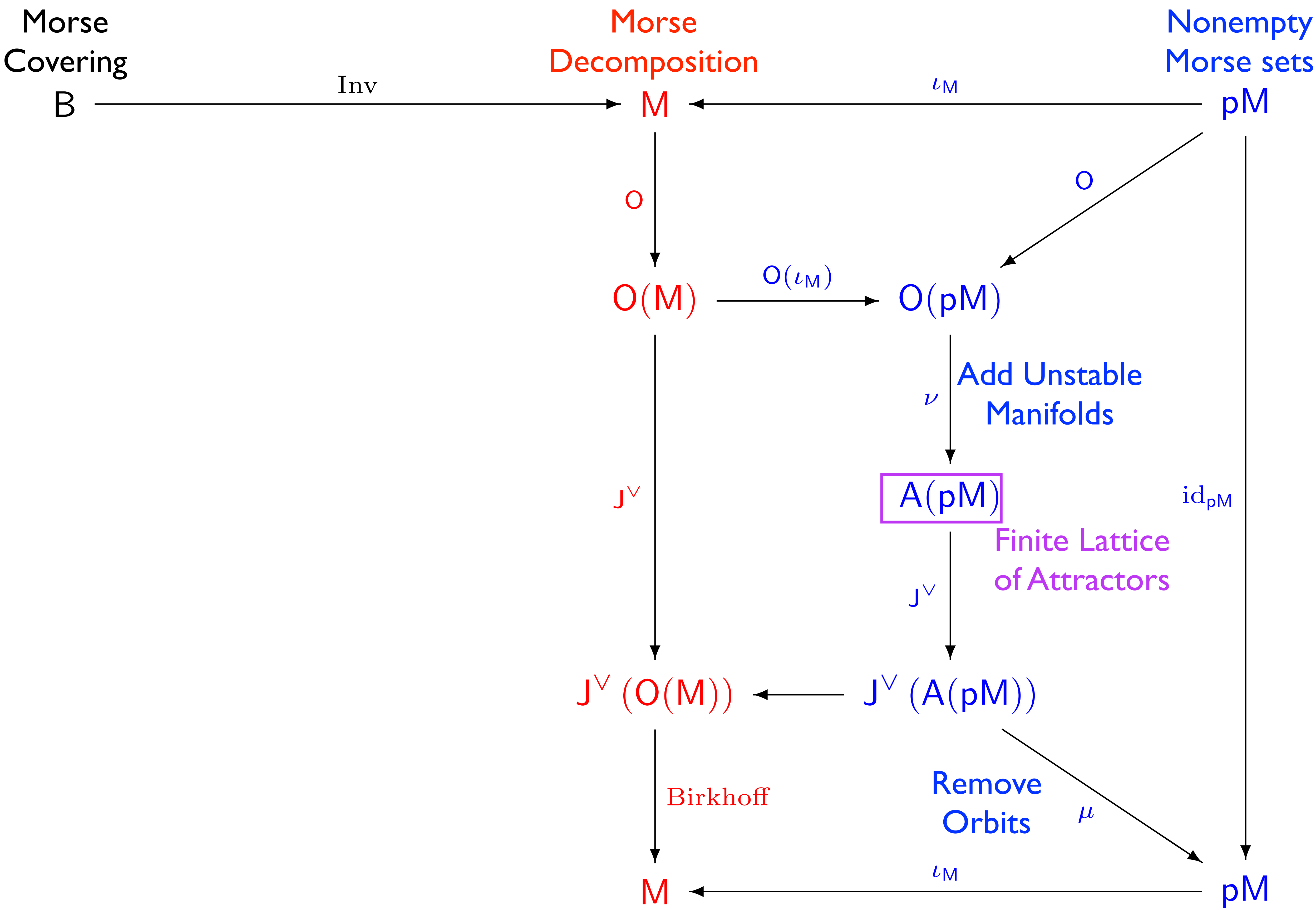
In the Computer

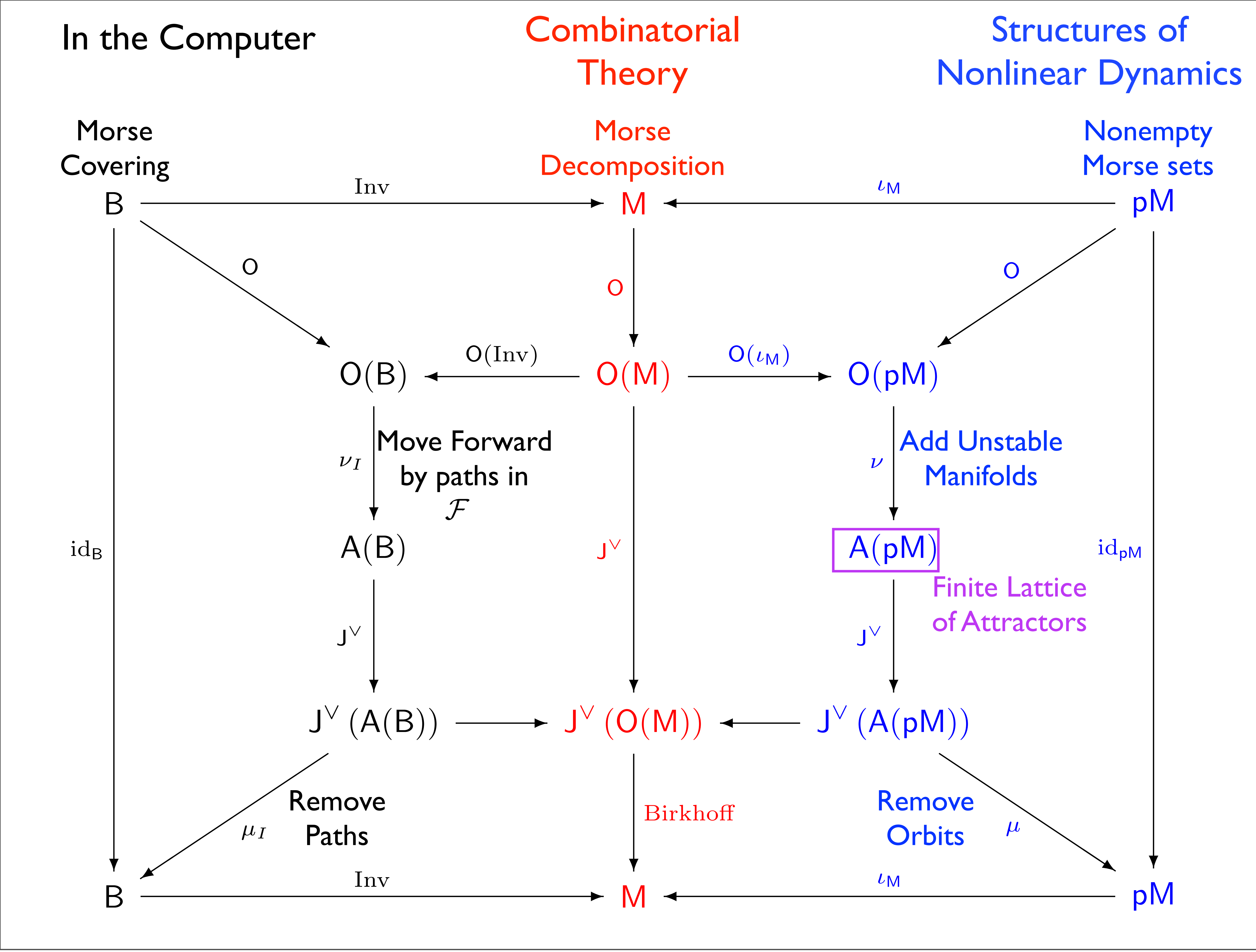
Combinatorial Theory

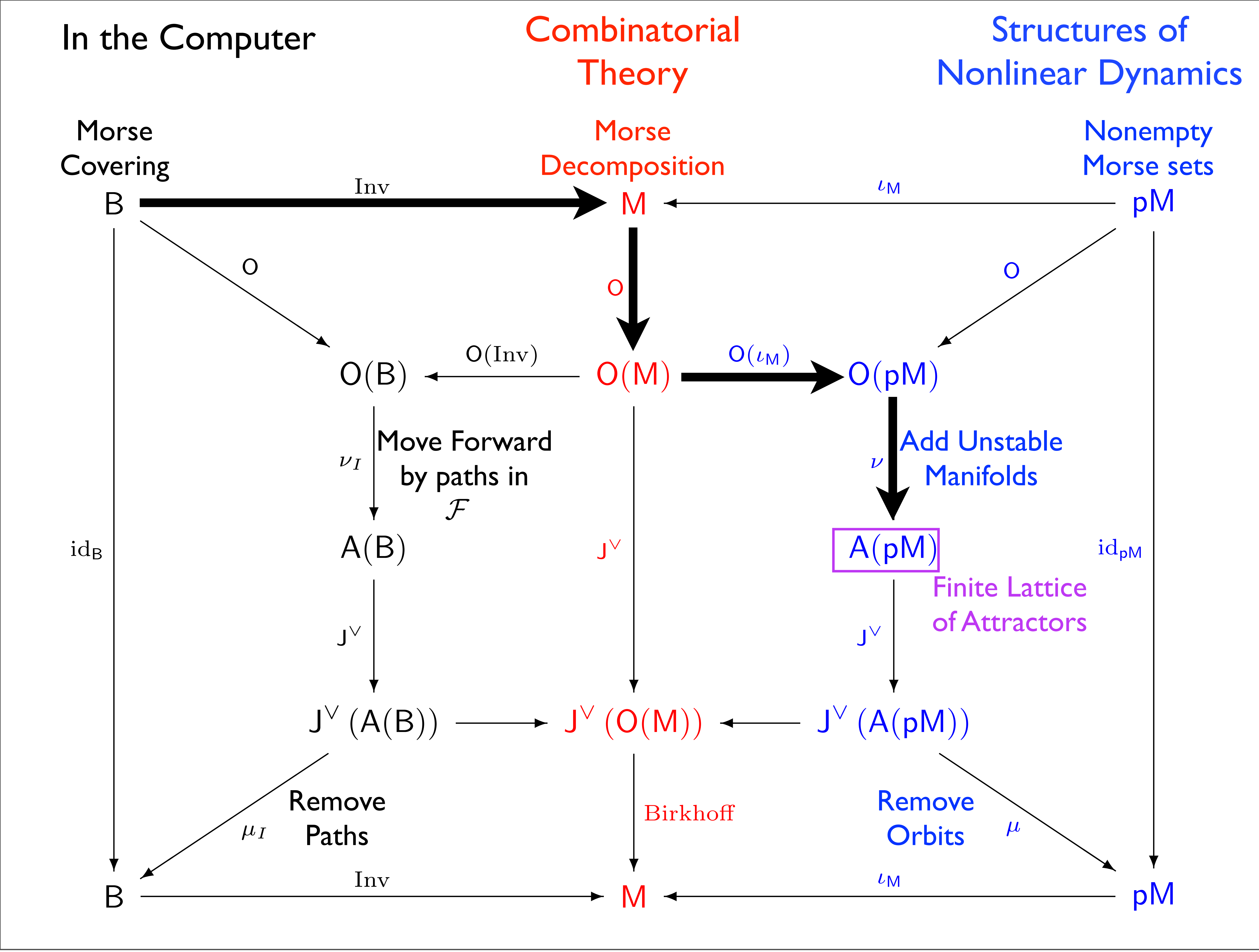
Structures of Nonlinear Dynamics



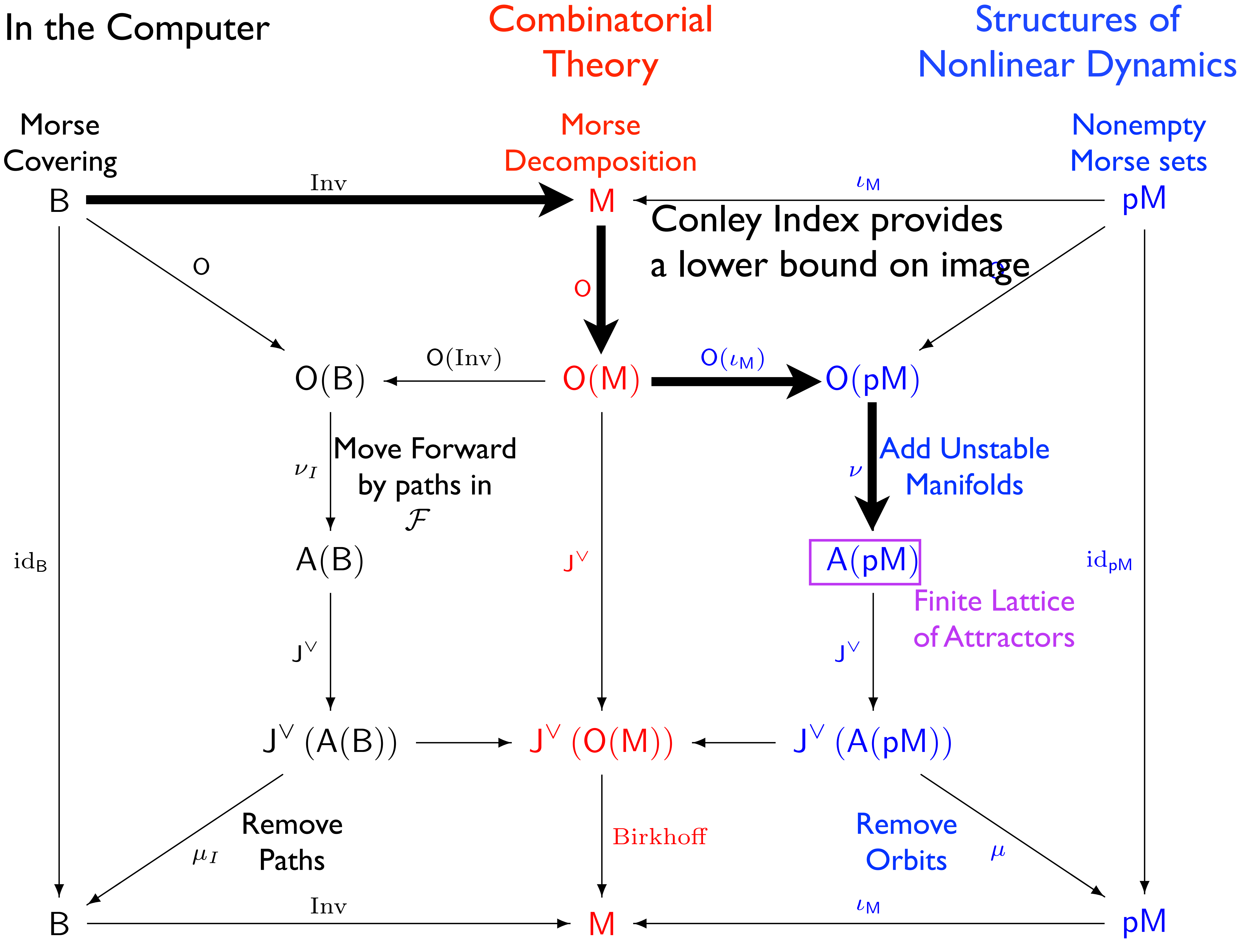
In the Computer | Combinatorial Theory | Structures of Nonlinear Dynamics




In the Computer
Combinatorial Theory
Structures of Nonlinear Dynamics
Morse Covering
Morse Decomposition
Nonempty Morse sets
B
Inv
M
ι_M
pM
O
O
O
O(B)
O(Inv)
O(M)
O(ι_M)
O(pM)
ν_I
Move Forward by paths in $\mathcal{F}$
ν
Add Unstable Manifolds
id_B
A(B)
$J^\vee$
A(pM)
id_{pM}
Finite Lattice of Attractors
$J^\vee$
$J^\vee$
$J^\vee$(A(B))
$J^\vee$(O(M))
$J^\vee$(A(pM))
Remove Paths
μ_I
Birkhoff
Remove Orbits
μ
B
Inv
M
ι_M
pM



Saturday, July 3, 2010



In the Computer
Combinatorial Theory
Structures of Nonlinear Dynamics
Morse Covering
Morse Decomposition
Nonempty Morse sets
B
Inv
M
ι_M
pM
Conley Index provides a lower bound on image
O
O
O(B)
O(Inv)
O(M)
$O(\iota_M)$
O(pM)
ν_I
Move Forward by paths in $\mathcal{F}$
ν
Add Unstable Manifolds
id_B
A(B)
$J^\vee$
A(pM)
id_{pM}
Finite Lattice of Attractors
$J^\vee$
$J^\vee$
$J^\vee(A(B))$
$J^\vee(O(M))$
$J^\vee(A(pM))$
Remove Paths
μ_I
Birkhoff
Remove Orbits
μ
B
Inv
M
ι_M
pM

Thank-you for your attention

http://chomp.rutgers.edu/

A Database Schema for the Analysis of Global Dynamics of Multiparameter Systems
SIADS, 8 (2009)

Z. Arai, Hokkaido
W. Kalies, Florida Atlantic
H. Kokubu, Kyoto
H. Oka, Ryukoku
P. Pilarczyk, Minho

W. Kalies, Florida Atlantic
R. Vandervorst, Amsterdam

J. Bush, Rutgers