ABSTRACT AND CLASSICAL HODGE/DE RHAM THEORY

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1 Introduction

We will discuss some results from two papers, the first [BSSS] "An Abstract Hodge Theory" is joint work with L. Bartholdi, T. Schick and S. Smale, and the second [SS] "Abstract and Classical Hodge-De Rham Theory is joint work with S. Smale.

In [BSSS], a version of Hodge and De Rham theory was proposed for metric spaces, which would describe the topology at a fixed scale. It was shown that the cohomology was isomorphic to the classical De Rham cohomology at small scales in the case of a Riemannian manifold M. In [SS], concrete chain maps were constructed between the classical differential forms on M and the chain complex constructed in [BSSS], which induce isomorphisms on cohomology.

2 Classical De Rham and Hodge Theory

Let M be a smooth, compact manifold of dimension n. We will denote by $\Omega^k(M)$, $k = 0, 1, \ldots, n$ the smooth differential k forms. Thus, $\Omega^0(M) = C^\infty(M)$ and for $x \in M$ and $\omega \in \Omega^k(M)$, $\omega(x)$ is an alternating k-linear function on the tangent space at x. Furthermore, let $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative. Then

$$d_{k+1} \circ d_k = 0$$

and we have the De Rham co-chain complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

Since $d^2 = 0$, $\operatorname{Im} d \subset \operatorname{Ker} d$ and we can form the quotient space, which is the De Rham cohomology:

$$H^k_{DR}(M) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}}$$

The De Rham theorem states that $H^k_{DR}(M)$ is a topological invariant of M and is in fact isomorphic to the singular cohomology of M.

Now suppose that M has a Riemannian metric $< \cdot, \cdot >_x$ (a smoothly varying inner product on $T_x M$). Then there is an induced inner product on the alternating k-tensors on $T_x M$, and thus on differential forms:

$$< \omega, \theta > = \int_M < \omega(x), \theta(x) >_x \, dx$$

for $\omega, \theta \in \Omega^k(M)$, where dx is the volume form on M. The exterior derivative

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

has a formal adjoint

$$d^* : \Omega^{k+1} \to \Omega^k(M)$$

also a first order linear differential operator. The Hodge Laplacian is the second order elliptic operator

$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M)$$

The classical Hodge theorem is

Hodge Theorem. *For $k = 0, \ldots, n$, we have the orthogonal direct sum decomposition*

$$\Omega^k(M) = Image\,(d) \oplus Image\,(d^*) \oplus Kernel\,\Delta$$

and $Kernel\,\Delta$ is isomorphic to $H^k_{DR}(M)$.

3 Abstract De Rham/Hodge Theory

In [BSSS] we propose, and partially develop an analogous Hodge-De Rham theory for metric spaces. Let

X, d be a compact metric space (even for example a finite data set). Also, let μ be a Borel probability measure on X. We will also fix a scale $\alpha > 0$. On the $k+1$ fold product of X

$$X^{k+1} = X \times \cdots \times X$$

we define the metric

$$d^{k+1}(x, y) = \max_{i=0,\ldots,k} d(x_i, y_i)$$

for $x, y \in X^{k+1}$. We denote by D^{k+1} the diagonal $\{(t, \ldots, t) : t \in X\}$ in X^{k+1}, and the α neighborhood of the diagonal

$$U_\alpha^{k+1} = \{x \in X^{k+1} : d^{k+1}(x, D^{k+1}) \leq \alpha\}$$

That is $x = (x_0, \ldots, x_k) \in U_\alpha^{k+1}$ if and only if there is a $t \in X$ such that $d(x_i, t) \leq \alpha$ for all $i = 0, \ldots, k$. Such a t is called a witness for x. The set of witnesses for x is called the witness set $w_\alpha(x)$, and $x \in U_\alpha^{k+1}$ precisely when

$$w_\alpha(x) = \cap_{i=0}^{k} B_\alpha(x_i) \neq \emptyset$$

The product measure μ^{k+1} induces a measure on U_α^{k+1} and the basic spaces of co-chains (analogous to k-forms) are the alternating real valued L^2 functions on U_α^{k+1}, $L_a^2(U_\alpha^{k+1})$. We define the co-boundary operator

$$\delta : L_a^2(U_\alpha^k) \to L_a^2(U_\alpha^{k+1})$$

by

$$\delta f(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^{i+1} f(x_0, \ldots, \hat{x}_i, \ldots, x_k)$$

Proposition. *The operator*

$$\delta : L_a^2(U_\alpha^k) \to L_a^2(U_\alpha^{k+1})$$

is a bounded operator of Hilbert spaces, and $\delta \circ \delta = 0$.

The following co-chain complex is analogous to the De Rham complex

$$0 \to L^2(X) \xrightarrow{\delta} L_a^2(U_\alpha^2) \xrightarrow{\delta} \cdots \xrightarrow{\delta} L_a^k(U_\alpha^2) \xrightarrow{\delta} \cdots$$

and we denote the corresponding cohomology by

$$H^k_{L^2,\alpha} = \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta}$$

This can be thought of as a cohomology at scale α. From the above proposition, δ has a bounded adoint

$$\delta^* : L^2_a(U^{k+1}_\alpha) \to L^2_a(U^k_\alpha)$$

and one can show that is given by

$$\delta^* f(x_0, \dots, x_{k-1}) = (k{+}2) \int_{S_{x_0,\dots,x_{k-1}}} f(t, x_0, \dots, x_{k-1})\, dt$$

where

$$S_{x_0,\dots,x_{k-1}} = \{t \in X : (t, x_0, \dots, x_{k-1}) \in U^{k+1}_\alpha\}$$

The corresponding Hodge operator at scale α

$$\Delta_{k,\alpha} = \delta\delta^* + \delta^*\delta : L^2(U^{k+1}_\alpha) \to L^2(U^{k+1}_\alpha)$$

is a bounded, self adjoint positive operator of Hilbert spaces. A natural question in this context is

Hodge Question at Scale α. *Under what conditions on X, d, μ, α do we have*

$$L^2(U_\alpha^{k+1}) = Im\,\delta \oplus Im\,\delta^* \oplus Ker\,\Delta_{k,\alpha}$$

and $Ker\,\Delta_{k,\alpha}$ is isomorphic to $H^k_{L^2,\alpha}$.

In [BSSS] some sufficient conditions are given on α and d. Roughly, the witness set

$$w_\alpha : U_\alpha^{k+1} \to \mathcal{K}(X)$$

must be continuous (here $\mathcal{K}(X)$ is the metric space of compact subsets of X with the Hausdorff metric), and the radius of intersections of α balls must be controlled. As a special case, it is shown to be consistent with the classical Hodge/De rham theory for Riemannian manifolds at small scales.

Theorem. *Let X, g be a compact Riemannian manifold. Then for $\alpha > 0$ sufficiently small, the answer to the Hodge question above is affirmative, and furthermore $Ker\,\Delta_{k,\alpha}$ is isomorphic to $H^k_{DR}(X)$ as well as $H^k_{L^2,\alpha}$.*

The proof is a bit lengthy and is carried out using a bi-complex argument.

4 An Explicit Isomorphism

The proof of the theorem in [BSSS] does not give an explicit isomorphism between $H^k_{DR}(X)$ and $H^k_{L^2,\alpha}$. In [SS], we construct a a co-chain map (in the case of a Riemannian manifold and small α) between the De Rham complex and the L^2, α complex which induces isomorphisms on cohomology. Let M be a compact Riemannian manifold, and let $\alpha > 0$ be small enough so that closed balls of radius 2α are strictly convex. We construct a co-chain map, that is for each k, a linear map

$$\Psi : \Omega^k(M) \to L^2_a(U^{k+1}_\alpha)$$

such that

$$\Psi \circ d = \delta \circ \Psi$$

For $(x_0, \ldots, x_k) \in U^{k+1}_\alpha$ we define a smooth k-simplex $S(x_0, \ldots, x_k)$ in M inductively on k. $S(x_0, x_1)$ is just the minimizing geodesic from x_0 to x_1. $S(x_0, x_1, x_2)$ is the union of geodesics from x_2 to points on $S(x_0, x_1)$, and so on. We then define

$$\Psi_0 : \Omega^k(M) \to L^2(U_\alpha^{k+1})$$

by

$$(\Psi_0\omega)(x_0, \ldots, x_k) = \int_{S(x_0,\ldots,x_k)} \omega$$

In general, $\Psi_0\omega$ will not be alternating, unless $k = 0, 1$ or M has constant curvature. We therefore alternate $\Psi_0\omega$ and define

$$\Psi\omega(x_0, \ldots, x_k) = \mathrm{Alt}\,(\Psi_0\omega)(x_0, \ldots, x_k)$$

Theorem. Ψ *is a co-chain map of co-chain complexes, and induces an isomorphism on cohomology.*

The proof that Ψ is a co-chain map is essentially Stoke's theorem. The proof that Ψ is an isomorphism on cohomology follows from essentially constructing a left inverse for Ψ, and using the fact from[BSSS] that the cohomology groups have the same dimension. To describe the left inverse, which we will call Φ, note that $\Psi\omega$ is actually a smooth alternating function. That is, Ψ is really a co-chain map into the sub-complex

$$0 \to C^\infty(X) \xrightarrow{\delta} C^\infty_a(U^2_\alpha) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^\infty_a(U^k_\alpha) \xrightarrow{\delta} \cdots$$

In [BSSS] it was shown that the inclusion map from this complex to the L^2 complex induces an isomorphism on cohomology, thus it suffices to define the left inverse Φ on smooth alternating functions. In fact, if we define $\Phi : C^\infty_a(U^{k+1}_\alpha) \to \Omega^k(M)$ by

$$(\Phi f)(p)(v_1, \ldots, v_k) = D^k f(p, t_1, \ldots, t_k)(v_1, \ldots, v_k)$$

for $p \in M$ and $v_1, \ldots, v_k \in T_pM$ (derivatives taken at $t_i = p$), then it can be shown that $\Phi\Psi = \mathrm{Id}$.

5 Further results

It is shown that in general, for a harmonic 1 form ω, $\Psi\omega$ is harmonic in the abstract sense, and so Ψ is an isomorhism between harmonic 1 forms, and harmonic functions on U_α^2. For the flat n-dimensional torus, Ψ takes harmonic k-forms to harmonic functions on U_α^{k+1} for all k. We conjecture that this might be true for constant curvature in general.

For $k = 0$, we can compare the classical and abstract Hodge Laplacian since both act on functions on M. When appropriately scaled, the α-Laplacian is close to the classical Laplacian.

Theorem. *There is a universal constant c_n such that for a C^3 function f on M, we have*

$$\|\Delta f - c_n \alpha^{-n-2} \Delta_\alpha f\|_\infty \leq C \|f\|_{C^3} \alpha$$