Hodge Theory

and

Sheaves

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What is Hodge theory?

$H^i(A^*) = \ker d^i / \operatorname{im} d^{i-1}$

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots$$

$$\Delta^i = d^{i-1} d^{i-1\,*} + d^{i\,*} d^i$$

$*$ ↔ inner prod.

(1) $\ker \Delta^i$ picks out $H^i(A^\bullet)$ rep

(2) $\Delta^i \geq 0$, gives algorithm:

$$|\Delta^i| \leq B$$

iterate $B - \Delta^i$

(3) etc.

Simplicial

Graphs

Categories

R-Mans

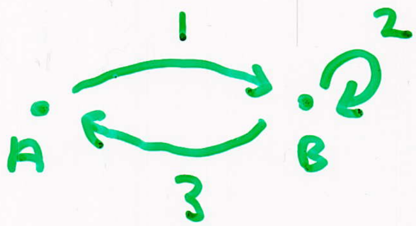
etc.

Heat, C^∞ forms, sparsity, etc.

Hodge on Graphs: $G = (V, E)$

$$\cdots 0 \to \mathbb{R}^E \xrightarrow{d} \mathbb{R}^V \to 0 \cdots$$

$d =$ incidence $= d_h - d_t$



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline A & & & \\ B & & & \end{array} \quad \begin{array}{c} A \\ B \end{array}\begin{bmatrix} -1 & 0 & +1 \\ +1 & 0 & -1 \end{bmatrix}$$

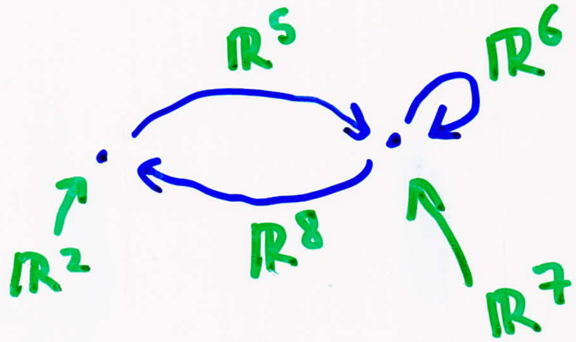
$$\Delta^V = dd^* = \underbrace{d_h d_h^* + d_t d_t^*}_{D^V} - (\underbrace{d_h d_t^* + d_t d_h^*}_{A^V})$$

$$\Delta^E = d^* d = D^E - A^E$$

$\beta_0(G) = \dim \operatorname{coker}(d) = \dim \ker(\Delta^V), \quad \beta_1(G) = \ldots$

$\mathbb{R}$ can be any field

What is a sheaf on a graph?



d_h, d_t

$\mathcal{F}$:

$\mathcal{F}(v)\ \forall v$

$\mathcal{F}(e)\ \forall e$

$\mathcal{F}(e,h)$:

$\mathcal{F}(e,t)$:

Formally:

$$\dots 0 \to \mathcal{F}(E) \xrightarrow{d_h - d_t} \mathcal{F}(V) \to 0 \to 0 \dots$$

$\forall\, v \in V,\ \mathcal{F}(v)$ $\qquad$ $\mathcal{F}(V) = \bigoplus_{v \in V} \mathcal{F}(v)$

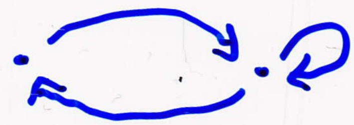
$\forall\, e \in E,\ \mathcal{F}(e)$ $\qquad$ $\mathcal{F}(E) = \bigoplus_{e \in E} \mathcal{F}(e)$

d_h: $\forall e \in E$ $\quad \mathcal{F}(e,h) : \mathcal{F}(e) \to \mathcal{F}(he)$

d_t: - - - - - - t - - - - - - t - -

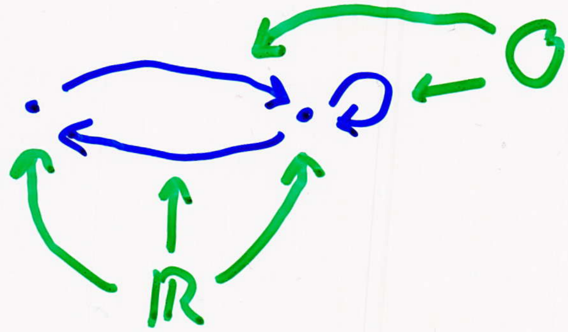
What is the point? (Intro to SHNC)

G'



G

$\underline{\mathbb{R}}_{G'}$:



$\left(\text{Any } K \to G \text{ gives rise to sheaf } \underline{\mathbb{R}}_K \text{ on } G \text{ s.t.} \right)$

$\underline{\mathbb{R}}_K, d_h, d_t \longleftrightarrow$ Usual d_h, d_t on K

$$\beta_i(K) = \dim \begin{Bmatrix} \text{coker} & (i=0) \\ \ker & (i=1) \end{Bmatrix} \quad d = d_h - d_t$$

<u>What is the point</u>?

$$0 \to \underline{\mathbb{R}}_{G_1 \cap G_2} \to \underline{\mathbb{R}}_{G_1} \oplus \underline{\mathbb{R}}_{G_2} \to \underline{\mathbb{R}}_{G_1 \cup G_2} \to 0$$

or

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

implies

$$0 \to H_1(\mathcal{F}_1) \to H_1(\mathcal{F}_2) \to H_1(\mathcal{F}_3) \xrightarrow{\delta} H_0(\mathcal{F}_1) \to \cdots$$

$$H_1(\mathcal{F}) = \ker d_{\mathcal{F}}, \quad H_0(\mathcal{F}) = \operatorname{coker} d_{\mathcal{F}}$$

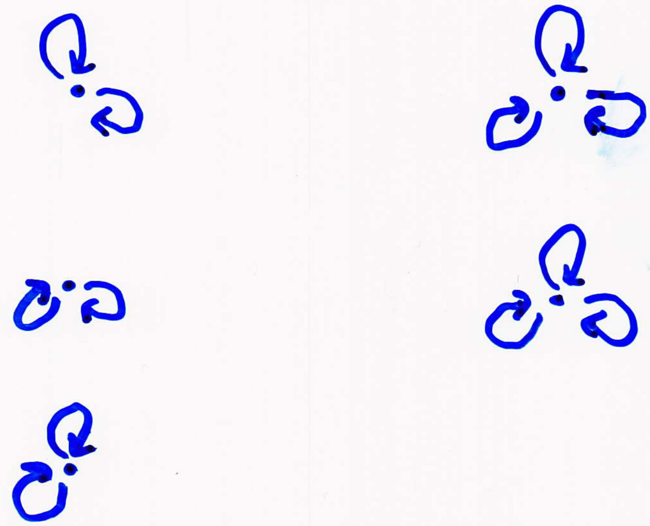
SHNC: $H_1(\mathcal{F}_1) = 0 \Rightarrow$

$$\dim H_1(\mathcal{F}_2) \le \dim H_1(\mathcal{F}_3)$$

$$\dim H_i^{\lim}(\mathcal{F}) = \lim_{\text{cov } \varphi : K \to G} \frac{\dim H_i(\overbrace{\varphi^*}^{\text{pullback to } K}\mathcal{F})}{\deg \varphi}$$

Sheaves give more morphisms:

G_1 G_2 G_3



G

G

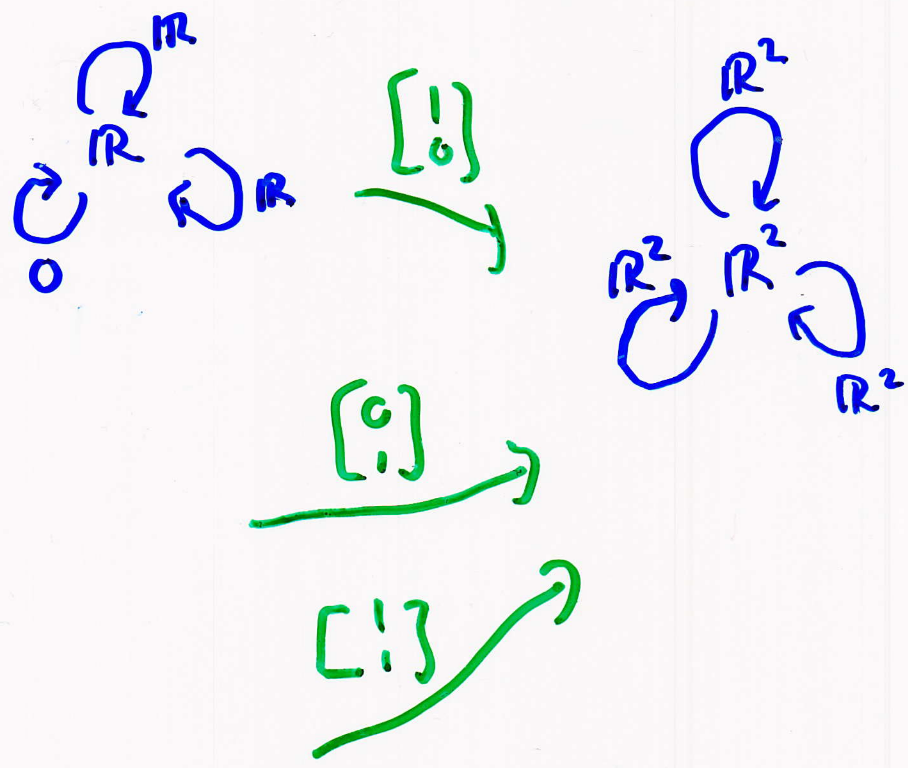
$$G_1 \amalg G_2 \amalg G_3 \xrightarrow{?} G \amalg G$$

$\underline{\mathbb{R}}_{G_1}$

$\oplus$

$\underline{\mathbb{R}}_{G_2}$

$\oplus$



$\underline{\mathbb{R}}_{G \amalg G}$

Hodge Theory:

$$d_{\mathcal{F}} = d_{h,\mathcal{F}} - d_{t,\mathcal{F}} : \mathcal{F}(E) \to \mathcal{F}(V)$$

$$\Delta_V = d_{\mathcal{F}} d_{\mathcal{F}}^* = D_{\mathcal{F}}^V - A_{\mathcal{F}}^V$$

$$\Delta_E = d_{\mathcal{F}}^* d_{\mathcal{F}} = D_{\mathcal{F}}^E - A_{\mathcal{F}}^E$$

β_0: $\mathrm{Ker}(\Delta_V) = \mathrm{Ker}(d_{\mathcal{F}}^*) \simeq \mathrm{Coker}(d_{\mathcal{F}})$

β_1: $\mathrm{Ker}(\Delta_E) = \mathrm{Ker}(d_{\mathcal{F}})$

Algebraic Graph Theory is special case

Hodge Theory for sheaves (char. 0)

- For sheaves on graphs
- For graphs as special case
- For more general complexes
- " " " categories
- Etc. ?