FPTAS for Computing a Symmetric Leontief Competitive Economy Equilibrium

Yinyu Ye

Department of Management Science and Engineering, and
Institute of Computational and Mathematical Engineering
Stanford University

Joint work with Chuangyin Dang and Zhisu Zhu

July 18, 2011

An Arrow-Debreu exchange market

- Each of n traders brings in one unit of a good and is equiped with a utility function on all goods;

An Arrow-Debreu exchange market

- Each of n traders brings in one unit of a good and is equiped with a utility function on all goods;
- They trade/exchange according to market prices.

An Arrow-Debreu exchange market

- Each of n traders brings in one unit of a good and is equiped with a utility function on all goods;
- They trade/exchange according to market prices.
- What would the prices and good allocations be?

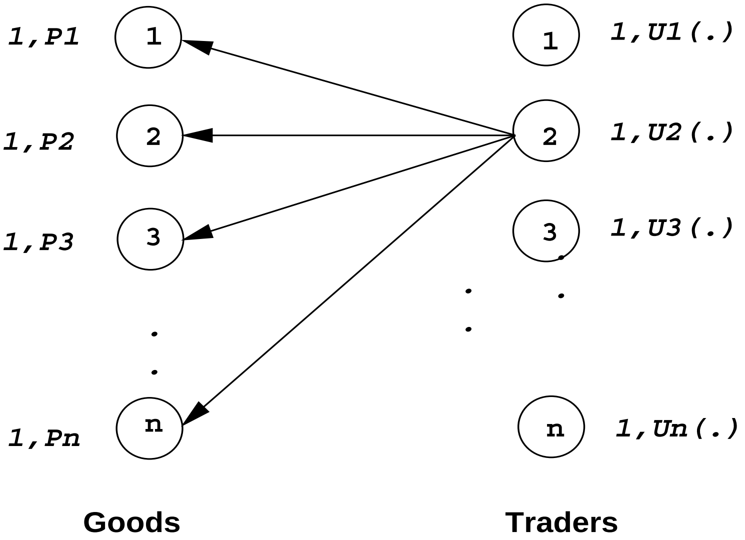

Figure: Pairing Exchange Market Model

Leontief economy

Leontief Utility:

$$u^j(\mathbf{x}_j) = \min_i \left\{ \frac{x_{ij}}{a_{ij}} \right\}$$

where a_{ij} represents the demand factor of trader j for the good of trader i ($\frac{*}{0} := \infty$).

Leontief economy

Leontief Utility:

$$u^j(\mathbf{x}_j) = \min_i \left\{ \frac{x_{ij}}{a_{ij}} \right\}$$

where a_{ij} represents the demand factor of trader j for the good of trader i ($\frac{*}{0} := \infty$).

Let the utility value for trader j be u_j. Then

$$x_{ij} = a_{ij} u_j, \ \forall i.$$

Leontief economy

Leontief Utility:

$$u^j(\mathbf{x}_j) = \min_i \left\{ \frac{x_{ij}}{a_{ij}} \right\}$$

where a_{ij} represents the demand factor of trader j for the good of trader i ($\frac{*}{0} := \infty$).

Let the utility value for trader j be u_j. Then

$$x_{ij} = a_{ij} u_j, \ \forall i.$$

Does the market has an equilibrium?

Market equilibrium principle I

Individual Rationality: Given market prices p_i for all i

$$\begin{array}{ll} \text{maximize} & u^j(\mathbf{x}_j) \\ \text{subject to} & \sum_i p_i x_{ij} \le p_j, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

where x_{ij} represents the amount of good i purchased by trader j.

Market equilibrium principle I

Individual Rationality: Given market prices p_i for all i

$$\begin{array}{ll} \text{maximize} & u^j(\mathbf{x}_j) \\ \text{subject to} & \sum_i p_i x_{ij} \leq p_j, \\ & x_{ij} \geq 0, \quad \forall j, \end{array}$$

where x_{ij} represents the amount of good i purchased by trader j.

Then at optimality it must be true $x^*_{ij} = a_{ij} u^*_j$ and

$$p_j = \sum_i p_i x^*_{ij} = u^*_j (\mathbf{a}_j^T \mathbf{p}) \ \forall j; \ \text{or}$$

Market equilibrium principle I

Individual Rationality: Given market prices p_i for all i

$$\begin{array}{ll} \text{maximize} & u^j(\mathbf{x}_j) \\ \text{subject to} & \sum_i p_i x_{ij} \le p_j, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

where x_{ij} represents the amount of good i purchased by trader j.

Then at optimality it must be true $x^*_{ij} = a_{ij} u^*_j$ and

$$p_j = \sum_i p_i x^*_{ij} = u^*_j (\mathbf{a}_j^T \mathbf{p}) \; \forall j; \text{ or}$$

$$U^* A^T \mathbf{p} = \mathbf{p}$$

where A is the the Leontief matrix formed by a_{ij}'s.

Market equilibrium principle II

Physical Constraint: The total purchase volume for good i should not exceed its available supply:

$$\sum_j a_{ij} u_j^* \leq 1 \ \forall i; \ \text{or}$$

Market equilibrium principle II

Physical Constraint: The total purchase volume for good i should not exceed its available supply:

$$\sum_j a_{ij} u_j^* \leq 1 \ \forall i; \text{ or}$$

$$A\mathbf{u}^* \leq \mathbf{e},$$

where $\mathbf{e}$ is the vector of all ones.

Market equilibrium principle III

Price "Cruelty" and "Fairness":

For every good i, $\sum_j a_{ij} u_j^* < 1$ implies $p_i = 0$;

Market equilibrium principle III

Price "Cruelty" and "Fairness":

For every good i, $\sum_j a_{ij} u_j^* < 1$ implies $p_i = 0$;

For every trader j, $\mathbf{a}_j^T \mathbf{p} > 0$.

Characterization of Leontief economy equilibrium I

At an equilibrium $\mathbf{u}^*, \mathbf{p}^*$, let $B = \{j : \; u_j^* > 0\}$ and the rest be N.

$$\mathbf{u}_B^* > \mathbf{0} \Longrightarrow \mathbf{p}_B^* > \mathbf{0} \Longrightarrow A_{BB}\mathbf{u}_B^* = \mathbf{e},$$

$$\mathbf{u}_N^* = \mathbf{0} \Longrightarrow \mathbf{p}_N^* = \mathbf{0} \Longrightarrow U_B^* A_{BB}^T \mathbf{p}_B^* = \mathbf{p}_B^* > \mathbf{0}.$$

Characterization of Leontief economy equilibrium I

At an equilibrium $\mathbf{u}^*, \mathbf{p}^*$, let $B = \{j : \; u_j^* > 0\}$ and the rest be N.

$$\mathbf{u}_B^* > \mathbf{0} \Longrightarrow \mathbf{p}_B^* > \mathbf{0} \Longrightarrow A_{BB}\mathbf{u}_B^* = \mathbf{e},$$

$$\mathbf{u}_N^* = \mathbf{0} \Longrightarrow \mathbf{p}_N^* = \mathbf{0} \Longrightarrow U_B^* A_{BB}^T \mathbf{p}_B^* = \mathbf{p}_B^* > \mathbf{0}.$$

Note that from the physical constraint $A_{NB}\mathbf{u}_B^* \leq \mathbf{e}$.

Characterization of Leontief economy equilibrium II

Theorem

(Y 2005) Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} be irreducible, and $\mathbf{u}_B$ satisfy

$$A_{BB}\mathbf{u}_B = \mathbf{e}, \quad A_{NB}\mathbf{u}_B \le \mathbf{e}, \quad \text{and} \quad \mathbf{u}_B > \mathbf{0}.$$

Then the (right) Perron-Frobenius eigenvector $\mathbf{p}_B$ of $U_B A_{BB}^T$ together with $\mathbf{u}_B, \mathbf{u}_N = \mathbf{p}_N = 0$ will be a Leontief economy equilibrium. And the converse is also true. Moreover, there is always a rational equilibrium for every such B, if the entries of A are rational. Furthermore, the size (bit-length) of the equilibrium is bounded polynomially by the size of A.

Leontief economy equilibrium and LCP

At a Leontief economy equilibrium, the utility value vector $\mathbf{u}$ is a solution of the linear complementarity system (LCP)

$$A\mathbf{u} + \mathbf{v} = \mathbf{e}, \;\; \mathbf{u}^T\mathbf{v} = 0, \;\; (\mathbf{u} \neq \mathbf{0}, \mathbf{v}) \geq \mathbf{0}.$$

Leontief economy equilibrium and LCP

At a Leontief economy equilibrium, the utility value vector $\mathbf{u}$ is a solution of the linear complementarity system (LCP)

$$A\mathbf{u}+\mathbf{v}=\mathbf{e},\ \ \mathbf{u}^T\mathbf{v}=0,\ \ (\mathbf{u}\neq\mathbf{0},\mathbf{v})\geq\mathbf{0}.$$

Note that $\mathbf{u}=\mathbf{0}$ and $\mathbf{v}=\mathbf{e}$ is a trivial complementary solution.

Relation to the Nash bimatrix game

Theorem

(Codenotti, Saberi, Varadarajan and Y 2005) Let (P,Q) *denote an arbitrary bimatrix game payoff matrix pair. Let*

$$A = \begin{pmatrix} \mathbf{0} & P \\ Q^T & \mathbf{0} \end{pmatrix}.$$

Then, there is a one-to-one correspondence between the Nash equilibria of the game (P,Q) *and the market equilibria of the Leontief economy described by Leontief matrix* A*.*

Symmetric Leontief economy I

That is $A = A^T$: "the demand factor of me from you is as the same as the demand factor of you from me."

Theorem

Let A be a real symmetric matrix. Then, it is NP-complete to decide whether or not the LCP has a complementary solution such that $\mathbf{u} \neq \mathbf{0}$.

Symmetric Leontief economy I

That is $A = A^T$: "the demand factor of me from you is as the same as the demand factor of you from me."

Theorem

Let A be a real symmetric matrix. Then, it is NP-complete to decide whether or not the LCP has a complementary solution such that $\mathbf{u} \neq \mathbf{0}$.

Another question: given symmetric A, is it easy to compute one if the LCP is known to have a complementary solution?

Symmetric Leontief economy II

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Three isolated non-trivial complementary solutions.

$$\mathbf{u}^1 = (1/2;\ 0), \quad \mathbf{u}^2 = (0;\ 1/2), \quad \mathbf{u}^3 = (1/3;\ 1/3).$$

A social utility maximization

- In the following, we assume that $\mathbf{e}^T A\mathbf{e} > 0$.

A social utility maximization

- In the following, we assume that $\mathbf{e}^T A\mathbf{e} > 0$.
- We consider a quadratic "social" utility function $\mathbf{u}^T A\mathbf{u}$, which is maximized over the simplex $\{\mathbf{u} : \; \mathbf{e}^T\mathbf{u} = 1, \; \mathbf{u} \geq \mathbf{0}\}$.

A social utility maximization

- In the following, we assume that $\mathbf{e}^T A\mathbf{e} > 0$.
- We consider a quadratic "social" utility function $\mathbf{u}^T A\mathbf{u}$, which is maximized over the simplex $\{\mathbf{u} : \ \mathbf{e}^T\mathbf{u} = 1, \ \mathbf{u} \geq \mathbf{0}\}$.
- Every KKT point of the social optimization problem, when $\mathbf{u}^T A\mathbf{u} > 0$, is a (non-trivial) complementarity solution (upon to scaling) to the LCP.

A social utility maximization

- In the following, we assume that $\mathbf{e}^T A\mathbf{e} > 0$.
- We consider a quadratic "social" utility function $\mathbf{u}^T A\mathbf{u}$, which is maximized over the simplex $\{\mathbf{u} : \ \mathbf{e}^T\mathbf{u} = 1, \ \mathbf{u} \geq \mathbf{0}\}$.
- Every KKT point of the social optimization problem, when $\mathbf{u}^T A\mathbf{u} > 0$, is a (non-trivial) complementarity solution (upon to scaling) to the LCP.
- How to compute an KKT point with $\mathbf{u}^T A\mathbf{u} > 0$?

A social utility maximization

- In the following, we assume that $\mathbf{e}^T A\mathbf{e} > 0$.
- We consider a quadratic "social" utility function $\mathbf{u}^T A\mathbf{u}$, which is maximized over the simplex $\{\mathbf{u} : \ \mathbf{e}^T\mathbf{u} = 1, \ \mathbf{u} \geq \mathbf{0}\}$.
- Every KKT point of the social optimization problem, when $\mathbf{u}^T A\mathbf{u} > 0$, is a (non-trivial) complementarity solution (upon to scaling) to the LCP.
- How to compute an KKT point with $\mathbf{u}^T A\mathbf{u} > 0$?
- Y (1998) "On The Complexity of Approximating a KKT Point of Quadratic Programming"

An interior-point potential reduction algorithm

The Karmarkar-Tenabe-Todd-Y type potential function

$$\phi(\mathbf{u}) = \rho \log \left(\bar{a} - \mathbf{u}^T A \mathbf{u} \right) - \sum_{j=1}^{n} \log(u_j);$$

where $\rho = (2n + \sqrt{n})/\epsilon$ and $\bar{a} = \max_{i,j}\{a_{ij}\} > 0$.

An interior-point potential reduction algorithm

The Karmarkar-Tenabe-Todd-Y type potential function

$$\phi(\mathbf{u}) = \rho \log\left(\bar{a} - \mathbf{u}^T A\mathbf{u}\right) - \sum_{j=1}^{n} \log(u_j);$$

where $\rho = (2n + \sqrt{n})/\epsilon$ and $\bar{a} = \max_{i,j}\{a_{ij}\} > 0$.

This potential function will be reduced by a constant each iteration from the initial point $\mathbf{u}^0 = \frac{1}{n}\mathbf{e}$, and the algorithm terminates in $\mathcal{O}(n(\frac{1}{\epsilon})\log(\frac{1}{\epsilon}))$ iterations at an ϵ-approximate KKT point.

It's a FPTAS

Note that

$$\phi(\mathbf{u}^0) = \rho \log\left(\bar{a} - \frac{1}{n^2}\mathbf{e}^T A\mathbf{e}\right) + n\log(n),$$

and for any $\mathbf{u}$ in the interior of the simplex,

$$-\sum_{j=1}^{n} \log(u_j) \geq n\log(n).$$

Thus, $\phi(\mathbf{u}) \leq \phi(\mathbf{u}^0)$ implies that

$$\rho\log\left(\bar{a} - \mathbf{u}^T A\mathbf{u}\right) \leq \rho\log\left(\bar{a} - \frac{1}{n^2}\mathbf{e}^T A\mathbf{e}\right)$$

or

$$\mathbf{u}^T A\mathbf{u} \geq \frac{1}{n^2}\mathbf{e}^T A\mathbf{e} > 0.$$

That is, any KKT point $\mathbf{u}$ generated by the algorithm must have $\mathbf{u}^T A\mathbf{u} > 0$ so that it is nontrivial.

That is, any KKT point $\mathbf{u}$ generated by the algorithm must have $\mathbf{u}^T A\mathbf{u} > 0$ so that it is nontrivial.

Theorem

There is a FPTAS to compute an ϵ-approximate non-trivial complementary solution when A is symmetric and $\mathbf{e}^T A\mathbf{e} > 0$ in $\mathcal{O}(n(\frac{1}{\epsilon})\log(\frac{1}{\epsilon}))$ iterations, and each iteration uses $\mathcal{O}(n^3\log(\log(\frac{1}{\epsilon})))$ arithmetic operations.

Preliminary computational results

n	mean_sup	mean_iter	mean_time	max_sup
100	5.3	48.2	0.3	7
200	5.5	53.5	1.2	6
400	5.7	55.1	5.9	7
800	5.8	62.6	33.8	8
1000	6.3	65.0	60.2	7
1500	6.1	71.5	187.2	8
2000	5.9	73.5	411.9	7
2500	6.4	74.6	774.5	8
3000	6.2	78.7	1404.2	8

Table: Social optimization for symmetric uniform matrix LCP

Non-symmetric Leontief matrix?

In this case, even all entries of A being non-negative may not guarantee the existence of a non-trivial complementary solution:

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

Corollary

(Y 2005) The LCP always has a non-trivial complementary solution if A has no all-zero column.

Summaries and Open Problems

- No complexity bound for solving non-symmetric LCP, although (finitely) convergent algorithms do exist.

Summaries and Open Problems

- No complexity bound for solving non-symmetric LCP, although (finitely) convergent algorithms do exist.
- It seems "symmetry" helps computation efficiency !

Summaries and Open Problems

- No complexity bound for solving non-symmetric LCP, although (finitely) convergent algorithms do exist.
- It seems "symmetry" helps computation efficiency !
- Why?