Specialized System Solvers for very large Systems; Theory and Practice

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Workshops on Algorithms for Modern Massive Data Sets
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Outline

1. Introduction
2. Symmetric Diagonally Dominate Systems
   - Graph Laplacians
3. Applications of SDD/Laplacians
4. Iterative Methods for Graph Laplacians
   - Combinatorial Preconditioners
5. Low Stretch Spanning Trees
6. Solver Code
7. Open Questions

Solving Linear Systems, a fundamental Problem

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & -5 & 4 \\ -1 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

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3. Goal: Find more easy cases that have applications.

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- Open: **Can we solve spd systems in near linear time?**
- Two approaches to solving: direct and iterative methods.

Direct Methods
Gaussian Elimination Matrices

- Goal: algorithms that minimize work and space.

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- Goal: algorithms that minimize work and space.
- Trick: View nonzero entries as an undirected graph and view pivoting as a graph operation.

Good Pivot Strategies

1970s and 1980s

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- 3D Systems: $O(n^2)$ work and $O(n^{3/2})$ fill/space, [M, Teng, Thurston, Vavasis] (EG: 3D images and 3D finite element).
- $O(n^{3/2})$ space is too big for 3D Image problems.

Pure Iterative Methods

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- CG: $O(nm)$, [Magnus, Eduard 52].

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- Goal: Minimize the number of iteration while minimizing the cost of the solve.

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- Gauss-Seidel: $B = UpperTriangular(A)$.
- SSOR: $B = (L + \frac{1}{\omega}D)\frac{1}{\omega}D(L + \frac{1}{\omega}D)$
- Still too slow and unreliable.

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- SDD can be reduce to Graph Laplacians, [Gremban M 96]

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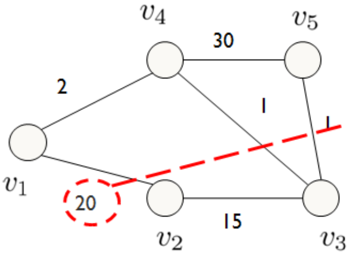
- Degree of v_i: $d_i = \sum_j w_{ij}$
- $$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$
- Laplacian: $L = D - A$

Example of Laplacian



$$L = \begin{pmatrix} 22 & -20 & 0 & -2 & 0 \\ -20 & 35 & -15 & 0 & 0 \\ 0 & -15 & 17 & -1 & -1 \\ -2 & 0 & -1 & 33 & -30 \\ 0 & 0 & -1 & -30 & 31 \end{pmatrix}$$

History for solving Laplacian

Recurrsive preconditioned iterative methods.

- $O(n^{1.2})$ for planar Laplacians, [Vaidya 91]
- $\tilde{O}(m^{1.5})$ for natural 3D graphs [Gremban, M 96].
- First near-linear time algorithm, $O(m \log^{15} n)$, [Spielman, Teng 04].
- $O(n)$ for planar Laplacians, [Koutis, M 07]
- $O(m \log^2 n)$ (ignoring $\log\log$ and lower terms), [Koutis, M, Peng 10].

Main Theorem

Theorem

Input:	*SDD system* $Ax = b$.
Output:	$\bar{x}$ *satisfying* $\|\|\bar{x} - A^{+}b\|\|_A < \epsilon \|\|A^{+}b\|\|_A$.
Expected Time:	$\tilde{O}(m \log^2 n \log(1/\epsilon))$.

[Koutis, M, Peng 10]

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- If $LV = c$ then c is the residual current needed to maintain the given voltages.

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- View each edge as a conductor with conductance w_{ij}.
- Let V be a column vector of temperatures.
- If $c = LV$ then c is the residual heat needed to maintain the given temperatures.
- The finite element heat equations can be preconditioned with a graph Laplacian and thus solved in $\tilde{O}(n + m)$ time. [Boman, Hendrickson, and Vavasis 06]

Graph Laplacian's and Random Walks

Transition Matrix: $A_G D^{-1}$, symmetric A. Mixing Rate-Fundamental Eigenvector:
$\tilde{O}(n+m)$ [Spielman Teng 04]
Trick: Inverse Powering only requires $O(\log n)$ iterations.

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  Eigen-pairs of $L_G x = \lambda M x$.

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- M is a diagonal matrix of mass constants
- Fact: Modes of vibration of Spring-Mass system G, M are:
  Eigen-pairs of $L_G x = \lambda M x$.
- Thus the fundamental mode can be found in $\tilde{O}(n + m)$ time.

Spring Mass System

Movie of a Simple Image

Graph Laplacian's and Linear Programming

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- Algorithm: Max-Flow is a LP problem so use log barrier interior point method.
- Fact: Each of $O(\sqrt{m})$ pivots requires the solution the graph Laplacian.
- Thus: Approximate Max-Flow is $\tilde{O}((m+n)^{3/2})$ [Daitch, Spielman 08]

Graph Laplacian's and Convex Programming

- Uniform TV Denoising:
  Input: image s
  Output: image $\arg\min ||x - s||_2^2 + \lambda|| \bigtriangledown x||_1$

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  Output: $\arg\min (x - s)^T (x - s) + Sum_{(i,j)\in G} |w_{ij}(x_i - x_j)|$

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  Output: $\arg\min (x - s)^T (x - s) + Sum_{(i,j)\in G} |w_{ij}(x_i - x_j)|$
- Use log-barrier interior point: pivots are low rank perturbation of Laplacian,
  Thus: $\tilde{O}((m + n)^{3/2})$ time. [Koutis M Peng Sinop Tolliver 09]

Condition Number

- **Def:** Condition number of A and B, $\kappa(B^{-1}A) = \lambda_{max}(B^{-1}A)/\lambda_{min}(B^{-1}A)$.
- OR: If $x^TAx \leq x^TBx \leq kx^TAx$ for all $x \in \mathbb{R}^n$, then $\kappa(B^{-1}A) \leq k$

Rate of Convergence

- Classical results, measured in number of iterations per bit of precision.
- Richardson iteration: $O(\kappa(B^{-1}A))$, too slow.
- Conjugate gradient: $O(\sqrt{\kappa(B^{-1}A)})$ or better, hard to analyze when B is called recursively and solved inexactly.
- Chebyshev iteration: $O(\sqrt{\kappa(B^{-1}A)})$, will use.

When is B a Good Recursive Preconditioner?

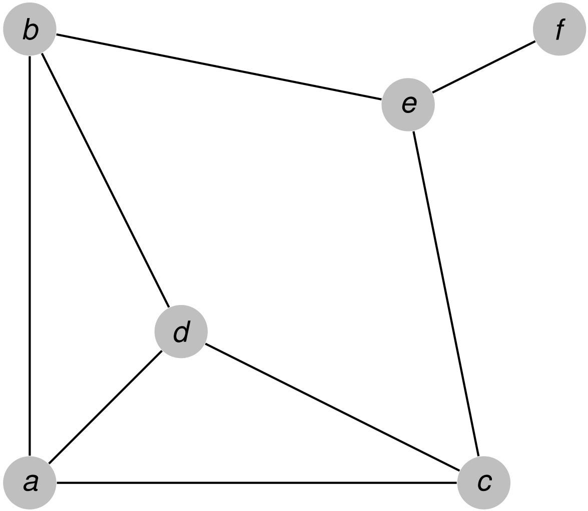
- Properties that Laplacian B should have:
  1. $B^{-1}A$ has low condition number.
  2. Quickly reduces to something that can be solved faster (smaller size).
- Examples:
  - [Vaidya 91] Spanning tree + a few edges.
  - [Gremban, M 96] Steiner tree.
  - [Boman, Hendrickson 03; Spielman, Teng 04] Low stretch spanning tree + a few edges.
  - [Koutis, M 07] Partition planar graphs into pieces of size k with $\sqrt{k}$ boundary, optimally precondition each piece.

Getting a Good Preconditioner.

Main steps:

- Find a sparse subgraph by random sampling.
- Use Gaussian elimination to remove degree 1 and 2 vertices.
- We need sampling to be fast and give good condition numbers.

Example: Pivoting out degree 1 and 2.



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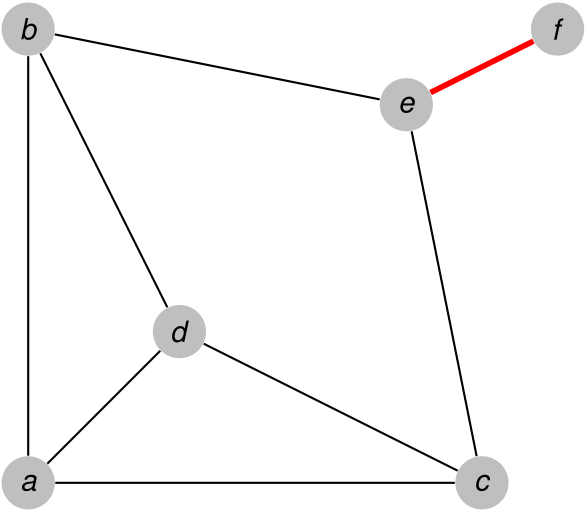
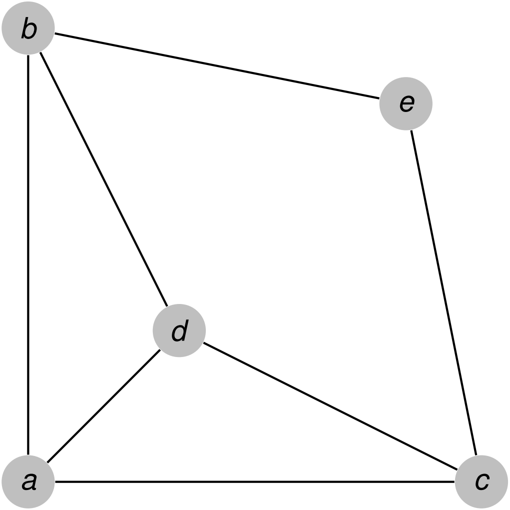


Pivot(f)



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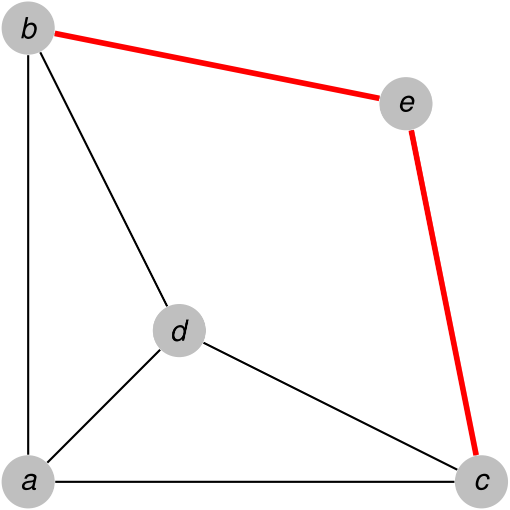
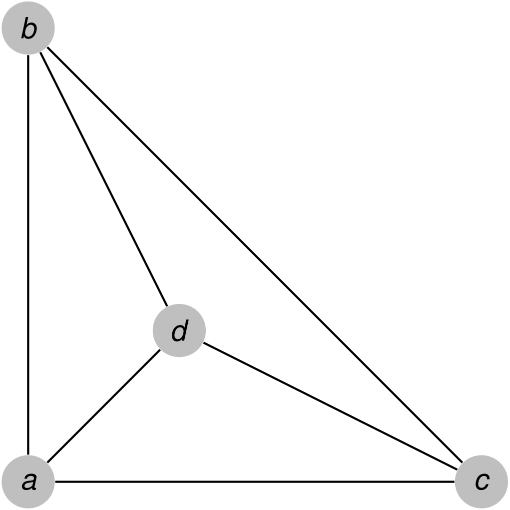


Pivot(e)



Ran out of degree 1 or 2 nodes, quit.

Graph Sparsifier

Given a graph G find H by: Remove most edges, increase weight of remaining edges.
Possible properties to be preserved:

- Spanners: distance, diameter
- Cut sparsifier: weight of cut for all $2^{|V|}$ subset of vertices
- Triangle sparsifiers: number of triangles in a subgraph
- We want spectral sparsifiers
- Also want H to be ultra-sparse for Gaussian elimination to make progress.
- Need to reduce to $n - 1 + m/c$ edges for in order to decrease edge count by factor of $c/3$.

Previous Work on Sparsification

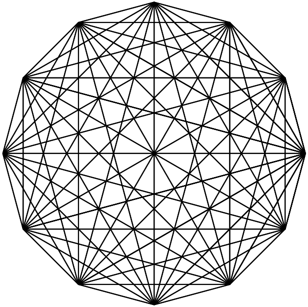
- Expanders: sparsifier for complete graph.
- Ramanujan graphs: optimal spectral sparsifiers for the complete graph.
- [Benczur, Karger 96] Cut sparsifiers, $O(n \log n)$ edges.
- [Kolountzakis, M, Tsourakakis 10] Edge sampling can give good triangle sparsifiers.

Previous Work on Spectral Sparsification

- [Spielman, Teng 04] H has $\tilde{O}(n)$ edges, constant condition number.
- [Spielman, Teng 04] Ultrasparsifier, H has $n - 1 + n/c$ edges, condition number $\tilde{O}(c)$.
- **[Spielman, Srivastava 08] Conceptually simple sampling algorithm for spectral sparsification.**
- [Batson, Spielman, Srivastava 09] and [Kolla, Makarychev, Saberi, Teng 10] gave better bounds, but their algorithms do not run in near-linear time.

Sparsifier for the Complete Graph

- For K_n, $\frac{1}{p}G_{(n,p)}$ is a good sparsifier when $p \geq \log n/n$.



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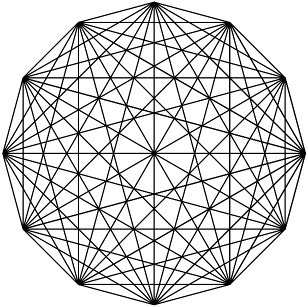

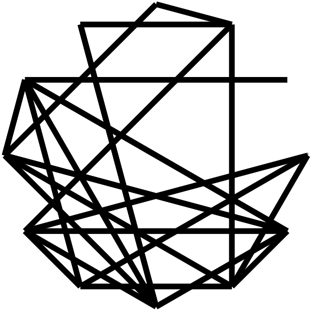

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Generalized Graph Sampling

- [Benczur, Karger 96] used this method for cut sparsifiers
- Near-linear time spectral sparsifiers use the same framework.
- Compute a probability p_e for each edge.
- For each edge e keep with probability p_e.
  If kept multiply weight by $1/p_e$.

What Sampling Gives

- Expected value: original graph
- Expected number of edges: $(\sum_e p_e) \log n$.
- Concentration?

Effective Resistance

- Consider each edge as a resistor with conductance w_e
- For edge $e = (u, v)$ let R_e be the effective resistance from u to v in G.

Sparsification by Effective Resistance

Theorem (Spielman, Srivastava 08)

Sampling a weighted graph G using edge probablities $p_e = w_e R_e$ to generate H with $O(n \log n)$ expected edges then $\kappa(G, H)$ is a constant with high probablity.

Calculating effective resistance efficiently?

Low Stretch Spanning Trees

We use low stretch spanning trees to approximate effective resistance.
Let T be a tree of G

Definition

Stretch(e) = $w_e \cdot ER_e^T$, the effective resistance in T.

$$stretch(T) = \sum_{e \in G} stretch(e)$$

Low Stretch Spanning Trees

- [Spielman, Teng 04] Fundamental for their solver and ultrasparsifier.
- [Kolla, Makarychev, Saberi, Teng 10] Fundamental their near-optimal sparsifier.

Known Results about Low Stretch Spanning Trees

- First studied in [Alon, Karp, Peleg, West 95] in the context of k server problem.
- [Elkin, Emek, Spielman, Teng 05] $O(m \log^2 n)$ stretch.
- [Abraham, Bartal & Neiman 08] roughly $O(m \log n)$ stretch.

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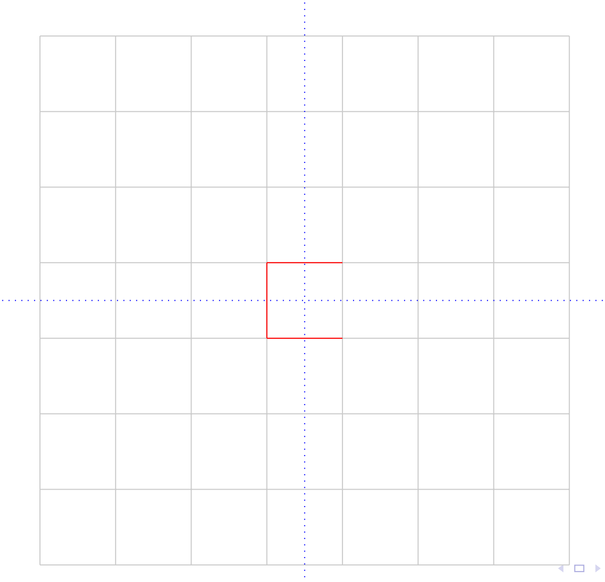

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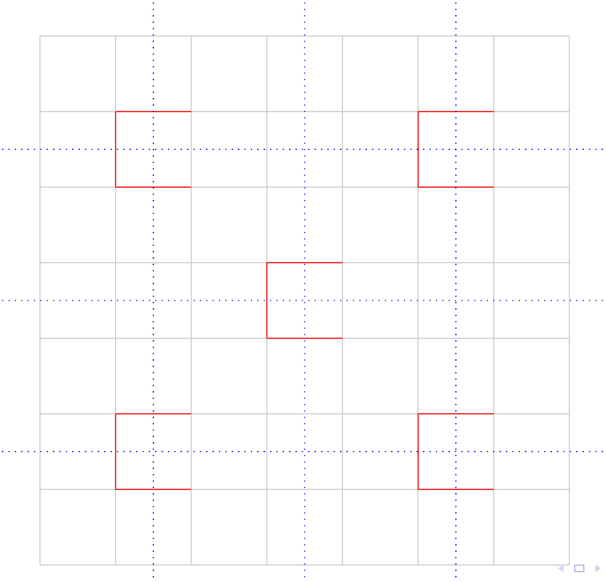

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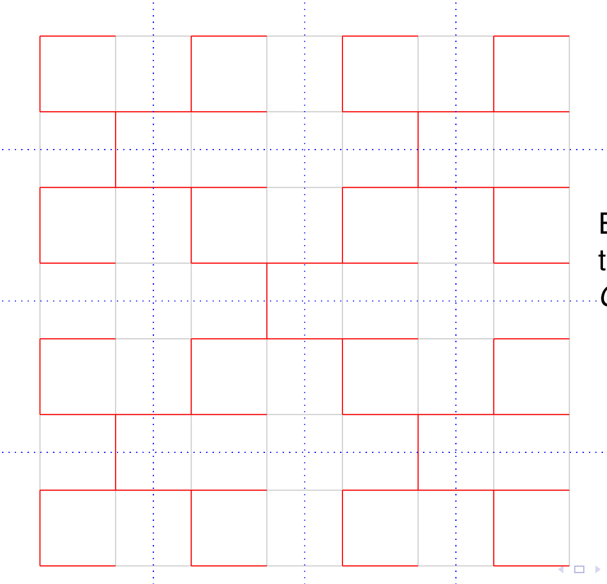


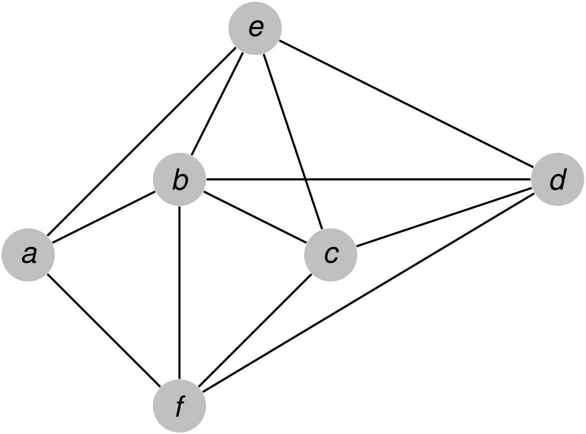
Exercise: show that stretch is $O(n \log n)$.

Pseudocode

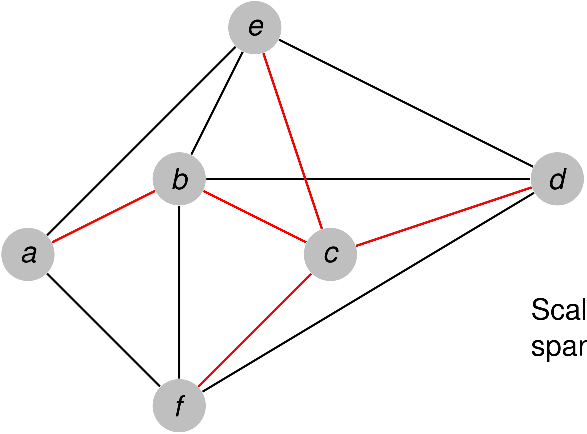
INCREMENTALSPARSIFY
Input: Graph G, real value $c = O(log^4 n)$.
Output: Graph H that's a sparsifier for G

1. $T \leftarrow$ LOWSTRETCHTREE(G)
2. Let T' be T scaled up by factor of c
3. Let G' be the graph obtained from G by replacing T with T'
4. FOR $e \in E$
5. Calculate $EffectiveResistance_{T'}(e)$
6. ENDFOR
7. $H \leftarrow$ SAMPLE(G', $EffectiveResistance_{T'}$)
8. RETURN H

Example: Original Graph



Example: Scale up a Good Spanning Tree



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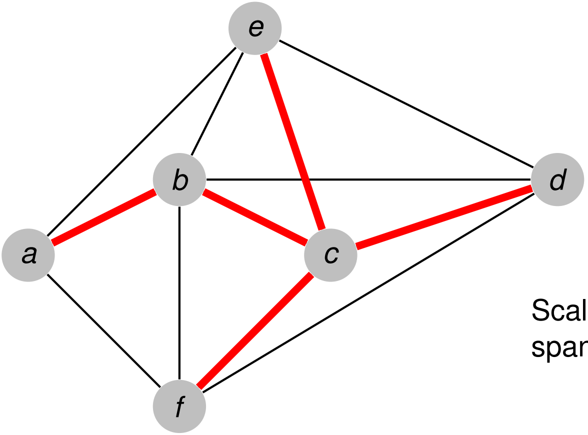

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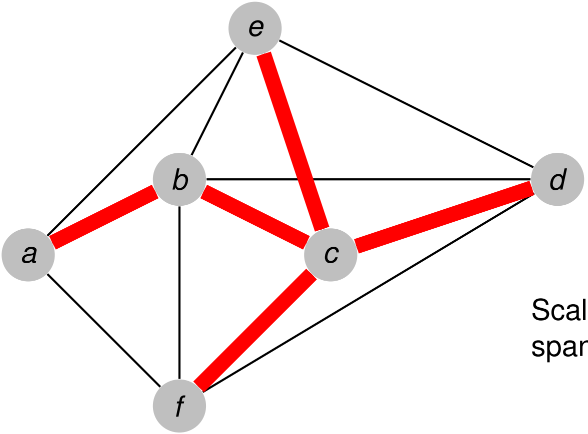

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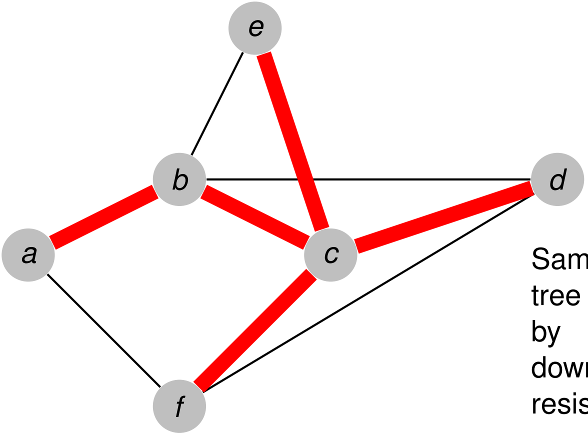
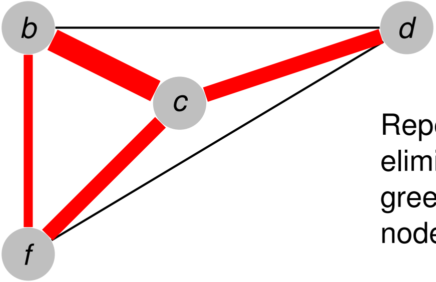
Example: Sample



Sample non-tree edges by (scaled down) effective resistance.

Example: Gaussian Elimination



Repeatedly eliminate degree 1 and 2 nodes.

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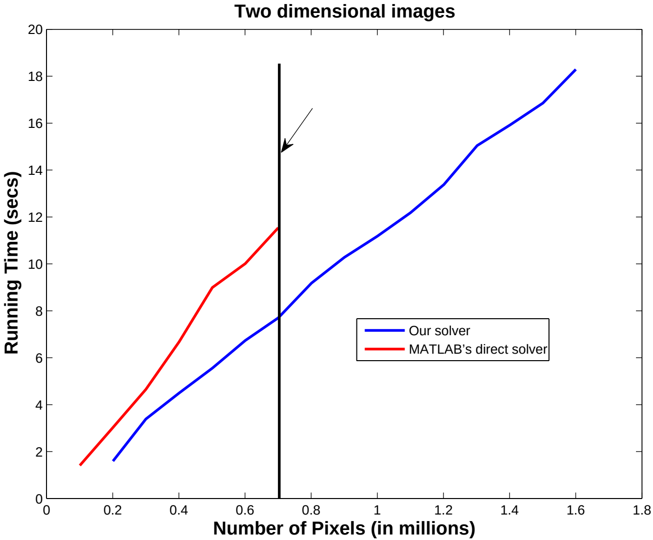
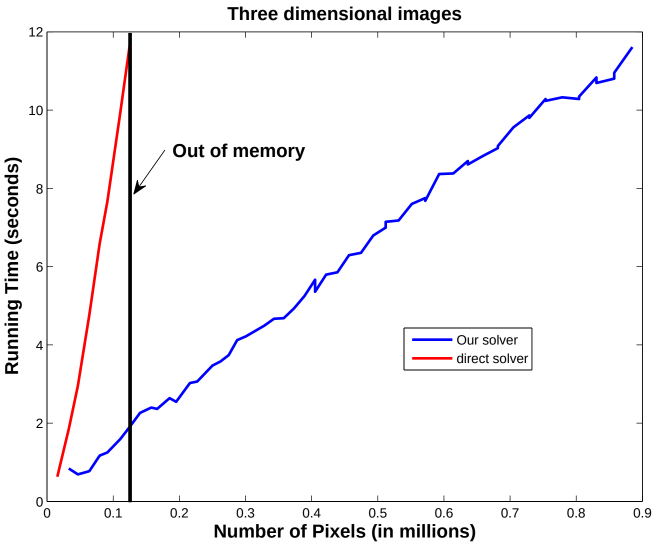
- 1990's $O(n^{1.2})$ (Combinatorial Preconditioners) (Vaidya)
- 2000's $O(n \log^2 n)$ (Low stretch spanning trees) (ST)

History of Planar Solvers

- 1950's $O(n^2)$ (Conjugate Gradient)
- 1970's $O(n^{1.5})$ (Nested Dissection) (LRT)
- 1990's $O(n^{1.2})$ (Combinatorial Preconditioners) (Vaidya)
- 2000's $O(n \log^2 n)$ (Low stretch spanning trees) (ST)
- 2006's $O(n)$ (separator based preconditioners) (KM)


Two dimensional images
Running Time (secs)
Number of Pixels (in millions)
Our solver
MATLAB's direct solver
0
2
4
6
8
10
12
14
16
18
20
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8


Three dimensional images
Running Time (seconds)
Number of Pixels (in millions)
0
2
4
6
8
10
12
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Out of memory
Our solver
direct solver

Open Questions

- Find fast methods for any SPD system.
- Find spectral methods that find better cuts by using more than one eigenvector.
- Find solvers that work in the L_2 norm.
- A implementable solver with near linear time guarantees. The low stretch spanning tree is the bottleneck!

Thank You