Flows and decomposition of games: harmonic and potential games

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ICIAM - July 2011

Motivation

- Potential games are games in which preferences of all players are aligned with a global objective.
  - easy to analyze
  - pure Nash equilibrium exists
  - simple dynamics converge to an equilibrium
- How "close" is a game to a potential game?
- What is the topology of the space of preferences?
- Are there "natural" decompositions of games?

Potential Games

- We consider finite games in strategic form:

$$\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}} \rangle$$

- $\mathcal{G}$ is an exact potential game if $\exists\, \Phi : E \to \mathbb{R}$ such that

$$u^m(x^m, x^{-m}) - u^m(y^m, x^{-m}) = \Phi(x^m, x^{-m}) - \Phi(y^m, x^{-m})$$

- Weaker notion: ordinal potential game, if the utility differences above agree only in sign.
- Potential Φ aggregates and explains incentives of all players.
- Examples: congestion games, etc.

Potential Games

- A global maximum of an ordinal potential is a pure Nash equilibrium.
- Every finite potential game has a pure equilibrium.
- Many learning dynamics (e.g., better-reply dynamics, fictitious play, spatial adaptive play) "converge" to a pure Nash equilibrium [Monderer and Shapley 96], [Young 98], [Hofbauer, Sandholm 00], [Marden, Arslan, Shamma 06, 07].

Potential Games

- When is a given game a potential game?
- More important, what are the obstructions, and what is the underlying structure?

Existence of Exact Potential

A path is a collection of strategy profiles $\gamma = (x_0, \ldots, x_N)$ such that x_i and x_{i+1} differ in the strategy of exactly one player where $x_i \in E$ for $i \in \{0, 1, \ldots, N\}$. For any path γ, let

$$I(\gamma) = \sum_{i=1}^{N} u^{m_i}(x_i) - u^{m_i}(x_{i-1}),$$

where m_i denotes the player changing its strategy in the ith step.

Theorem ([Monderer and Shapley 96])

A game $\mathcal{G}$ is an exact potential game iff for all simple closed paths γ, $I(\gamma) = 0$. Moreover, it is sufficient to check closed paths of length 4.

A linear condition, thus the set of exact potential games is a *subspace*.

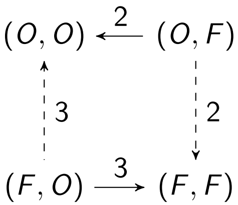
Game Flows

A key reformulation: instead of utilities, a flow on a graph

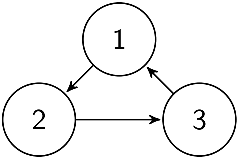
- Nodes are strategy profiles
- Edges between comparable strategy profiles
- Labeled by utility differences
- Isomorphic to a direct product of M cliques (one per player)
- E.g., for (modified) battle-of-the-sexes:

	O	F
O	4, 2	0, 0
F	1, 0	2, 3

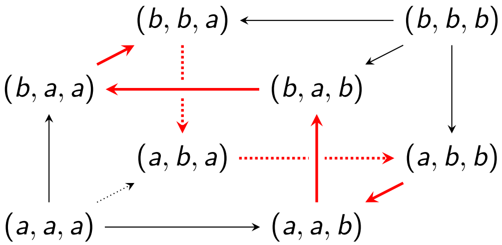
$\Longrightarrow$



Game Flows: 3-Player Example



- $E^m = \{a, b\}$ for all $m \in \mathcal{M}$, and payoff of player i be -1 if its strategy is the same with its successor, 0 otherwise.
- This game is neither an exact nor an ordinal potential game.



Global Structure of Preferences

- What is the global structure of these cycles?
- Equivalently, topological structure of aggregated preferences.
- Conceptually similar to structure of (continuous) vector fields.
- A well-developed theory from algebraic topology, we need the combinatorial analogue (e.g., [Jiang-Lim-Yao-Ye 08])

Helmholtz (Hodge) Decomposition

The Helmholtz Decomposition allows orthogonal decomposition of a vector field into three vector fields:

- Gradient flow (globally acyclic component)
- Harmonic flow (locally acyclic but globally cyclic component)
- Curl flow (locally cyclic component).

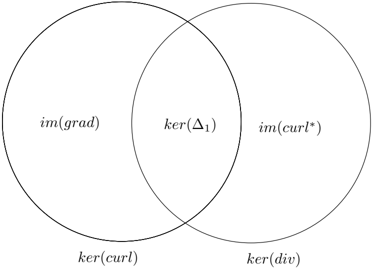
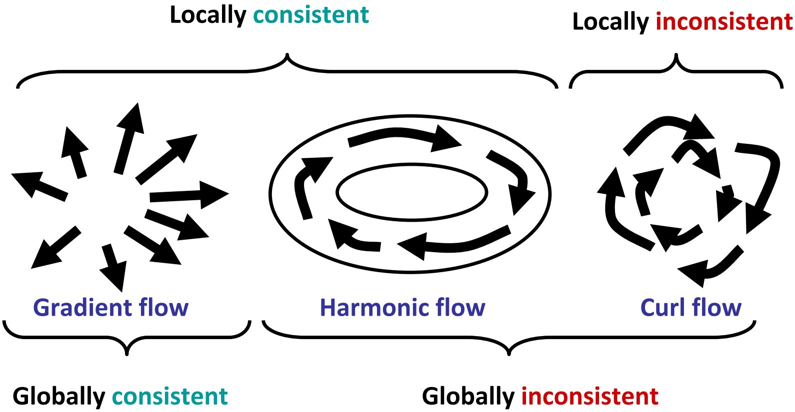
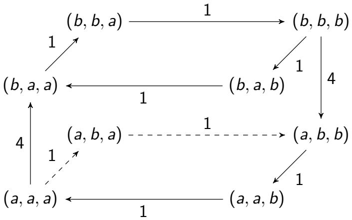


Figure: Helmholtz Decomposition

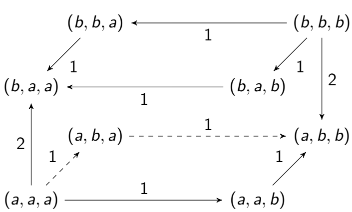
Helmholtz decomposition (a cartoon)



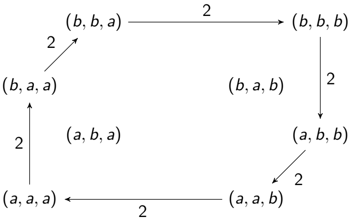
Decomposition example



(a) Original game.

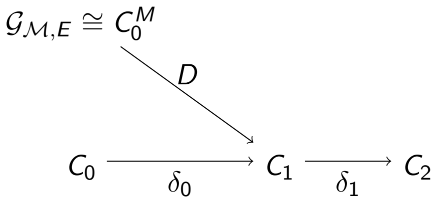


(b) Potential Component.



(c) Harmonic Component.

Decomposition



Pull-back through D the Helmholtz decomposition of the flows (C_1):

$$\begin{aligned}
\mathcal{P} &\triangleq \left\{ u \in C_0^M \mid u = \Pi u \text{ and } Du \in \text{im } \delta_0 \right\} \\
\mathcal{H} &\triangleq \left\{ u \in C_0^M \mid u = \Pi u \text{ and } Du \in \ker \delta_0^* \right\} \\
\mathcal{N} &\triangleq \left\{ u \in C_0^M \mid u \in \ker D \right\}.
\end{aligned}$$

where $\Pi = D^\dagger D$.

Decomposition: Potential, Harmonic, and Nonstrategic

Decomposition of game flows induces a similar partition of the space of games:

- When going from utilities to flows, the nonstrategic component is removed.
- If we start from utilities (not preferences), always locally consistent.
- Therefore, only two flow components: potential and harmonic

Thus, the space of games has a canonical direct sum decomposition:

$$\underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{\text{Potential games}}\!\!\!\overbrace{\quad \oplus \quad \mathcal{H}}^{\text{Harmonic games}}$$

where the components are orthogonal subspaces.

Bimatrix games

For two-player games, simple explicit formulas.
Assume the game is given by matrices (A, B), and (for simplicity), the non-strategic component is zero (i.e., $\mathbf{1}^T A = 0, B\mathbf{1} = 0$). Define

$$S := \frac{1}{2}(A + B), \quad D := \frac{1}{2}(A - B), \quad \Gamma := \frac{1}{2n}(A\mathbf{1}\mathbf{1}^T - \mathbf{1}\mathbf{1}^T B).$$

- Potential component:

$$(S + \Gamma, \quad S - \Gamma)$$

- Harmonic component:

$$(D - \Gamma, \quad -D + \Gamma)$$

Notice that the harmonic component is zero sum.

Harmonic games

Very different properties than potential games.
Agreement between players is never a possibility!

- Simple examples: rock-paper-scissors, cyclic games, etc.
- Essentially, sums of cycles.
- Generically, *never* have pure Nash equilibria.
- Uniformly mixed profile (for all players) is mixed Nash.

Other interesting static and dynamic properties (e.g., correlated equilibria, best-response dynamics, etc.)

Potential vs. harmonic

	Potential Games	Harmonic Games
Subspaces	$\mathcal{P} \oplus \mathcal{N}$	$\mathcal{H} \oplus \mathcal{N}$
Flows	Globally consistent	Locally consistent but globally inconsistent
Pure NE	Always exists	Generically does not exist
Mixed NE	Always exists	- Uniformly mixed strategy is always a mixed NE -Players do not strictly prefer their equilibrium strategies.
Special cases		-(two players) Set of mixed Nash equilibria coincides with the set of correlated equilibria -(two players & equal number of strategies) Uniformly mixed strategy is the unique mixed NE

Consequences

Nice and beautiful. But (if that's not enough!) why should we care?

- Provides classes of games with simpler structures, for which stronger results can be proved.
- Yields natural mechanisms for approximation, for both static and dynamical properties.

Let's see this...

Projection onto the Set of Exact Potential Games

- Since the set of exact of exact potential games is a subspace, can easily find "closest" potential game $\hat{\mathcal{G}}$ to a given game $\mathcal{G}$:

$$\hat{\mathcal{G}} := \arg\min_{h \in \mathcal{H}} ||\mathcal{G} - h||$$

- For L_2-type distances, closed-form expressions, in terms of a Laplacian-like operator.

Equilibria of a Game and its Projection

Theorem

Let $\mathcal{G}$ be a game and $\hat{\mathcal{G}}$ be its projection. Any equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ for some $\epsilon \leq \sqrt{2} \cdot d(\mathcal{G})$ (and viceversa).

- If projection distance is small, equilibria of the projected game are "close" to the equilibria of the initial game.
- Thus, near-potential games have pure ϵ-equilibria
- Similar results for dynamics: for "near-potential" games, natural game dynamics will converge to "near-equilibria".

Summary

- Analysis of the global structure of preferences
- Decomposition: nonstrategic, potential and harmonic components
- Projection to "closest" potential game
- Preserves ϵ-approximate equilibria and dynamics
- Enables extension of many tools to non-potential games

Want to know more?

- Candogan, Menache, Ozdaglar, P., "Flow representations of games: harmonic and potential games," *Math. of OR*, to appear. `arXiv:1005.2405`.
- Candogan, Menache, Ozdaglar, P., "Near-optimal power control in wireless networks: a potential game approach," INFOCOM 2010.
- Candogan, Ozdaglar, P., "Dynamics in near-potential games," in preparation.