Graph Sparsification by Effective Resistances

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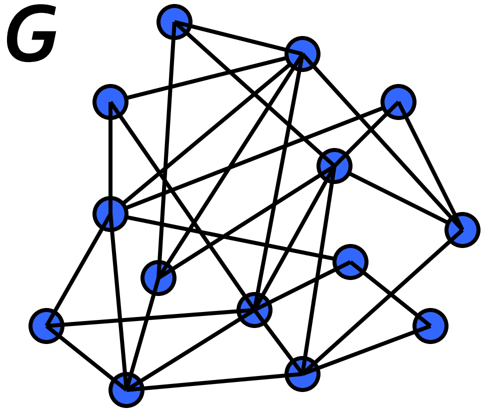
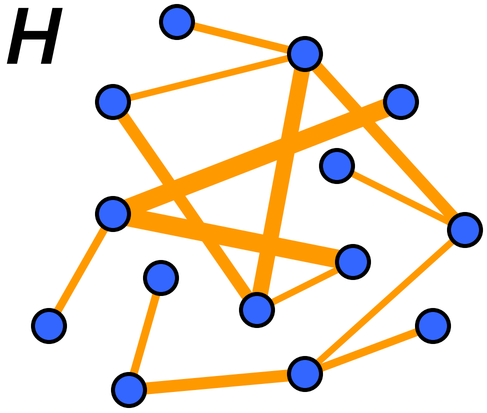
Yale

Sparsification

Approximate any graph ***G*** by a sparse graph ***H***.





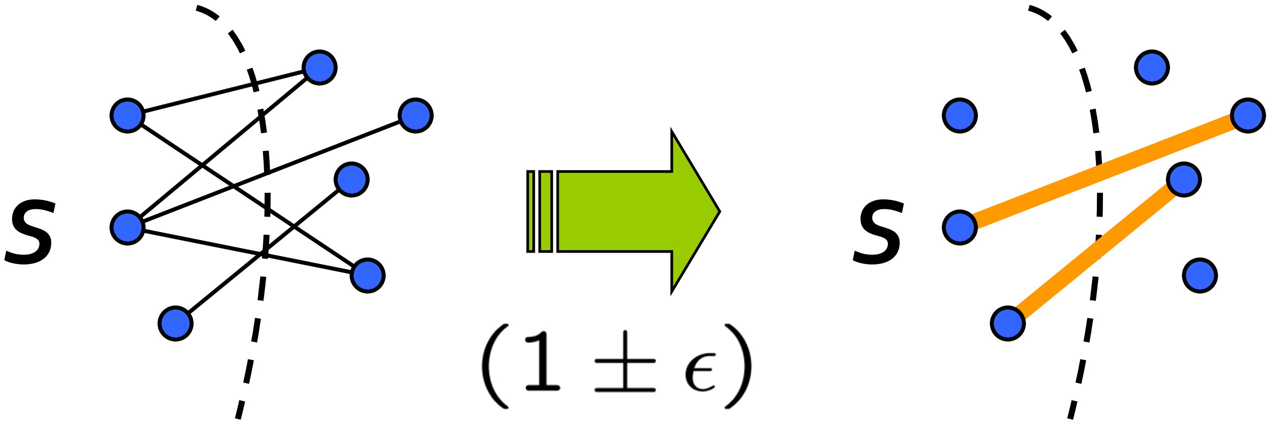
- Nontrivial statement about ***G***
- ***H*** is faster to compute with than ***G***

Cut Sparsifiers [Benczur-Karger'96]

H approximates ***G*** if

for every cut ***S*½*V***

sum of weights of edges leaving ***S*** is preserved



Can find ***H*** with O(nlogn/ε^2) edges in $\tilde{O}(m)$ time

The Laplacian (quick review)

$$L_G = D_G - A_G$$

Quadratic form

$$x : V \to \mathbb{R}$$

$$x^T L_G x = \sum_{uv \in E} c_{uv}(x(u) - x(v))^2$$

Positive semidefinite

Ker(L_G)=span(**1**) if ***G*** is connected

Cuts and the Quadratic Form

For characteristic vector $x_S \in \{0,1\}^n$ of $S \subseteq V$

$$\begin{aligned} x_S^T L_G x_S &= \sum_{uv \in E} c_{uv}(x(u) - x(v))^2 \\ &= \sum_{uv \in (S,\overline{S})} c_{uv} \\ &= wt_G(S, \overline{S}) \end{aligned}$$

So BK says:

$$1 - \epsilon \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon \quad \forall x \in \{0,1\}^n$$

A Stronger Notion

For characteristic vector $x_S \in \{0,1\}^n, S \subseteq V$

$$\begin{aligned} x_S^T L_G x_S &= \sum_{uv \in E} c_{uv}(x(u) - x(v))^2 \\ &= \sum_{uv \in (S,\overline{S})} c_{uv} \\ &= wt_G(S,\overline{S}) \end{aligned}$$

So BK says:

$$\forall x \in \mathbb{R}^n$$

$$1 - \epsilon \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon \quad \forall x \in \{0,1\}^n$$

Why?

1. All eigenvalues are preserved

By Courant-Fischer,

$$(1-\epsilon)\lambda_i(G) \leq \lambda_i(H) \leq (1+\epsilon)\lambda_i(G)$$

G and ***H*** have similar eigenvalues.

For spectral purposes, ***G*** and ***H*** are equivalent.

1. All eigenvalues are preserved

By Courant-Fischer,

$$(1-\epsilon)\lambda_i(G) \leq \lambda_i(H) \leq (1+\epsilon)\lambda_i(G)$$

G and ***H*** have similar e values.

For spectral purpo nt.

cf. matrix sparsifiers
[AM01,FKV04,AHK05]

$$\|L_G - L_H\|_2 \leq \epsilon$$

2. Linear System Solvers

Conj. Gradient solves $Ax = b$ in

$$\sqrt{\underbrace{\kappa(A)}_{\frac{\max(x^TAx)}{\min(x^TAx)}}\underbrace{m}_{\mathtt{nnz}(A)}}\,\underbrace{\log(1/\epsilon)}_{\text{ignore}}$$

(time to mult. by A)

2. Preconditioning

Find easy B that approximates A .

Solve $B^{-1}Ax = B^{-1}b$ instead.

$$\sqrt{\underline{\kappa(B^{-1}A)}}(m + \underline{\texttt{solve}(B)})\log(1/\epsilon)$$

$$\frac{\max \frac{x^T Ax}{x^T Bx}}{\min \frac{x^T Ax}{x^T Bx}}$$

Time to solve
$By = c$
(mult.by B^{-1})

2. Preconditioning

Find easy B s A .

Solve B^{-1} nstead.

Use B=L$_H$?

$$\sqrt{\underline{\kappa(B^{-1}A)}(m+\underline{\texttt{solve}(B)})}\,\log(1/\epsilon)$$

$$\kappa = \frac{1+\epsilon}{1-\epsilon} = O(1)$$

?

2. Preconditioning

Find easy B A .

Solve B^{-1} stead.

Spielman-Teng
[STOC '04]
Nearly linear time.

$$\sqrt{\underline{\kappa(B^{-1}A)}}(m + \underline{\texttt{solve}(B)})\log(1/\epsilon)$$

$\kappa = \log^{O(1)} n$

$O(m \log^{O(1)} n)$

Examples

Example: Sparsify Complete Graph by Ramanujan Expander

G is complete on n vertices. $\lambda_i(L_G) = n$

H is d-regular Ramanujan graph. $\lambda_i(L_H) \sim d$

$$\lambda_i(\frac{n}{d} L_H) \sim n$$

Example: Sparsify Complete Graph by Ramanujan Expander

G is complete on n vertices. $\lambda_i(L_G) = n$

H is d-regular Ramanujan graph. $\lambda_i(L_H) \sim d$

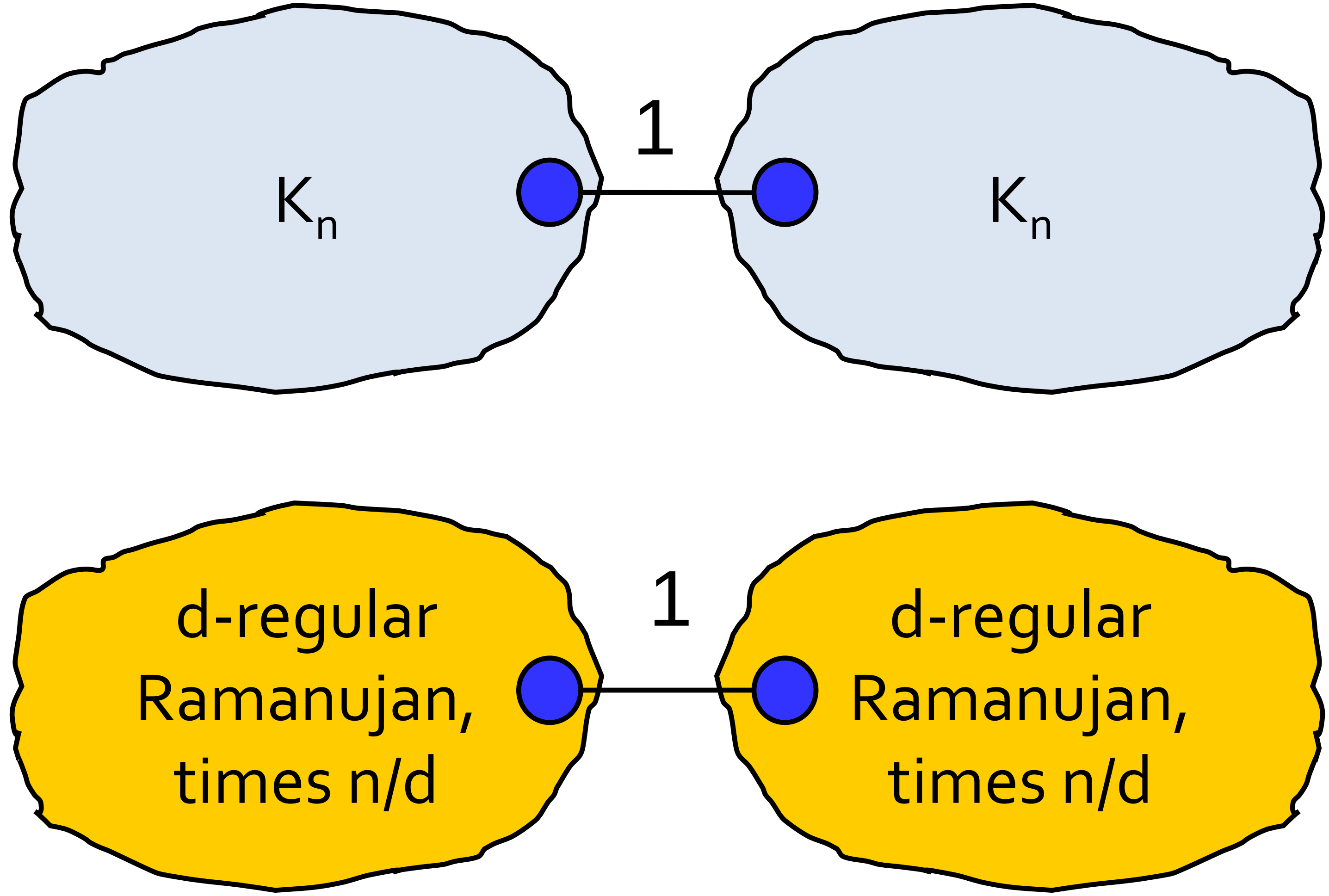
$$\lambda_i(\tfrac{n}{d} L_H) \sim n$$

$$\frac{x^T(\frac{n}{d}L_H)x}{x^T L_G x} \sim 1$$

Each edge has weight (n/d)

So, $\frac{n}{d}H$ is a good sparsifier for ***G***.

Example: Dumbell


K_n
1
K_n
d-regular Ramanujan, times n/d
1
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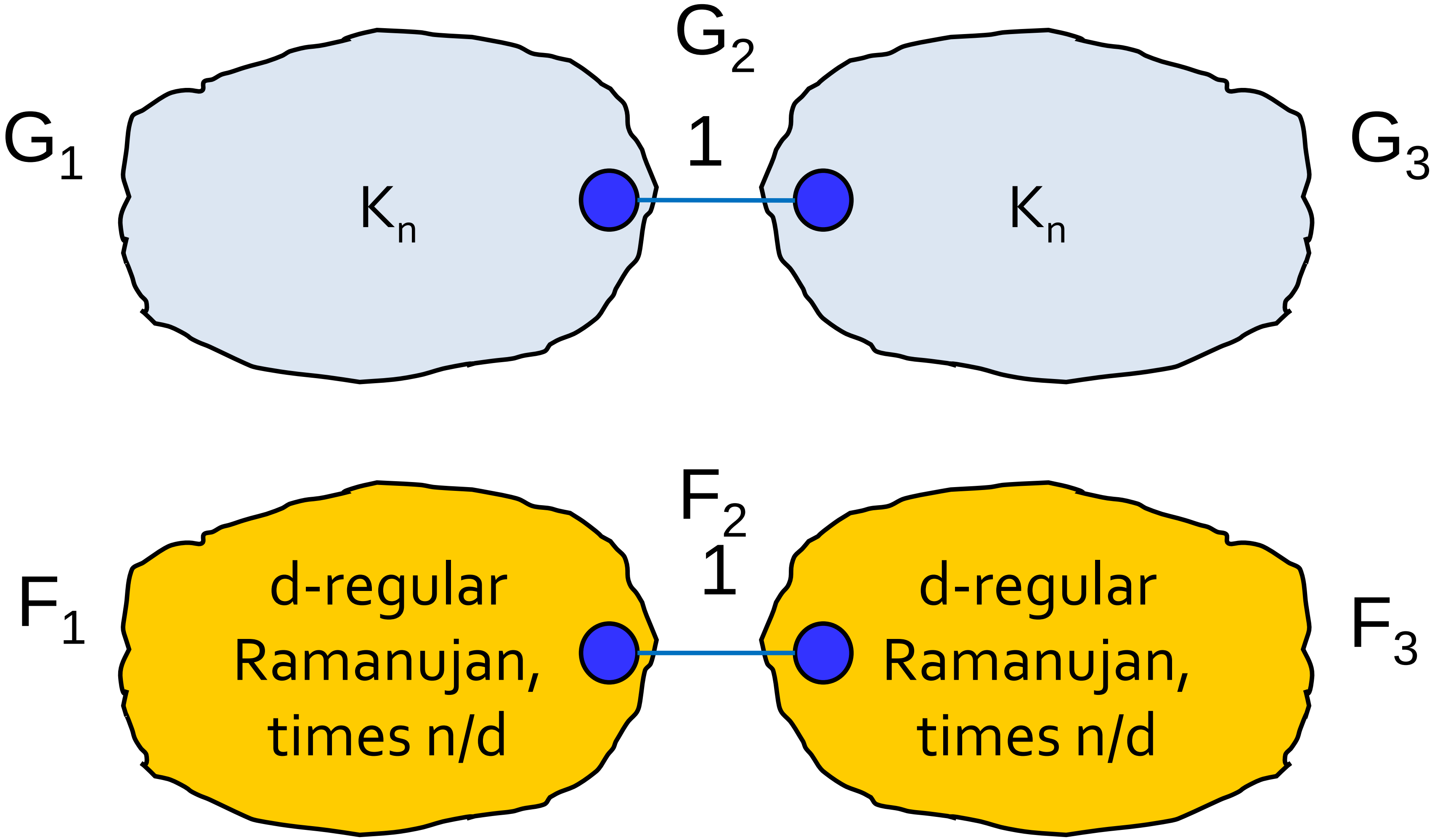
Example: Dumbell



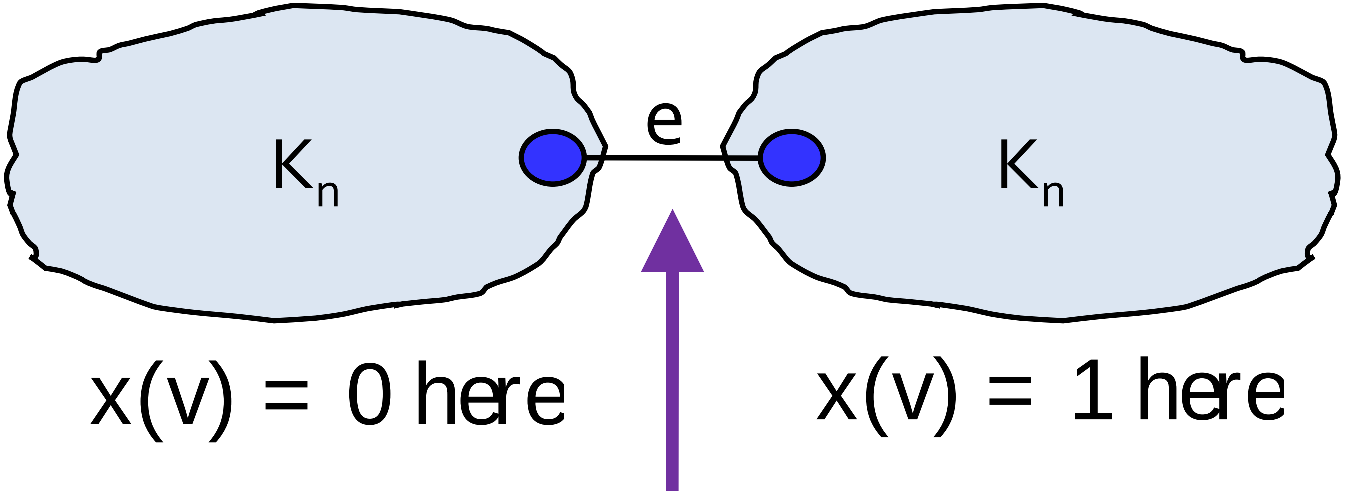
d-regular Ramanujan, times n/d

d-regular Ramanujan, times n/d

$$G = G_1 + G_2 + G_3$$

$$x^T G x = x^T G_1 x + x^T G_2 x + x^T G_3 x$$

Example: Dumbell. Must include cut edge



x(v) = 0 here

x(v) = 1 here

Only this edge contributes to

$$x^T L_G x = \sum_{(u;v) \in E} c_{(u;v)} (x(u) - x(v))^2$$

If $e \notin H;$ $x^T L_H x = 0$

Results

Main Theorem

Every ***G=(V,E,c)*** contains ***H=(V,F,d)*** with O(nlogn/ε²) edges such that:

$$(1-\epsilon)x^T L_G x \leq x^T L_H x \leq (1+\epsilon)x^T L_G x \quad \forall x \in \mathbb{R}^n$$

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Can find ***H*** in $\tilde{O}(m)$ time by random sampling.

Main Theorem

Every ***G=(V,E,c)*** contains ***H=(V,F,d)*** with $O(n\log n/\varepsilon^2)$ edges such that:

$$(1-\epsilon)x^T L_G x \leq x^T L_H x \leq (1+\epsilon)x^T L_G x \quad \forall x \in \mathbb{R}^n$$

Can find ***H*** in $\tilde{O}(m)$ time by random sampling.

Improves [BK'96]

Improves $O(n\log^c n)$ sparsifiers [ST'04]

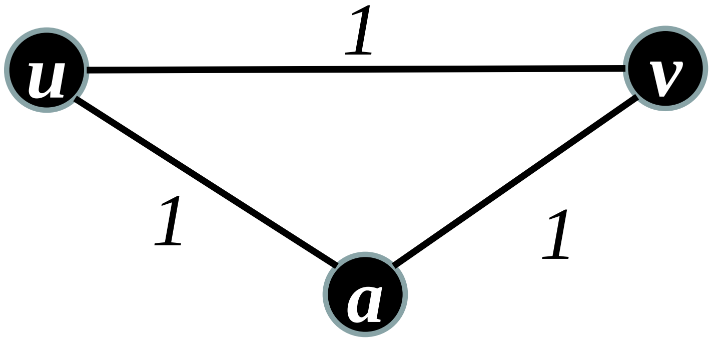
How?

Electrical Flows.

Effective Resistance

Identify each edge of G with a unit resistor

$R_{\mathsf{eff}}(e)$ is resistance between endpoints of e



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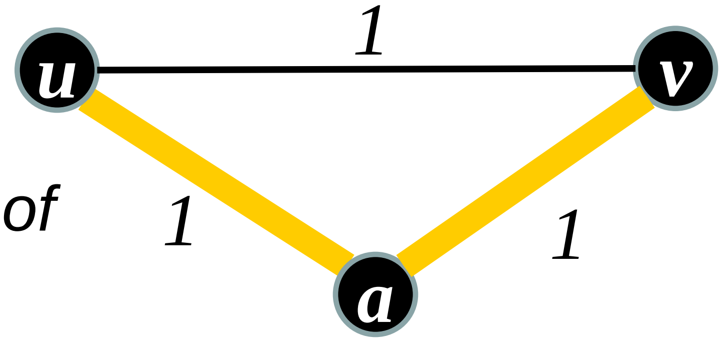

Resistance of path is 2

Effective Resistance

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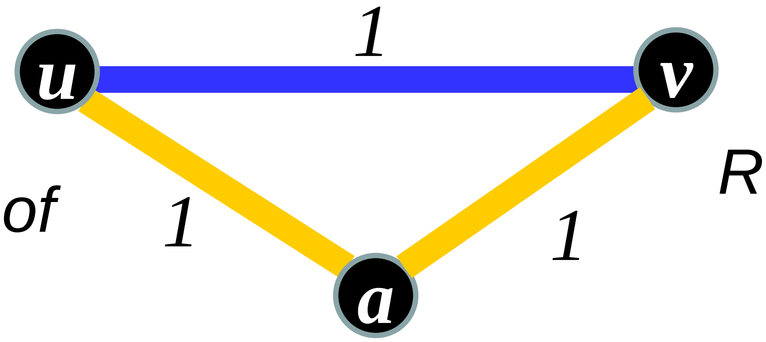

Resistance of path is 2

Resistance from u to v is

$$\frac{1}{1=2 + 1=1} = 2=3$$

Effective Resistance

Identify each edge of G with a unit resistor

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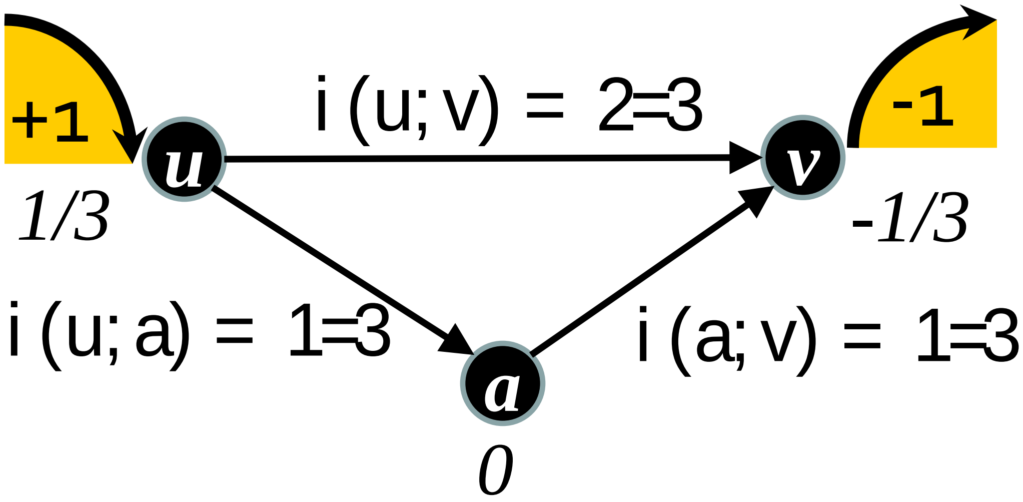

$$v = i\,r$$

Effective Resistance

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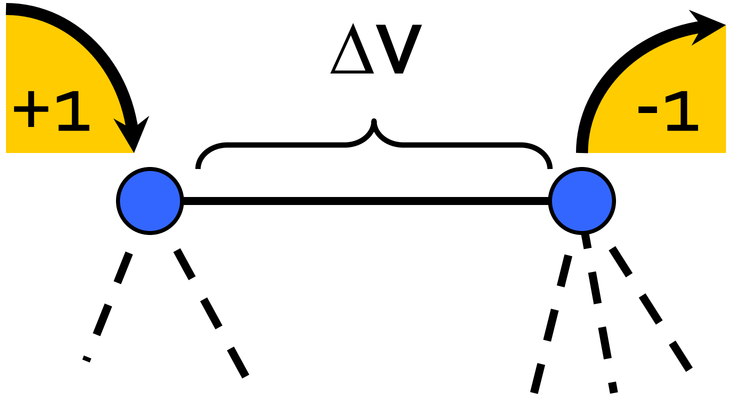
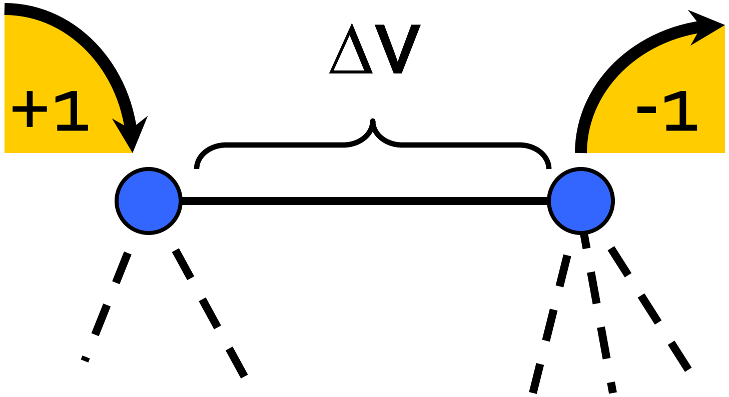


= potential difference between endpoints when flow one unit from one endpoint to other

Effective Resistance

$$R_{\text{eff}}(e) =$$



$$R_{\text{eff}}(e) = \mathbb{P}_{\text{spanning T}}[e \in T]$$

$$R_{\text{eff}}(uv) \propto \mathbb{E}_v T_u + \mathbb{E}_u T_v$$

[Chandra et al. STOC ’89]

The Algorithm

Sample edges of G with probability

$$p_e \propto R_{\mathsf{eff}}(e)$$

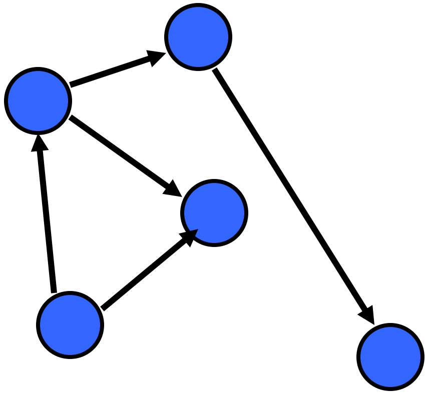
If chosen, include in H with weight $\frac{1}{p_e}$

Take q=O(nlogn/ε^2) samples with replacement

Divide all weights by *q*.

An algebraic expression for R_{eff}

Orient G arbitrarily.



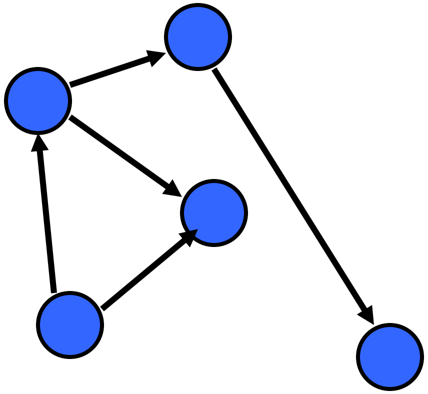
An algebraic expression for R_{eff}

Orient G arbitrarily.

Signed incidence matrix $B_{m£ n}$:

$$B(e,v) = \begin{cases} +1 & \text{if } v \text{ is head of } e \\ -1 & \text{if } v \text{ is tail of } e \\ 0 & \text{otherwise} \end{cases}$$

$$\text{i.e.,} B(uv,\cdot) = \chi_u - \chi_v.$$



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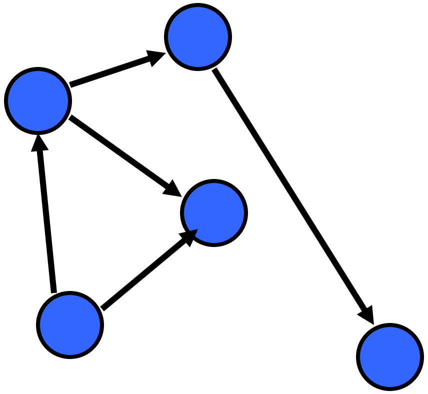
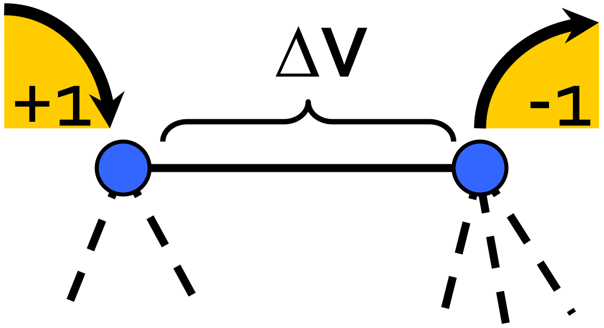


Write Laplacian as $L = B^T B$

An algebraic expression for R_{eff}



$$\boxed{\begin{aligned} R_{\text{eff}}(uv) &= (\chi_u - \chi_v)^T L^{-1} (\chi_u - \chi_v) \\ &= B(uv, \cdot) L^{-1} B(uv, \cdot)^T \end{aligned}}$$

An algebraic expression for R_{eff}

$$\text{Let } \Pi = BL^{-1}B^T.$$

Then

$$\begin{aligned} R_{\mathsf{eff}}(e) &= B(e,\cdot)L^{-1}B(e,\cdot)^T \\ &= BL^{-1}B^T(e,e). \end{aligned}$$

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$$\boxed{\Pi\Pi = \Pi}$$

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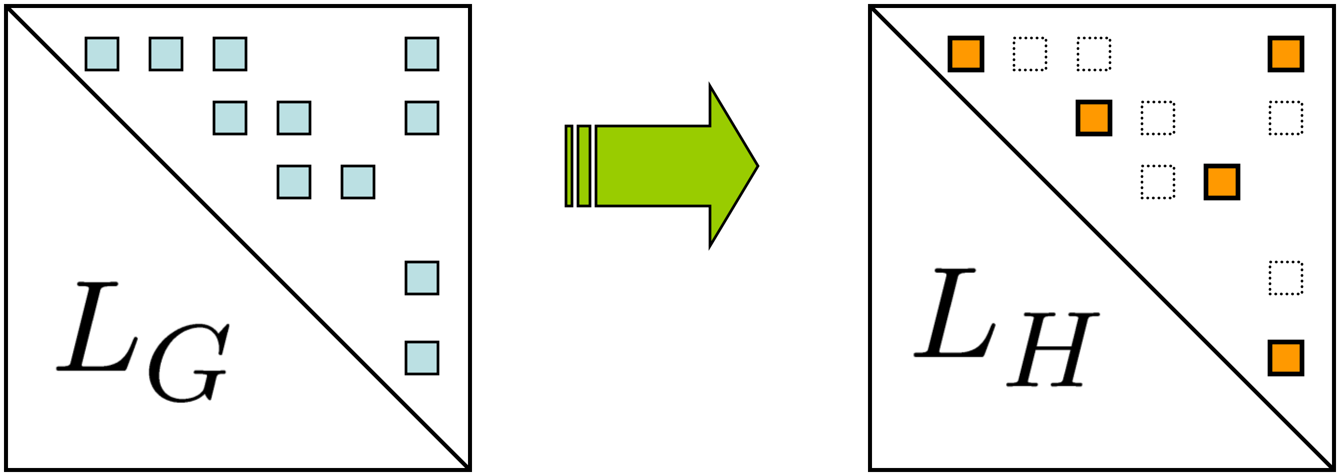
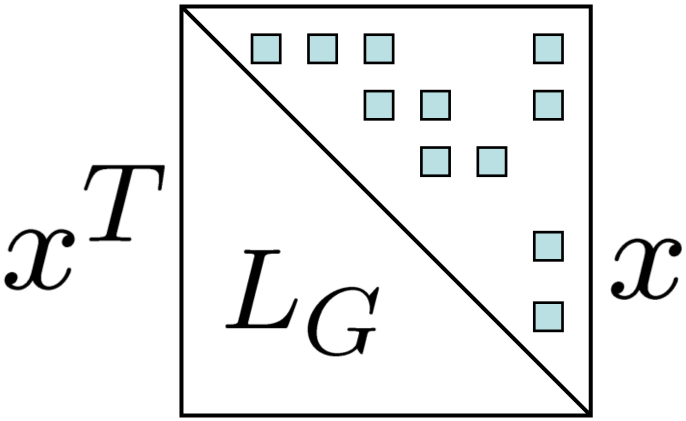
Reduce thm. to statement about Π

Goal



Want



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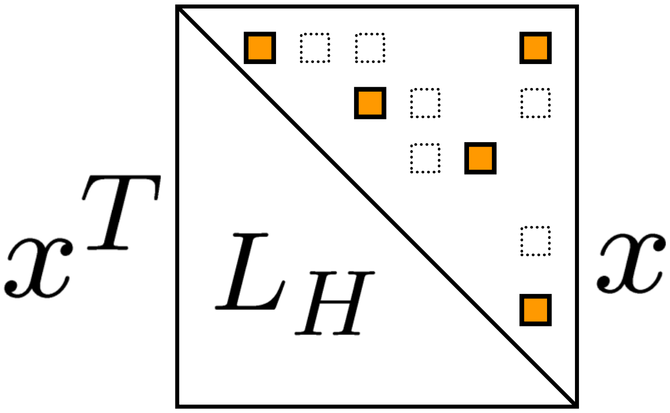
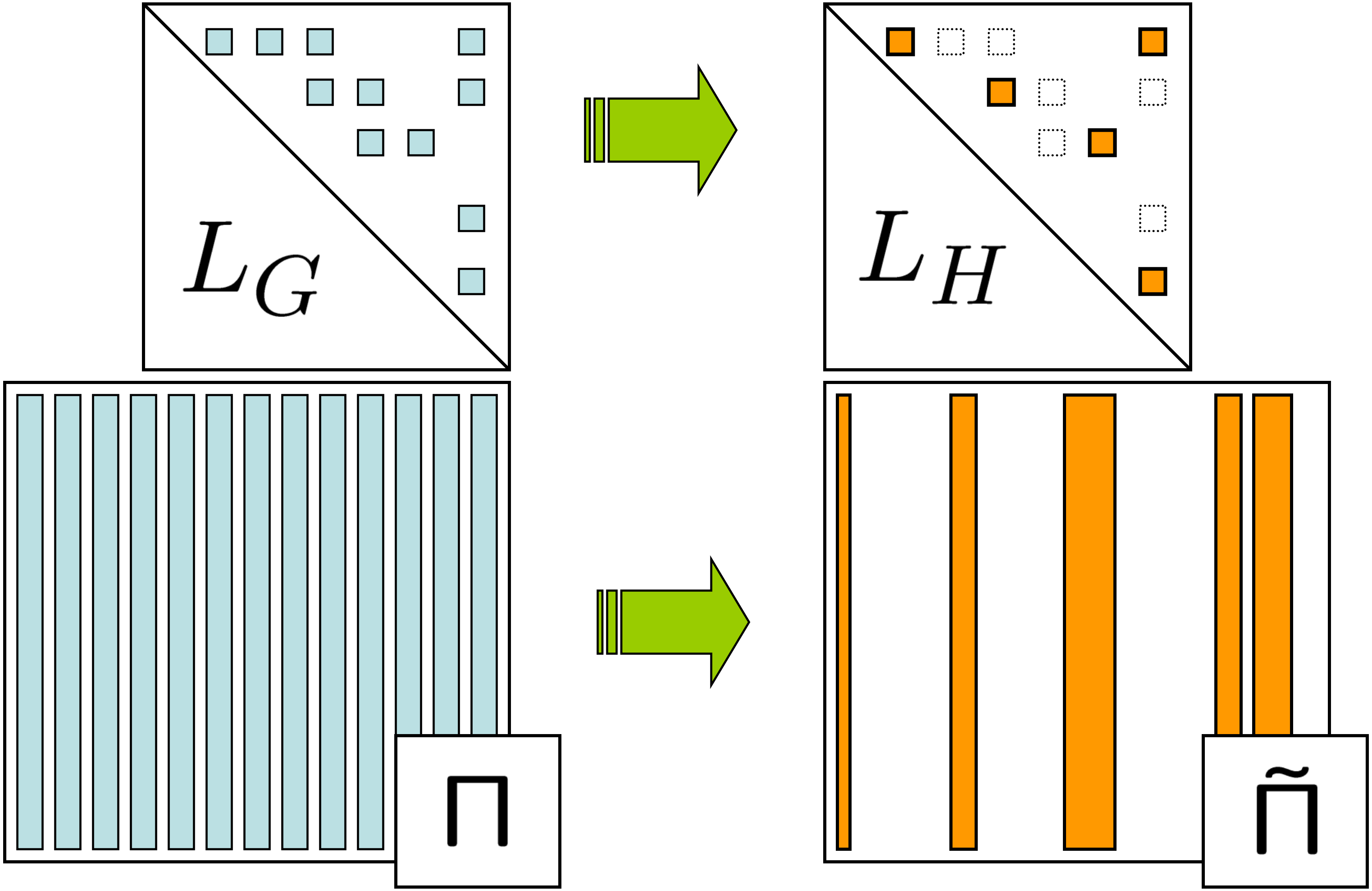


Sampling in Π



Reduction to Π

Lemma.

$$1 - \epsilon \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon \quad \forall x \in \mathbb{R}^n$$

$$\iff \|\tilde{\Pi}\tilde{\Pi}^T - \Pi\Pi^T\|_2 \leq \epsilon$$

New Goal

Lemma.

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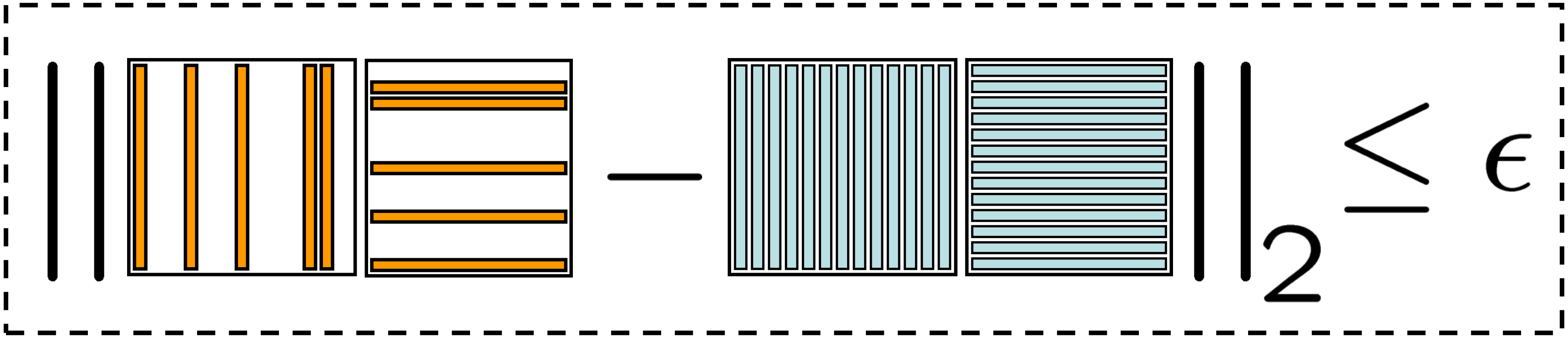

The Algorithm

Sample edges of ***G*** with probability

$$p_e \propto R_{\text{eff}}(e)$$

If chosen, include in ***H*** with weight $\frac{1}{p_e}$

Take q=O(nlogn/ε^2) samples with replacement

Divide all weights by *q*.

The Algorithm

Sample columns of Π with probability

$$p_e \propto R_{\text{eff}}(e)$$

If chosen, include in $\tilde{\Pi}$ with weight $\frac{1}{p_e}$

Take $q=O(n\log n/\varepsilon^2)$ samples with replacement

Divide all weights by *q*.

The Algorithm

Sample columns of Π with probability

$$p_e \propto \Pi(e,e)$$

If chosen, include in $\tilde{\Pi}$ with weight $\frac{1}{p_e}$

Take q=O(nlogn/ε^2) samples with replacement

Divide all weights by *q*.

The Algorithm

Sample columns of Π with probability

$$p_e \propto \Pi(e,e) = \|\Pi(\cdot,e)\|^2$$

If chosen, include in $\tilde{\Pi}$ with weight $\frac{1}{p_e}$

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$$\Pi^T\Pi = \Pi$$

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Di

cf. low-rank approx.
[FKV04,RV07]

$$\Pi^T \Pi = \Pi$$

A Concentration Result

Lemma.(Rudelson ’99)

If we sample $n \log n/\epsilon^2$ cols of Π with $p_e \propto \|\Pi(\cdot, e)\|^2$, then

$$\mathbb{E}\|\tilde{\Pi}\tilde{\Pi} - \Pi\Pi\|_2 \leq \epsilon.$$

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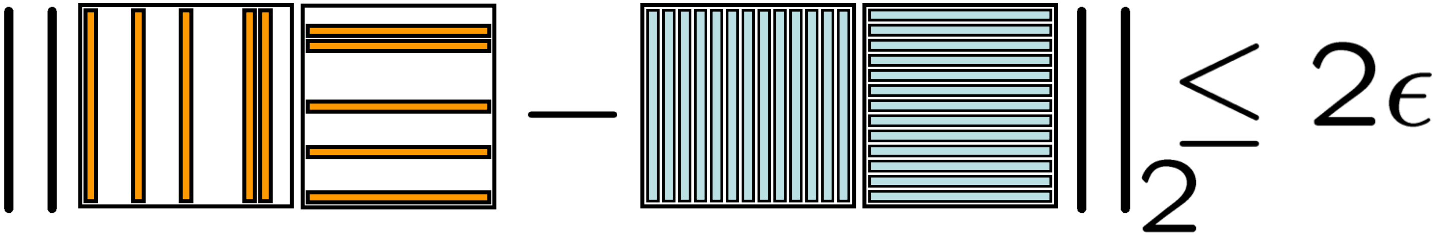
$$\mathbb{E}\|\tilde{\Pi}\tilde{\Pi} - \Pi\Pi\|_2 \leq \epsilon.$$

So with prob. ½:



$$\|\ \ -\ \ \|_2 \leq 2\epsilon$$

A Concentration Result

Lemma. (Rudelson '99)
If we sample $n \log n/\epsilon^2$ cols of Π with $p_e \propto \|\Pi(\cdot, e)\|^2$, then

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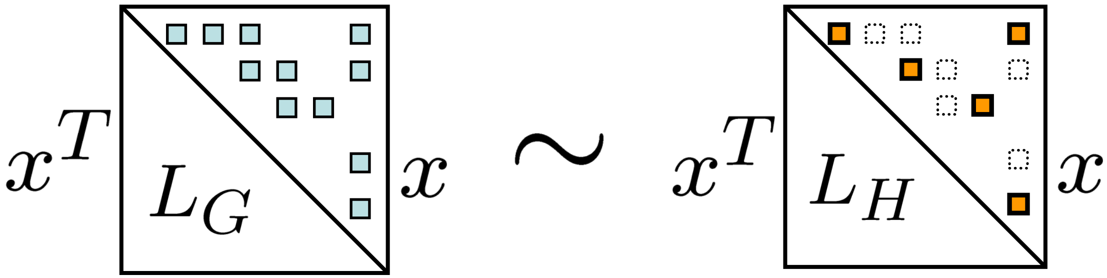

□

Nearly Linear Time

The Algorithm

Sample edges of G with probability

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Divide all weights by *q*.

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Divide all weights by *q*.

Nearly Linear Time

$$R_{\mathsf{eff}}(uv) = \|BL^{-1}(\chi_u - \chi_v)\|_{\mathsf{2}}^{\mathsf{2}}$$

Nearly Linear Time

$$\boxed{R_{\text{eff}}(uv) = \|BL^{-1}(\chi_u - \chi_v)\|_2^2}$$

So care about distances between cols. of $\boldsymbol{BL^{-1}}$

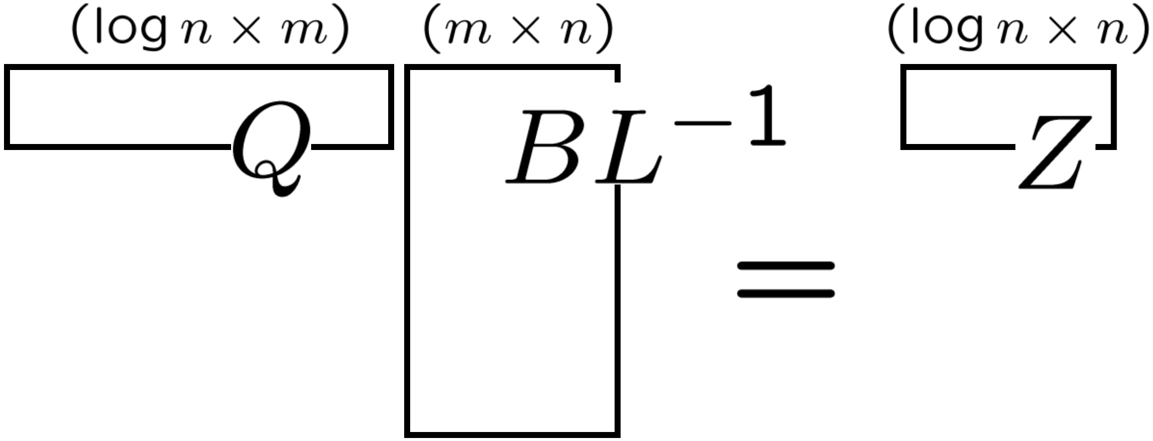
Nearly Linear Time

$$\boxed{R_{\text{eff}}(uv) = \|BL^{-1}(\chi_u - \chi_v)\|_2^2}$$

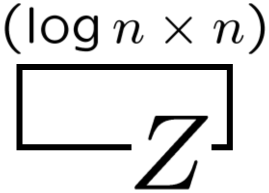
So care about distances between cols. of ***BL*** $^{-1}$

Johnson-Lindenstrauss! Take random $\boldsymbol{Q}_{\boldsymbol{logn£\ m}}$

Set ***Z=QBL*** $^{-1}$

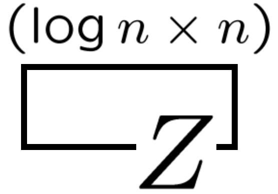


Nearly Linear Time



$$R_{\mathsf{eff}}(uv) \sim \|Z(\chi_u - \chi_v)\|^2$$

Nearly Linear Time



Find **rows** of $\boldsymbol{Z}_{\boldsymbol{\log n£ n}}$ by

$\boldsymbol{Z=QBL^{-1}}$

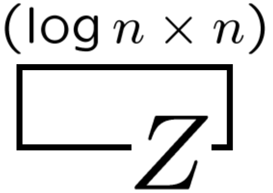
$$R_{\mathsf{eff}}(uv) \sim \|Z(\chi_u - \chi_v)\|^2$$

$\boldsymbol{ZL=QB}$

$\boldsymbol{z_iL=(QB)_i}$

Nearly Linear Time

Find **rows** of $Z_{log\ n£\ n}$ by



$Z=QBL^{-1}$

$ZL=QB$

$z_iL=(QB)_i$

$$\boxed{R_{\text{eff}}(uv) \sim \|Z(\chi_u - \chi_v)\|^2}$$

Solve ***O(logn)*** linear systems in ***L*** using Spielman-Teng ’04 solver

which uses combinatorial O(nlog^{c}n) sparsifier.

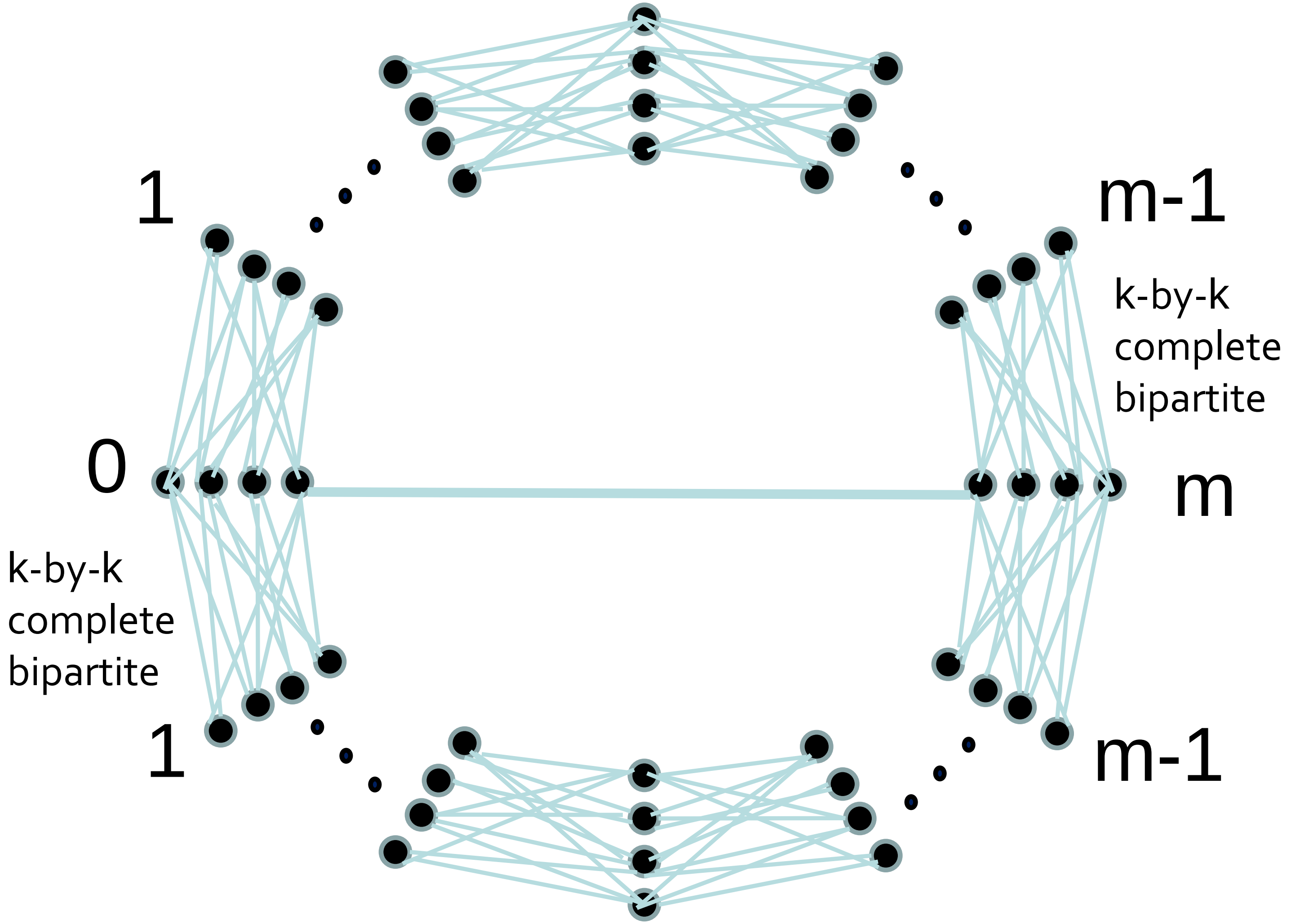
Can show approximate R_{eff} suffice.

□

Main Conjecture

Sparsifiers with O(n) edges.

Example: Another edge to include $(k^2 < m)$



$$x^T L_G x = m^2 + 2mk^2$$

The Projection Matrix

Lemma.

1. Π is a projection matrix

$$\begin{aligned} \Pi\Pi &= B^T L^+ B^T B L^+ B^T \\ &= B L^+ L L^+ B^T \\ &= B L^+ B^T \end{aligned}$$

2. im(Π)=im(B)
3. Tr(Π)=n-1
4. Π(e,e)=||Π(e,-)||^2

Last Steps

$$\Pi S \Pi = \sum_e S(e,e) \Pi_e \Pi_e^T$$

Last Steps

$$\begin{aligned}\Pi S \Pi &= \sum_e S(e,e)\Pi_e \Pi_e^T \\ &= \sum_e \frac{(\#\ \text{times } e \text{ sampled})}{q p_e}\Pi_e \Pi_e^T\end{aligned}$$

Last Steps

$$\Pi S \Pi = \sum_e S(e,e) \Pi_e \Pi_e^T$$

$$= \sum_e \frac{(\#\text{ times } e \text{ sampled})}{q p_e} \Pi_e \Pi_e^T$$

$$= \frac{1}{q} \sum_e (\#\text{ times } e \text{ sampled}) \frac{\Pi_e}{\sqrt{p_e}} \frac{\Pi_e^T}{\sqrt{p_e}} = \frac{1}{q} \sum_{i=1}^{q} y_i y_i^T$$

for y_i drawn i.i.d. from

$$y = \frac{\Pi_e}{\sqrt{p_e}} \quad \text{with prob. } p_e$$

Last Steps

$$\Pi S \Pi = \sum_e S(e,e) \Pi_e \Pi_e^T$$

$$= \sum_e \frac{(\#\text{ times } e \text{ sampled})}{q p_e} \Pi_e \Pi_e^T$$

$$= \frac{1}{q} \sum_e (\#\text{ times } e \text{ sampled}) \frac{\Pi_e}{\sqrt{p_e}} \frac{\Pi_e^T}{\sqrt{p_e}} = \frac{1}{q} \sum_{i=1}^{q} y_i y_i^T$$

for y_i drawn i.i.d. from

$$y = \frac{\Pi_e}{\sqrt{p_e}} \quad \text{with prob. } p_e = \frac{R_{\text{eff}}(e)}{n-1}.$$

since $\sum_e R_{\text{eff}}(e) = Tr(\Pi) = n - 1$.

Last Steps

We also have

$$\mathbb{E} y y^T = \sum_e p_e \frac{1}{p_e} \Pi_e \Pi_e^T = \Pi \Pi = \Pi$$

and

$$\|y\| = \frac{1}{\sqrt{p_e}} \|\Pi_e\| = \sqrt{\frac{n-1}{R_{\mathsf{eff}}(e)}} \sqrt{R_{\mathsf{eff}}(e)}$$

since $\|\Pi_e\|^2 = \Pi(e,e)$.

Reduction to Π

Goal: $1 - \epsilon \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon \quad \forall x \in \mathbb{R}^n$

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$$\boxed{\|\Pi S \Pi - \Pi \Pi\|_2 \leq \epsilon}$$

Reduction to Π

Lemma.

$$1 - \epsilon \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon \quad \forall x \in \mathbb{R}^n$$

$$\iff \|\tilde{\Pi}\tilde{\Pi}^T - \Pi\Pi^T\|_2 \leq \epsilon$$

Proof. Π is the projection onto im(B).

□