Numerical Methods for Hodge Decomposition

Anil N. Hirani *
K. Kalyanaraman, H. Wang, S. Watts
(Some with A. Demlow)

* Department of Computer Science
University of Illinois at Urbana-Champaign

Applied Hodge Theory Minisymposium
ICIAM 2011, Vancouver
Tuesday, July 19, 2011

Triangle meshes: 1-cochains

Tetrahedral meshes: 1- and 2-cochains

Manifolds: Differential forms and vector fields

Graphs: 1-cochains

$$\text{gradients} \oplus \text{curls} \oplus \text{harmonics}$$

Computing Hodge Decomposition:

> Find two and infer third by subtraction
> Curl and harmonic part are harder
> Find gradient part, and curl OR harmonic part

Elementary in principle, but in practice:

> Functional Analysis
> Numerical Linear Algebra
> Computational Geometry
> Computational Topology

are all connected to the problem.

Functional Analysis :

Convergence and stability

Numerical Linear Algebra :

Least squares for theory and practice

On graphs conjugate gradient beats multigrid

Computational Geometry :

Meshing affects choice of method

Mesh properties for near-optimal solve time

Computational Topology :

Use (co)homology basis for harmonics

Experiments on clique complexes

1-norm and linear programming

Functional Analysis :

Convergence and stability

Numerical Linear Algebra :

Least squares for theory and practice

On graphs conjugate gradient beats multigrid

Computational Geometry :

Meshing affects choice of method

Mesh properties for near-optimal solve time

Computational Topology :

Use (co)homology basis for harmonics

Experiments on clique complexes

1-norm and linear programming

Functional Analysis :

Convergence and stability

Numerical Linear Algebra :

Least squares for theory and practice

On graphs conjugate gradient beats multigrid

Computational Geometry :

Meshing affects choice of method

Mesh properties for near-optimal solve time

Computational Topology :

Use (co)homology basis for harmonics

Experiments on clique complexes

1-norm and linear programming

Only the highlighted topics will be discussed

Hodge Decomposition in Finite Dimensions

A pattern that appears in Hodge decomposition on graphs and meshes

U, V, W finite-dimensional inner product spaces

$$\begin{array}{ccccc} U & \xrightarrow{A} & V & \xrightarrow{B} & W \\ \downarrow & & \downarrow & & \downarrow \\ U^* & \xleftarrow{A^T} & V^* & \xleftarrow{B^T} & W^* \end{array} \qquad B \circ A = 0$$

$$V = \operatorname{im} A \oplus \operatorname{im} A^\perp = \operatorname{im} A \oplus \ker A^T$$

$$V = \operatorname{im} A \oplus \operatorname{im} B^T \oplus (\ker B \cap \ker A^T)$$

$$V = \operatorname{im} A \oplus \operatorname{im} B^T \oplus \ker \Delta$$

$$\Delta = A A^T + B^T B$$

Laplace-deRham Operators on Graphs

Chain and cochain complexes on 2-dimensional clique complex of graph G

$$
\begin{array}{ccccc}
C^0(G) & \xrightarrow{\partial_1^T} & C^1(G) & \xrightarrow{\partial_2^T} & C^2(G) \\
\downarrow & & \downarrow & & \downarrow \\
C_0(G) & \xleftarrow{\partial_1} & C_1(G) & \xleftarrow{\partial_2} & C_2(G)
\end{array}
$$

$$
\begin{aligned}
\Delta_0 &= \partial_1 \, \partial_1^T \\
\Delta_1 &= \partial_1^T \, \partial_1 + \partial_2 \, \partial_2^T \\
\Delta_2 &= \partial_2^T \, \partial_2
\end{aligned}
$$

Hodge Decomposition on Graphs

Useful for least squares ranking on graphs

$$\omega = \partial_1^T \alpha + \partial_2 \beta + h$$

$$\partial_1 \partial_1^T a = \partial_1 \omega \qquad \partial_2^T \partial_2 b = \partial_2^T \omega$$

$$\partial_1^T a \simeq \omega$$
$$\partial_2 \; b \simeq \omega$$

$$\min_a \|r\|_2 \;\; \text{such that} \;\; r = \omega - \partial_1^T a$$
$$\min_b \|s\|_2 \;\; \text{such that} \;\; s = \omega - \partial_2 \; b$$

Hodge Decomposition on Graphs

Useful for least squares ranking on graphs

$$\omega = \partial_1^T \alpha + \partial_2 \beta + h$$

$$\partial_1 \partial_1^T a = \partial_1 \omega \qquad \partial_2^T \partial_2 b = \partial_2^T \omega$$

$$\Delta_0 a = \partial_1 \omega \qquad \Delta_2 b = \partial_2^T \omega$$

$$\partial_1^T a \simeq \omega$$

$$\partial_2 b \simeq \omega$$

$$\min_a \|r\|_2 \text{ such that } r = \omega - \partial_1^T a$$

$$\min_b \|s\|_2 \text{ such that } s = \omega - \partial_2 b$$

Laplace-deRham Operators on Meshes

Discretization of weak form

$$(\Delta\, u, v) = (\mathrm{d}\, u, \mathrm{d}\, v) + (\delta\, u, \delta\, v)$$

Laplace-deRham Operators on Meshes

Discretization of weak form

$$(\Delta\, u, v) = (\mathrm{d}\, u, \mathrm{d}\, v) + (\delta\, u, \delta\, v)$$

$$\begin{tikzcd}
C^0 \arrow[r, "\mathrm{d}_0"] \arrow[d, "*_0"] & C^1 \arrow[r, "\mathrm{d}_1"] \arrow[d, "*_1"] & C^2 \arrow[d, "*_2"] \\
D^2 & D^1 \arrow[l, "\mathrm{d}_0^T"'] & D^0 \arrow[l, "\mathrm{d}_1^T"']
\end{tikzcd}$$

$$\begin{aligned}
\Delta_0 &= \mathrm{d}_0^T *_1 \mathrm{d}_0 \\
\Delta_1 &= \mathrm{d}_1^T *_2 \mathrm{d}_1 + *_1 \mathrm{d}_0 *_0^{-1} \mathrm{d}_0^T *_1 \\
\Delta_2 &= - *_2 \mathrm{d}_1 *_1^{-1} \mathrm{d}_1^T *_2
\end{aligned}$$

Laplace-deRham Operators on Meshes

Discretization of weak form

$$(\Delta\, u, v) = (\mathrm{d}\, u, \mathrm{d}\, v) + (\delta\, u, \delta\, v)$$

$$\begin{array}{ccccc}
C^0 & \xrightarrow{\mathrm{d}_0} & C^1 & \xrightarrow{\mathrm{d}_1} & C^2 \\
\downarrow{\scriptstyle *_0} & & \downarrow{\scriptstyle *_1} & & \downarrow{\scriptstyle *_2} \\
D^2 & \xleftarrow{\mathrm{d}_0^T} & D^1 & \xleftarrow{\mathrm{d}_1^T} & D^0
\end{array}$$

$$\begin{aligned}
\Delta_0 &= \mathrm{d}_0^T *_1 \mathrm{d}_0 \\
\Delta_1 &= \mathrm{d}_1^T *_2 \mathrm{d}_1 + *_1 \mathrm{d}_0 *_0^{-1} \mathrm{d}_0^T *_1 \\
\Delta_2 &= - *_2 \mathrm{d}_1 *_1^{-1} \mathrm{d}_1^T *_2
\end{aligned}$$

Highlighted part problematic unless inverse Hodge star can be handled easily

Laplace-deRham Operators on Meshes

Discretization of weak form

$$(\Delta\, u, v) = (\mathrm{d}\, u, \mathrm{d}\, v) + (\delta\, u, \delta\, v)$$

$$\begin{tikzcd}
C^0 \arrow[r, "\mathrm{d}_0"] \arrow[d, "*_0"] & C^1 \arrow[r, "\mathrm{d}_1"] \arrow[d, "*_1"] & C^2 \arrow[r, "\mathrm{d}_2"] \arrow[d, "*_2"] & C^3 \arrow[d, "*_3"] \\
D^3 & D^2 \arrow[l, "\mathrm{d}_0^T"'] & D^1 \arrow[l, "\mathrm{d}_1^T"'] & D^0 \arrow[l, "\mathrm{d}_2^T"']
\end{tikzcd}$$

$$\begin{aligned}
\Delta_0 &= \mathrm{d}_0^T *_1 \mathrm{d}_0 \\
\Delta_1 &= \mathrm{d}_1^T *_2 \mathrm{d}_1 + *_1 \mathrm{d}_0 *_0^{-1} \mathrm{d}_0^T *_1 \\
\Delta_2 &= \mathrm{d}_2^T *_3 \mathrm{d}_2 + *_2 \mathrm{d}_1 *_1^{-1} \mathrm{d}_1^T *_2 \\
\Delta_3 &= *_3 \mathrm{d}_2 *_2^{-1} \mathrm{d}_2^T *_3
\end{aligned}$$

Laplace-deRham Operators on Meshes

Discretization of weak form

$$(\Delta\, u, v) = (\mathrm{d}\, u, \mathrm{d}\, v) + (\delta\, u, \delta\, v)$$

$$\begin{tikzcd}
C^0 \arrow[r, "\mathrm{d}_0"] \arrow[d, "*_0"] & C^1 \arrow[r, "\mathrm{d}_1"] \arrow[d, "*_1"] & C^2 \arrow[r, "\mathrm{d}_2"] \arrow[d, "*_2"] & C^3 \arrow[d, "*_3"] \\
D^3 & D^2 \arrow[l, "\mathrm{d}_0^T"'] & D^1 \arrow[l, "\mathrm{d}_1^T"'] & D^0 \arrow[l, "\mathrm{d}_2^T"']
\end{tikzcd}$$

$$\begin{aligned}
\Delta_0 &= \mathrm{d}_0^T *_1 \mathrm{d}_0 \\
\Delta_1 &= \mathrm{d}_1^T *_2 \mathrm{d}_1 + *_1 \mathrm{d}_0 *_0^{-1} \mathrm{d}_0^T *_1 \\
\Delta_2 &= \mathrm{d}_2^T *_3 \mathrm{d}_2 + *_2 \mathrm{d}_1 *_1^{-1} \mathrm{d}_1^T *_2 \\
\Delta_3 &= *_3 \mathrm{d}_2 *_2^{-1} \mathrm{d}_2^T *_3
\end{aligned}$$

Highlighted part problematic unless inverse Hodge star can be handled easily

Naive Hodge Decomposition on Meshes

Linear system for curl (β) part has inverse Hodge star

$$\omega = \mathsf{d}\,\alpha + \delta\,\beta + h$$

$$\begin{aligned} \delta\;\omega &= \delta\;\mathsf{d}\;\alpha \\ *^{-1}\,\mathsf{d}^T * \mathsf{d}\;\alpha &= *^{-1}\,\mathsf{d}^T * \omega \\ \mathsf{d}^T * \mathsf{d}\;\alpha &= \mathsf{d}^T * \omega \end{aligned}$$

$$\begin{aligned} \mathsf{d}\;\omega &= \mathsf{d}\;\delta\;\beta \\ \mathsf{d}\;*^{-1}\,\mathsf{d}^T * \beta &= \mathsf{d}\;\omega \end{aligned}$$

Strategy for Computing Hodge Decomposition

Different for graphs and meshes

Graphs (no metric involved – Hodge star is Identity):

- Use least squares for gradient and curl
- Find harmonic component by subtraction

Manifold simplicial complexes (Hodge star is not Identity):

- Use weighted least squares for gradient part. In addition:
- Find harmonic basis as eigenvectors and project

  OR
- Find harmonic basis by discrete Hodge-deRham and project

Eigenvector Method for Harmonic Basis

Use of weak form to avoid inverse Hodge stars

Consider the linear system for σ and u :

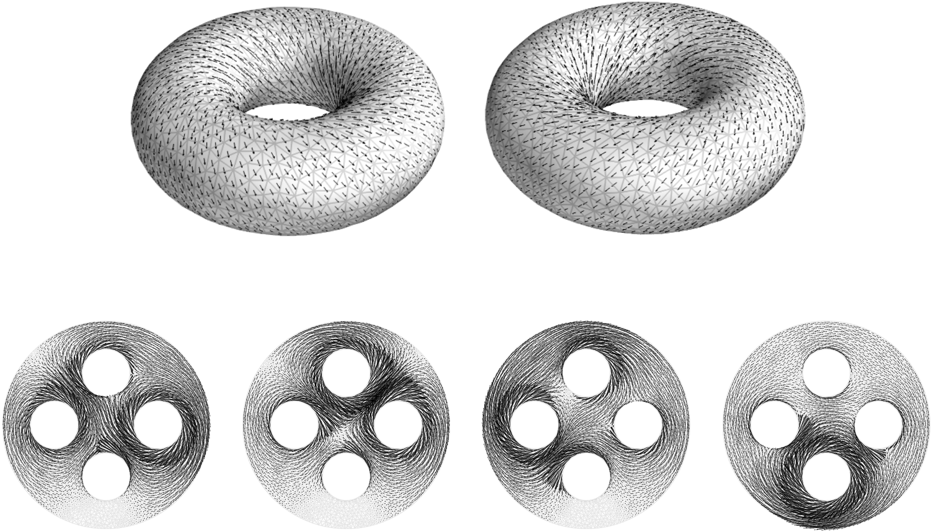
$$
\begin{aligned}
(\sigma, \tau) - (\mathrm{d}_{p-1}\, \tau, u) &= 0\,,\\
(\mathrm{d}_{p-1}\, \sigma, v) + (\mathrm{d}_p\, u, \mathrm{d}_p\, v) &= 0\,,
\end{aligned}
$$

for all τ and v. Then (σ, u) is a solution if and only if $\sigma = 0$ and u is a harmonic p-form.

$$
\begin{bmatrix}
*_{p-1} & -\,\mathrm{d}_{p-1}^T *_p \\
*_p\, \mathrm{d}_{p-1} & \mathrm{d}_p^T *_{p+1} \mathrm{d}_p
\end{bmatrix}
$$

Eigenvectors of 0 eigenvalue are harmonic cochains.

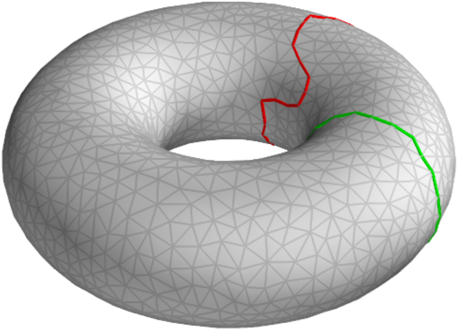
Harmonic cochains using eigenvector method



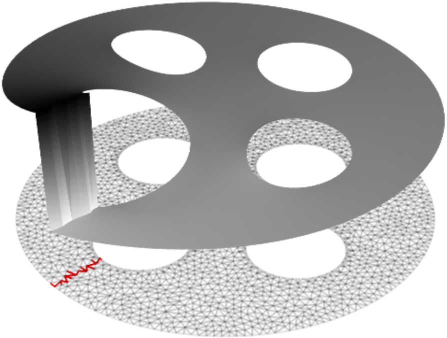
Discrete Hodge-deRham Method for Harmonic Basis

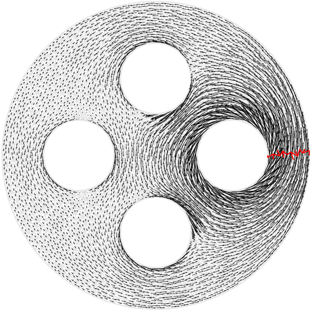
Avoids inverse Hodge stars and in addition provides topological control

- Find a homology basis and its dual cohomology basis
- Find harmonic cochains cohomologous to cohomology basis

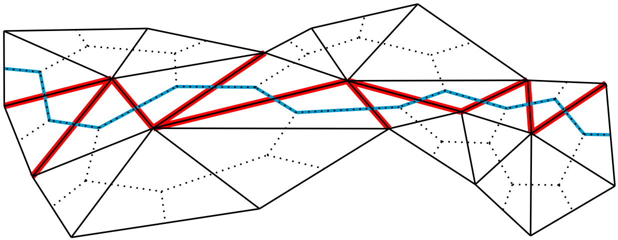


Chain a *homologous* to b if $a - b \in \operatorname{im} \partial$





Cochain h *cohomologous* to ω if $h - \omega \in \operatorname{im} \mathrm{d}$



Poincaré-Lefschetz duality:

M manifold of dimension n

If $\partial M = \emptyset$, then $H^p(M;\mathbb{R}) \cong H_{n-p}(M;\mathbb{R})$

If $\partial M \neq \emptyset$, then $H^p(M;\mathbb{R}) \cong H_{n-p}(M,\partial M;\mathbb{R})$

Naive Cohomologous Harmonic Cochain

Given $[\omega] \in H^p(M; \mathbb{R})$. Find α such that

$$\Delta\left(\omega + \mathrm{d}\ \alpha\right) = 0$$

Naive Cohomologous Harmonic Cochain

Given $[\omega] \in H^p(M;\mathbb{R})$. Find α such that

$$\Delta\,(\omega + \mathsf{d}\ \alpha) = 0$$

Since $\omega \in \ker \mathsf{d}$, above is equivalent to

$$\mathsf{d}\ \delta\ \mathsf{d}\ \alpha = -\,\mathsf{d}\ \delta\ \omega$$
$$\mathsf{d} *^{-1} \mathsf{d}^T * \mathsf{d}\ \alpha = -\,\mathsf{d} *^{-1} \mathsf{d}^T * \omega$$

Naive Cohomologous Harmonic Cochain

Given $[\omega] \in H^p(M;\mathbb{R})$. Find α such that

$$\Delta\,(\omega + \mathrm{d}\ \alpha) = 0$$

Since $\omega \in \ker \mathrm{d}$, above is equivalent to

$$\mathrm{d}\ \delta\ \mathrm{d}\ \alpha = -\,\mathrm{d}\ \delta\ \omega$$
$$\mathrm{d} *^{-1} \mathrm{d}^T * \mathrm{d}\ \alpha = -\,\mathrm{d} *^{-1} \mathrm{d}^T * \omega$$

Again, the inverse Hodge stars could be a problem

Discrete Hodge-deRham Method

Smallest cochain in each cohomology class is harmonic

Given $[\omega] \in H^p(M;\mathbb{R})$. Find α such that

$$\min_{\alpha \in C^{p-1}} (\omega + \mathrm{d}\,\alpha\,,\, \omega + \mathrm{d}\,\alpha)$$

$$\min_{\alpha \in C^{p-1}} \left(\omega + \mathrm{d}\,\alpha\right)^T * \left(\omega + \mathrm{d}\,\alpha\right)$$

$$\mathrm{d}_{p-1}^T *_p \mathrm{d}_{p-1}\,\alpha = -\,\mathrm{d}_{p-1}^T *_p \omega$$

Discrete Hodge-deRham Method

Smallest cochain in each cohomology class is harmonic

Given $[\omega] \in H^p(M;\mathbb{R})$. Find α such that

$$\min_{\alpha \in C^{p-1}} (\omega + \mathrm{d}\,\alpha\,,\, \omega + \mathrm{d}\,\alpha)$$

$$\min_{\alpha \in C^{p-1}} \big(\omega + \mathrm{d}\,\alpha\big)^T * \big(\omega + \mathrm{d}\,\alpha\big)$$

$$\mathrm{d}_{p-1}^T *_p \mathrm{d}_{p-1}\,\alpha = -\,\mathrm{d}_{p-1}^T *_p \omega$$

Note : no inverse Hodge stars

Discrete Hodge-deRham Method

Smallest cochain in each cohomology class is harmonic

Given $[\omega] \in H^p(M;\mathbb{R})$. Find α such that

$$\min_{\alpha \in C^{p-1}} (\omega + \mathrm{d}\,\alpha \,,\, \omega + \mathrm{d}\,\alpha)$$

$$\min_{\alpha \in C^{p-1}} \big(\omega + \mathrm{d}\,\alpha\big)^T * \big(\omega + \mathrm{d}\,\alpha\big)$$

$$\mathrm{d}_{p-1}^T *_p \mathrm{d}_{p-1}\,\alpha = -\,\mathrm{d}_{p-1}^T *_p \omega$$

Inspired by a result in the smooth case that smallest differential form cohomologous to a given form is harmonic. Precise statement of theorem in discrete case given later.

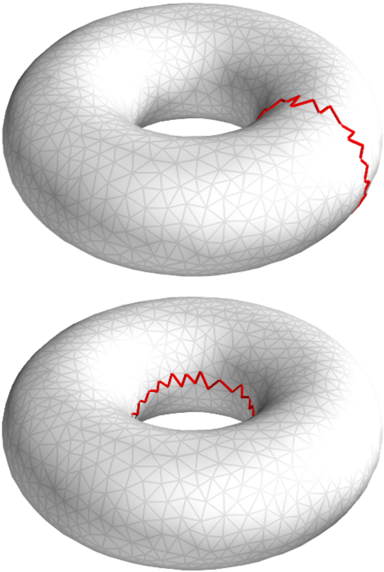
Sample Code for Harmonic Cochain Computation

Using discrete Hodge-deRham method

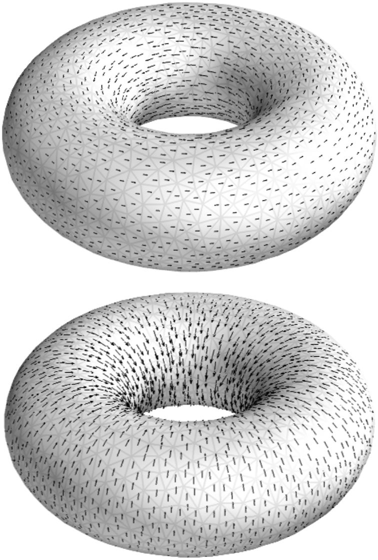
```
from numpy import zeros, loadtxt
from numpy.linalg import inv
from scipy.sparse import csr_matrix
from scipy.sparse.linalg import cg
from dlnyhdg.simplicial_complex import simplicial_complex
from pydec import whitney_innerproduct

tol = 1e-8; scale = 5; width = 0.001
vertices = loadtxt('vertices.txt')
triangles = loadtxt('triangles.txt', dtype=int)
sc = simplicial_complex((vertices, triangles))
edge_list, edge_orientation = loadtxt('12cocycle.txt', dtype=int)
omega = zeros(sc[1].num_simplices)
for i, e in enumerate(edge_list):
    omega[e] += edge_orientation[i]
d0 = sc[0].d
hodge1 = whitney_innerproduct(sc, 1)
A = d0.T * hodge1 * d0; b = -d0.T * hodge1 * omega
alpha = cg(A, b, tol=tol)[0]
harmonic = omega + d0 * alpha
```

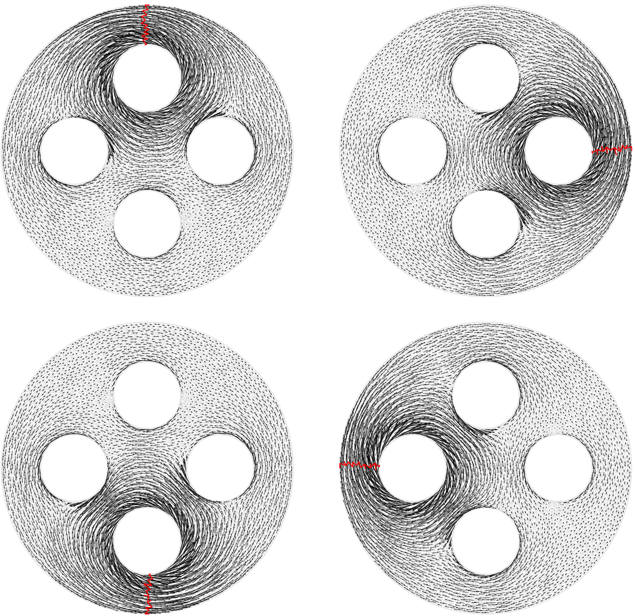
Cohomologous harmonic cochains using discrete Hodge-deRham method



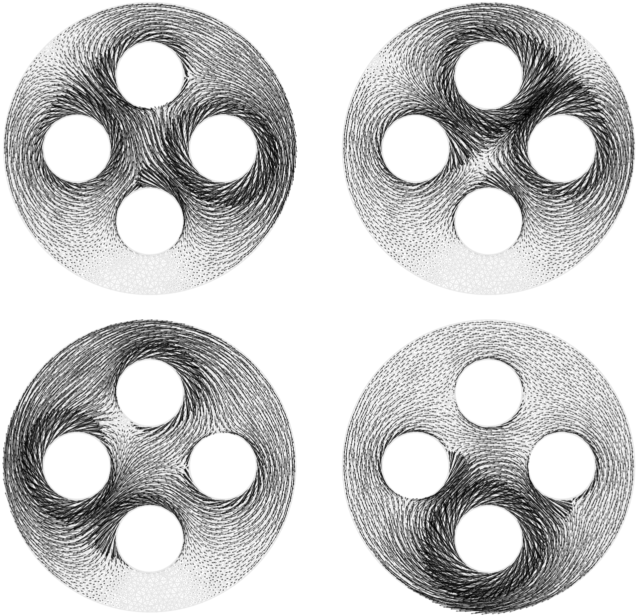
Cohomologous harmonic cochains using discrete Hodge-deRham method



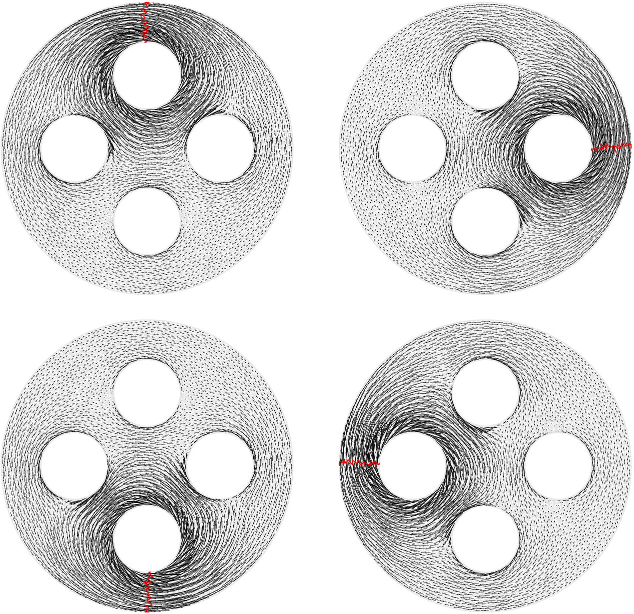
Cohomologous harmonic cochains using discrete Hodge-deRham method



Compare with harmonic cochains using eigenvector method



Cohomologous harmonic cochains using discrete Hodge-deRham method



Isomorphism Theorem in Smooth Case

Inspiration for the discrete Hodge-deRham method

Theorem (Hodge-deRham Isomorphism)

For a boundaryless manifold M ($\partial M = \emptyset$), $H^p(M;\mathbb{R}) \cong \mathcal{H}^p(M) = \ker \Delta_p$. For manifolds with boundary, $H^p(M;\mathbb{R}) \cong \mathcal{H}^p_N(M)$ and $H^p(M,\partial M;\mathbb{R}) \cong \mathcal{H}^p_D(M)$.

Harmonic forms : $\ker \Delta$
Harmonic fields : $\mathcal{H}(M) = \ker \mathrm{d} \cap \ker \delta$
Harmonic Neumann fields : $\mathcal{H}_N(M)$ normal component 0
Harmonic Dirichlet fields : $\mathcal{H}_D(M)$ tangential component 0

Isomorphism Theorem in Discrete Case

Theorem (Discrete Hodge-deRham Isomorphism, H-K-W-W)

Let $[\omega] \in H^p(K\,;\mathbb{R})$. *Then*

1. *There exists a cochain* $\alpha \in C^{p-1}(K\,;\mathbb{R})$, *not necessarily unique, such that* $\delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0$;
2. *There is a unique cochain* $\mathrm{d}_{p-1}\,\alpha$ *satisfying* $\delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0$ *; and*
3. $\delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0 \Rightarrow \Delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0$.

Theorem (H-Demlow)

Let $[\omega] \in H^p(\mathcal{T}_h;\mathbb{R})$ *and let* $\alpha \in C^{p-1}$ *be such that* $(\omega + \mathrm{d}_{p-1}\,\alpha, \mathrm{d}_{p-1}\,\tau) = 0$ *for all* $\tau \in C^{p-1}$. *Then* $\mathrm{W}(\omega + \mathrm{d}_{p-1}\,\alpha) \in \mathfrak{H}_h^p$.

Isomorphism Theorem in Discrete Case

Theorem (Discrete Hodge-deRham Isomorphism, H-K-W-W)

Let $[\omega] \in H^p(K\,;\mathbb{R})$*. Then*

1. *There exists a cochain* $\alpha \in C^{p-1}(K\,;\mathbb{R})$*, not necessarily unique, such that* $\delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0$*;*
2. *There is a unique cochain* $\mathrm{d}_{p-1}\,\alpha$ *satisfying* $\delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0$ *; and*
3. $\delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0 \Rightarrow \Delta_p\,(\omega + \mathrm{d}_{p-1}\,\alpha) = 0$.

Theorem (H-Demlow)

Let $[\omega] \in H^p(\mathcal{T}_h;\mathbb{R})$ *and let* $\alpha \in C^{p-1}$ *be such that* $(\omega + \mathrm{d}_{p-1}\,\alpha, \mathrm{d}_{p-1}\,\tau) = 0$ *for all* $\tau \in C^{p-1}$*. Then* $\mathrm{W}(\omega + \mathrm{d}_{p-1}\,\alpha) \in \mathfrak{H}_h^p$.

If Whitney Hodge star is used, the method produces a solution to the finite element exterior calculus equations for harmonic cochains.

Back to Graphs

Comparison of algebraic multigrid and conjugate gradient

- Results from experiments on graphs will be shown next
- 2-dimensional clique complexes for Erdős-Rényi random graphs and Barabási-Albert scale-free graphs were used
- N_p is number of p-simplices
- Conjugate gradient easily beats algebraic multigrid in these experiments
- Smoothed aggregation and Lloyd aggregation were used
- Entries for algebraic multigrid show setup time, solve time, and total time
- Pictures of system matrices hint at reason for poor performance of multigrid

Back to Graphs

Comparison of algebraic multigrid and conjugate gradient

- Results from experiments on graphs will be shown next
- 2-dimensional clique complexes for Erdős-Rényi random graphs and Barabási-Albert scale-free graphs were used
- N_p is number of p-simplices
- Conjugate gradient easily beats algebraic multigrid in these experiments
- Smoothed aggregation and Lloyd aggregation were used
- Entries for algebraic multigrid show setup time, solve time, and total time
- Pictures of system matrices hint at reason for poor performance of multigrid

Numerical experiments on Erdős-Rényi graphs

Case	N_0	N_1	N_2	Edge Density	Triangle Density
a	100	1212	2359	2.45e-01	1.46e-02
b	100	2530	21494	5.11e-01	1.33e-01
c	100	3706	67865	7.49e-01	4.20e-01
d	500	1290	21	1.03e-02	1.01e-06
e	500	12394	20315	9.94e-02	9.81e-04

Case	Algebraic Multigrid		Conjugate Gradient	
	α	β	α	β
a	0.0001	0.5708	**0.0023**	**0.0191**
	0.0078	0.0551		
	0.0079	**0.6259**		
b	0.0030	0.866	**0.0017**	**0.1033**
	0.0111	1.236		
	0.0141	**2.102**		
c	0.0303	5.66	**0.0015**	**0.4759**
	0.0386	11.08		
	0.0689	**16.74**		
d	0.1353	0.0071	**0.0072**	**0.0007**
	0.5760	0.0289		
	0.7113	**0.0360**		
e	0.0001	0.49	**0.0030**	**0.2155**
	0.3303	2.22		
	0.3305	**2.71**		

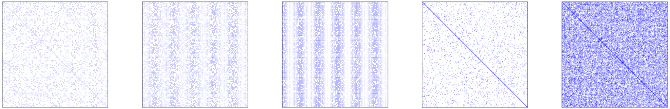
Numerical experiments on Barabási-Albert graphs

Case	N_0	N_1	N_2	Edge Density	Triangle Density
a	100	900	1701	1.82e-01	1.05e-02
b	100	1600	8105	3.23e-01	5.01e-02
c	100	2400	24497	4.85e-01	1.51e-01
d	500	9600	25016	7.70e-02	1.21e-03
e	1000	19600	37365	3.92e-02	2.25e-04

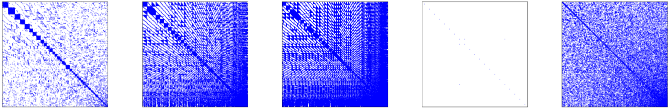
Case	Algebraic Multigrid		Conjugate Gradient	
	α	β	α	β
a	0.0005	0.0242		
	0.0012	0.0753	**0.0033**	**0.0281**
	0.0017	**0.0995**		
b	0.0001	0.1842		
	0.0008	0.1720	**0.0032**	**0.0967**
	0.0009	**0.3562**		
c	0.0005	1.111		
	0.0012	1.521	**0.0030**	**0.2435**
	0.0017	**2.632**		
d	0.0002	1.018		
	0.0140	3.608	**0.0056**	**1.043**
	0.0142	**4.626**		
e	0.0174	2.40		
	0.0219	8.21	**0.0088**	**2.521**
	0.0393	**10.61**		

Pictures of system matrices for graph experiments

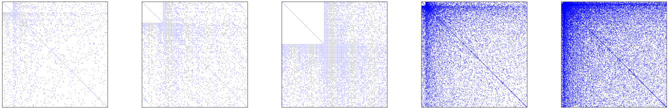
ER Δ_0



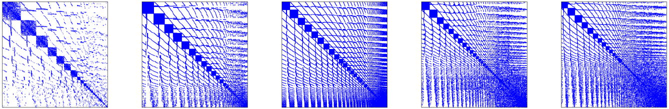
ER Δ_2



BA Δ_0

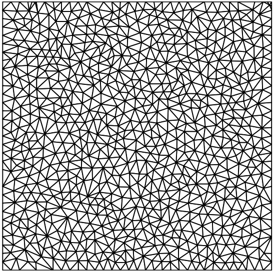


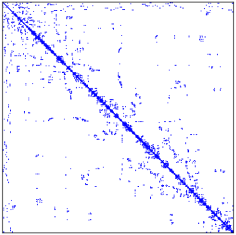
BA Δ_2

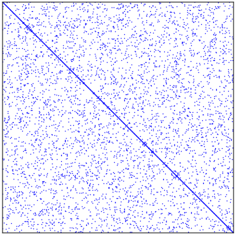


Meshes versus Graphs

A mesh, its Δ_0, and Δ_0 for a graph with same number of vertices and same edge density

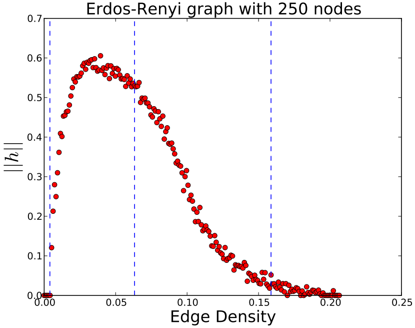


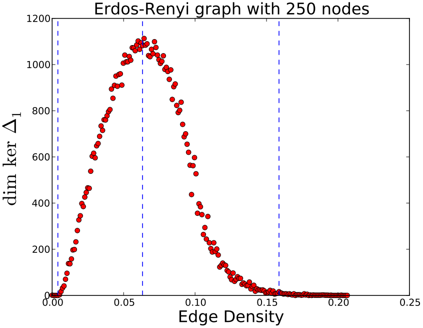




Experiments with Random Clique Complexes

Graph Hodge decomposition code can be used to formulate conjectures





Dashed lines are Kahle's bounds [Kahle, Discrete Math., Vol. 309, pp. 1658–1671]. Betti number is almost always zero before the first bound and after third bound. It is almost always nonzero between first and second bounds. Theory is silent about the region between second and third bounds and about harmonic norm.

Optimal (Co)homologous (Co)chains

Common threads

Cohomologous harmonic cochains:

$$\min_{\alpha \in C^{p-1}} (\omega + \mathrm{d}\,\alpha\,,\, \omega + \mathrm{d}\,\alpha)$$

$$\min_{\alpha} \|h\|_{*_p} \quad \text{subject to} \quad h = \omega + \mathrm{d}\,\alpha \quad \text{(all vectors real)}$$

Least squares ranking on graphs:

$$\min_{\alpha} \|r\|_2 \quad \text{subject to} \quad r = \omega - \partial_1^T\,\alpha \quad \text{(all vectors real)}$$

$$\min_{\beta} \|s\|_2 \quad \text{subject to} \quad s = \omega - \partial_2\,\beta \quad \text{(all vectors real)}$$

Optimal (Co)homologous (Co)chains

Common threads

Cohomologous harmonic cochains:

$$\min_{\alpha \in C^{p-1}} (\omega + \mathrm{d}\,\alpha\,,\, \omega + \mathrm{d}\,\alpha)$$

$$\min_{\alpha} \|h\|_{*_p} \quad \text{subject to} \quad h = \omega + \mathrm{d}\,\alpha \quad \text{(all vectors real)}$$

Least squares ranking on graphs:

$$\min_{\alpha} \|r\|_2 \quad \text{subject to} \quad r = \omega - \partial_1^T\,\alpha \quad \text{(all vectors real)}$$

$$\min_{\beta} \|s\|_2 \quad \text{subject to} \quad s = \omega - \partial_2\,\beta \quad \text{(all vectors real)}$$

Optimal homologous chains:
[See Dey-Hirani-Krishnamoorthy and Dunfield-Hirani]

$$\min_{x,y} \|x\|_1 \quad \text{subject to} \quad x = c + \partial\,y \quad \text{(all vectors integer)}$$

1. Hirani, A. N., Kalyanaraman, K., Wang, H., and Watts, S. Cohomologous harmonic cochains, 2011. arXiv:1012.2835
2. Hirani, A. N., Kalyanaraman, K., and Watts, S. Least squares ranking on graphs, 2011. arXiv:1011.1716
3. Dey, T. K., Hirani, A. N., and Krishnamoorthy, B. Optimal homologous cycles, total unimodularity, and linear programming. *SIAM Journal on Computing 40*, 4 (2011), 1026–1044. doi:10.1137/100800245
4. Dunfield, N. M., and Hirani, A. N. The least spanning area of a knot and the optimal bounding chain problem. In *ACM Symposium on Computational Geometry (SoCG 2011)* (2011), pp. 135–144. doi:10.1145/1998196.1998218

Acknowledgement

NSF Grant DMS-0645604
Algebraic Topology and Exterior Calculus in Numerical Analysis

NSF Grant CCF 10-64429
Optimality in Homology – Algorithms and Applications