Hodge theory, Hilbert complexes, and finite element differential forms

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Outline

1. Motivating examples
2. Hilbert complexes and their discretization
3. Finite element differential forms
4. The elasticity complex

Joint with R. Falk and R. Winther. Primary reference:
Finite element exterior calculus: From Hodge theory to numerical stability, Bull. AMS 2010, pp. 281-354

FEEC:
Hilbert complex framework + finite element diff'l forms + applications

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Motivating example

Vector Poisson equation on a plane domain

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \quad \text{in } \Omega \ (\text{mod } \mathfrak{H}),$$
$$u \cdot n = 0,\ \operatorname{curl} u \times n = 0,\ u \perp \mathfrak{H} \quad \text{on } \partial\Omega$$

Just the Hodge Laplacian for 1-forms: $d^* d\, u + d\, d^*\, u = f$ (mod $\mathfrak{H}$)

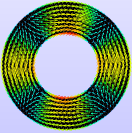
Weak formulation: find $u \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div}) \cap \mathfrak{H}^{\perp}$ such that

$$\int_\Omega (\operatorname{curl} u \cdot \operatorname{curl} v + \operatorname{div} u \ \operatorname{div} v)\, dx = \int_\Omega f \cdot v\, dx, \quad v \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div}) \cap \mathfrak{H}^{\perp}$$

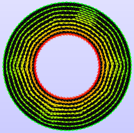
Variational formulation:

$$u = \underset{H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div}) \cap \mathfrak{H}^{\perp}}{\arg\min} \left(\frac{1}{2}\int_\Omega |\operatorname{curl} u|^2 + |\operatorname{div} u|^2\, dx - \int_\Omega f \cdot u\, dx\right)$$

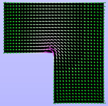
Standard finite elements do *not* work



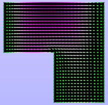
$f = (0, x)$



$\mathcal{P}_1$ elements



$f = (-1, 0)$



$\mathcal{P}_1$ elements

Hilbert complexes and their discretization

Hilbert complexes

We view the exterior derivative d as a closed unbounded operator $L^2\Lambda^k \to L^2\Lambda^{k+1}$ with domain

$$H\Lambda^k = \{\, u \in L^2\Lambda^k \mid du \in L^2\Lambda^{k+1} \,\}.$$

Resulting structure is a *closed Hilbert complex*, which abstracts the de Rham complex:

- Hilbert spaces $W^0, W^1, \dots, W^n$;
- Densely defined closed operators $W^k \xrightarrow{d^k} W^{k+1}$ with domain $V^k \subset W^k$ and *closed range* $\mathfrak{B}^{k+1}$, satisfying:
- $d^{k-1} \circ d^k = 0$ (i.e., $\mathfrak{B}^k \subset \mathfrak{Z}^k := \ker d^k \subset V^k$)

Defining $\|v\|^2_{V^k} = \|v\|^2_{W^k} + \|dv\|^2_{W^{k+1}}$, we get a *complex of Hilbert spaces*

$$0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \cdots \xrightarrow{d} V^n \to 0$$

with associated cohomology spaces $\mathfrak{Z}^k / \mathfrak{B}^k$

Properties of closed Hilbert complexes

Adjoint complex: d_k^* is densely-defined, closed, w/ closed range

$$0 \leftarrow V_0^* \xleftarrow{d^*} V_1^* \xleftarrow{d^*} \cdots \xleftarrow{d^*} V_n^* \leftarrow 0$$

Abstract Hodge Laplacian: $d\,d^* + d^*d : W^k \to W^k$

Harmonic forms: $\mathfrak{Z}^k / \mathfrak{B}^k \equiv \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} = \ker(d\,d^* + d^*d) := \mathfrak{H}^k$

Hodge decomposition: $W^k = \underbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k}_{\mathfrak{Z}^k} \oplus \underbrace{\mathfrak{B}_k^*}_{\mathfrak{Z}^{k\perp}}$

Poincaré inequality: $\exists\, c$ such that $\|u\| \le c\|du\| \quad \forall u \in \mathfrak{Z}^{k\perp} \cap V^k$

Mixed formulation of the (abstract) Hodge Laplacian

$$\sigma = d^*u, \quad d\sigma + d^*du = f \ (\mathrm{mod}\ \mathfrak{H}), \quad u \perp \mathfrak{H}$$

Weak formulation: Given $f \in W^k$, find $\sigma \in V^{k-1}$, $u \in V^k$, $p \in \mathfrak{H}^k$:

$$\begin{aligned} &\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 && \forall \tau \in V^{k-1} \\ &\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle && \forall v \in V^k \\ &\langle u, q \rangle = 0 && \forall q \in \mathfrak{H}^k \end{aligned}$$

Variational formulation:

$$\frac{1}{2}\langle \sigma, \sigma \rangle - \frac{1}{2}\langle du, du \rangle - \langle d\sigma, u \rangle - \langle u, p \rangle + \langle f, u \rangle \to \textit{saddle point}$$

Well-posedness of the mixed formulation

Theorem: $\forall\, (\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k \quad \exists\, (\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$

1. *Bounded:* $\|\tau\|_V + \|v\|_V + \|q\| \leq C(\|\sigma\|_V + \|u\|_V + \|p\|)$
2. *Coercing:* $B \geq c(\|\sigma\|_V^2 + \|u\|_V^2 + \|p\|^2)$

where $B := \langle \sigma, \tau \rangle - \langle d\tau, u \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle - \langle u, q \rangle$.

Hodge decompose $u = d\eta + s + z$ with $\eta \in (\mathfrak{Z}^{k-1})^\perp, s \in \mathfrak{H}^k, z \in (\mathfrak{Z}^k)^\perp$
Choose $\tau = \sigma - \epsilon\eta,\ v = u + d\sigma + p,\ q = p - s$:

$$B = \|\sigma\|^2 + \|d\sigma\|^2 + \epsilon\|d\eta\|^2 + \|s\|^2 + \|du\|^2 + \|p\|^2 - \epsilon\langle \sigma, \eta \rangle.$$

$\epsilon\langle \sigma, \eta \rangle \leq \frac{1}{2}\|\sigma\|^2 + \frac{\epsilon^2}{2}\|\eta\|^2$; and, by Poincaré ineq, $\|\eta\| \leq c_P\|d\eta\|$, so $\epsilon = c_P^{-2} \implies$

$$B(\sigma, u, p; \tau, v, q) \geq c(\|\sigma\|^2 + \|d\sigma\|^2 + \|d\eta\|^2 + \|s\|^2 + \|du\|_V^2 + \|p\|^2).$$

But, Poincaré ineq also gives $\|z\| \leq c_P\|dz\| = c_P\|du\|$.

Discretization

We now want to discretize the mixed formulation with f.d. subspaces $V_h^k \subset V^k$ indexed by h (Galerkin). Of course we assume

$$\inf_{v_h \in V_h^k} \|v - v_h\|_V \to 0 \text{ as } h \to 0 \quad \forall v \in V \qquad \text{(A)}$$

It turns out that there are *two more key assumptions*.

Subcomplex assumption (SC): $d(V_h^k) \subset V_h^{k+1}$

The subcomplex

$$\cdots \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1} \xrightarrow{d^{k+1}} \cdots$$

"structure-preserving discretization"

is itself a Hilbert complex so we have discrete harmonic forms $\mathfrak{H}_h^k$, discrete Hodge decomp, and discrete Poincaré ineq with constant $c_{P,h}$.

Bounded Cochain Projection assumption (BCP): $\exists\, \pi_h^k : V^k \to V_h^k$

$$\begin{tikzcd} \cdots \arrow[r] & V^k \arrow[r, "d^k"] \arrow[d, "\pi_h^k"] & V^{k+1} \arrow[r] \arrow[d, "\pi_h^{k+1}"] & \cdots \\ \cdots \arrow[r] & V_h^k \arrow[r, "d^k"] & V_h^{k+1} \arrow[r] & \cdots \end{tikzcd}$$

- π_h^k V-bounded, uniform in h
- π_h^k a projection
- $\pi_h^{k+1} d^k = d^k \pi_h^k$

Stability theorem

Theorem

Let (V^k, d^k) be a Hilbert complex and V_h^k finite dimensional subspaces satisfying A, SC, and BCP. Then

- *π_h induces an isomorphism on cohomology for h small*
- $\mathrm{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \to 0$
- *The discrete Poincaré inequality* $\|\omega\| \leq c\|d\omega\|, \quad \omega \in \mathfrak{Z}_h^{k\perp}$, *holds with c independent of h*
- *Galerkin's method is stable*

Proof of discrete Poincaré inequality: Given $\omega \in \mathfrak{Z}_h^{k\perp}$, define $\eta \in \mathfrak{Z}^{k\perp} \subset V^k$ by $d\eta = d\omega$. By the Poincaré inequality, $\|\eta\| \leq c_P\|d\omega\|$, whence $\|\eta\|_V \leq c'\|d\omega\|$, so it is enough to show that $\|\omega\| \leq c''\|\eta\|_V$. Now, $\omega - \pi_h\eta \in V_h^k$ and, by SC and BCP, $d(\omega - \pi_h\eta) = d\omega - \pi_h d\omega = 0$, so $\omega - \pi_h\eta \in \mathfrak{Z}_h^k$. Thus $\omega \perp (\omega - \pi_h\eta)$, so $\|\omega\| \leq \|\pi_h\eta\|$ by Pythagoras. Result follows since π_h is bounded. Note $c_{P,h} + \leq (c_P^2 + 1)^{1/2}\|\pi_h\|$.

Convergence of Galerkin's method

Since we have uniform control of the Poincaré constant $c_{P,h}$, the well-posedness argument applied on the discrete level gives stability. From stability we get the basic energy estimate, which is quasi-optimal plus a small consistency error term if there are harmonic forms.

Notation: for $w \in V^k$, $\quad E(w) := \inf_{w \in V_h^k} \|w - v\|_V$

Theorem

Assume SC and BCP. Then

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\|_V \leq c[\, E(\sigma) + E(u) + E(p) + \epsilon\,]$$

where

$$\epsilon \leq \inf_{v \in V_h^k} E(P_{\mathfrak{B}} u) \times \sup_{\substack{r \in \mathfrak{H}^k \\ \|r\|=1}} E(r).$$

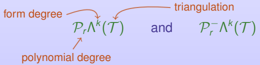
Improved estimates can then be derived using duality techniques.

Finite element differential forms

FE differential forms, the concrete realization of FEEC

How to construct FE subspaces of $H\Lambda^k$ satisfying SP and BCP?

FEEC reveals that *there are precisely two natural families:*



form degree

triangulation

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) \quad \text{and} \quad \mathcal{P}_r^-\Lambda^k(\mathcal{T})$$

polynomial degree

Like all finite element spaces they are constructed from *shape functions* and *degrees of freedom* on each simplex T.

Shape fns for $\mathcal{P}_r\Lambda^k$: polynomial k-forms of degree $\leq r$.

Shape fns for $\mathcal{P}_r^-\Lambda^k$ defined via *Koszul* differential $\kappa : \Lambda^{k+1} \to \Lambda^k$:

$$(\kappa\omega)_x(v_1, \ldots, v_k) = \omega_x(x, v_1, \cdots, v_k)$$

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa\,\mathcal{P}_{r-1}\Lambda^{k+1}(T)$$

Degrees of freedom

DOF for $\mathcal{P}_r\Lambda^k(T)$: to a subsimplex f of dimension d we associate

$$\omega \mapsto \int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f)$$

Theorem. These DOFs are unisolvent and the resulting finite element space satisfies

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\, \omega \in H\Lambda^k(\Omega) \ : \ \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T} \,\}$$

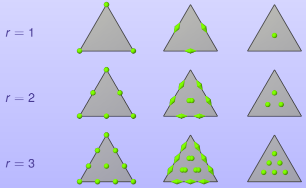
DOF for $\mathcal{P}_r^-\Lambda^k(T)$ (Hiptmair '99):

$$\omega \mapsto \int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f)$$

+ similar theorem. . .

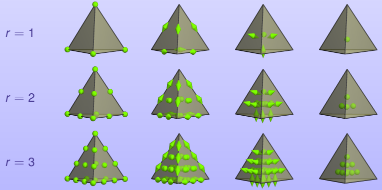
The $\mathcal{P}_r^-\Lambda^k$ family in 2D

$\mathcal{P}_r^-\Lambda^0$	$\mathcal{P}_r^-\Lambda^1$	$\mathcal{P}_r^-\Lambda^2$
Lagrange	Raviart-Thomas	DG



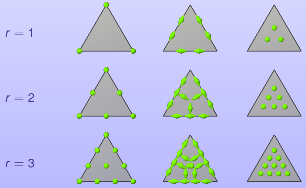
The $\mathcal{P}_r^-\Lambda^k$ family in 3D

$\mathcal{P}_r^-\Lambda^0$	$\mathcal{P}_r^-\Lambda^1$	$\mathcal{P}_r^-\Lambda^2$	$\mathcal{P}_r^-\Lambda^3$
Lagrange	Nédélec edge I	R-T-N face	DG



The $\mathcal{P}_r\Lambda^k$ family in 2D

$\mathcal{P}_r\Lambda^0$	$\mathcal{P}_r\Lambda^1$	$\mathcal{P}_r\Lambda^2$
Lagrange	BDM	DG

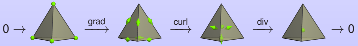


Finite element de Rham subcomplexes

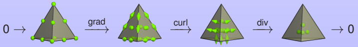
We don't only want spaces, we also want them to fit together into discrete de Rham complexes with BCP.

- One such FEdR subcomplex uses $\mathcal{P}_r^-\Lambda^k$ spaces of constant degree r:

$$0 \to \mathcal{P}_r{}^-\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r{}^-\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r{}^-\Lambda^n(\mathcal{T}) \to 0$$



Whitney '57



- Another uses $\mathcal{P}_r\Lambda^k$ spaces with decreasing degree:

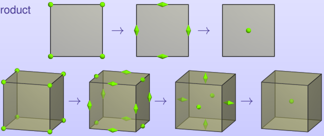
$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}) \to 0$$

Sullivan '75

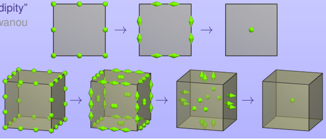
- These are extreme cases. For every r $\exists\ 2^{n-1}$ such FEdR subcomplexes.

Finite element de Rham subcomplexes on cubes

Tensor-product



"Serendipity" (DNA-Awanou '11)



The elasticity complex

Mixed finite elements for elasticity

Find stress $\sigma : \Omega \to \mathbb{S} = \mathbb{R}^{n\times n}_{\text{sym}}$, displacement $u : \Omega \to \mathbb{V} = \mathbb{R}^n$ such that

$$A\sigma = \epsilon(u), \quad \operatorname{div}\sigma = f$$

$$\int_\Omega \left(\frac{1}{2}A\sigma:\sigma + \operatorname{div}\sigma\cdot u + f\cdot u\right)\,dx \xrightarrow[H(\mathrm{div};\mathbb{S})\times L^2(\mathbb{V})]{\sigma,u} \text{saddle point}$$

Search for stable finite elements dates back to the '60s, very limited success.

> "[Mixed finite elements were] achieved by Raviart and Thomas in the case of the heat conduction problem discussed previously. Extension to the full stress problem is difficult and as yet such elements have not been successfully noted."
>
> –Zienkiewicz, Taylor, Zhu
> *The Finite Element Method: Its Basis & Fundamentals*, 6th ed., 2005

With FEEC, this is no longer true. Now, lots of elements.

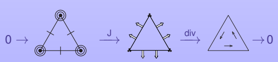
The elasticity complex in 2D

The elasticity system is the *n*-Hodge Laplacian associated to a complex:

$$\begin{array}{ccccccccc} & & \text{Airy potential} & & \text{stress} & & \text{load} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^2(\Omega) & \xrightarrow{J} & H(\mathrm{div},\Omega;\mathbb{S}) & \xrightarrow{\mathrm{div}} & L^2(\Omega;\mathbb{R}^2) & \to & 0 \end{array}$$

$$J\phi = \begin{pmatrix} \partial_y^2\phi & -\partial_{xy}\phi \\ -\partial_{xy}\phi & \partial_x^2\phi \end{pmatrix},$$ the Airy stress function, is second order!

The question is: how to discretize this sequence? This simplest element (DNA-Winther '02) involves 21 stress degrees of freedom. It provides a Hilbert subcomplex with bounded cochain projections.



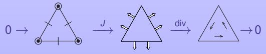
The elasticity complex in 3D

$$\begin{tikzcd}
& \text{displacement} \arrow[d] & \text{strain} \arrow[d] & \text{stress} \arrow[d] & \text{load} \arrow[d] & \\
0 \arrow[r] & H^1(\Omega;\mathbb{R}^3) \arrow[r, "\epsilon"] & H(J,\Omega;\mathbb{S}) \arrow[r, "J"] & H(\mathrm{div},\Omega;\mathbb{S}) \arrow[r, "\mathrm{div}"] & L^2(\Omega;\mathbb{R}^3) \arrow[r] & 0
\end{tikzcd}$$

$J = \mathrm{curl}\, T\, \mathrm{curl}$

The simplest FE subcomplex involves 162 DOFs for stress (DNA-Awanou-Winther '08).

Much simpler *nonconforming* elements can be devised: 12 stress DOF in 2D (DNA-Winter '03), 36 in 3D (DNA-Awanou-Winther '11)



The elasticity complex with weak symmetry

To obtain simpler conforming elements we went back to an old idea, enforcing the symmetry of the stress tensor, skw $\sigma = 0$, via a Lagrange multiplier (Fraeijs de Veubeke '65, Amara-Thomas '79, DNA-Brezzi-Douglas '84).

$$\int_\Omega \left(\frac{1}{2}A\sigma:\sigma + \mathrm{div}\,\sigma\cdot u + f\cdot u\right)\,dx \xrightarrow[H(\mathrm{div};\mathbb{S})\times L^2(\mathbb{V})]{\sigma,u} \text{S.P.}$$

$$\int_\Omega \left(\frac{1}{2}A\sigma:\sigma + \mathrm{div}\,\sigma\cdot u + \sigma:p + f\cdot u\right)\,dx \xrightarrow[H(\mathrm{div};\mathbb{M})\times L^2(\mathbb{V})\times L^2(\mathbb{K})]{\sigma,u,p} \text{S.P.}$$

The associated elasticity complex:

$$\begin{tikzcd}
& \text{displacement} \arrow[d] & \text{rotation} \arrow[d] & & \text{strain} \arrow[d] & \\
0 \arrow[r] & H^1(\Omega;\mathbb{R}^3) \times & L^2(\Omega,\mathbb{K}) \arrow[rr, "(\mathrm{grad},-I)"] & & H(J,\Omega;\mathbb{M}) \arrow[r, "J"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "J"] & H(\mathrm{div},\Omega;\mathbb{M}) \arrow[r, "\binom{\mathrm{div}}{\mathrm{skw}}"] & L^2(\Omega;\mathbb{R}^3) \times & L^2(\Omega;\mathbb{K}) \arrow[r] & 0 \\
& \text{stress} \arrow[u] & \text{load} \arrow[u] & \text{couple} \arrow[u] &
\end{tikzcd}$$

The key to discretizations of the elasticity complex

The elasticity complex can be derived from the de Rham complex through a form of the *BGG resolution*.

BGG can be applied to finite element de Rham subcomplexes, to get finite element subcomplexes of the elasticity complex.

Theorem (AFW '06, 07):

Choose *two* subcomplexes of the de Rham complex satisfying BCP:

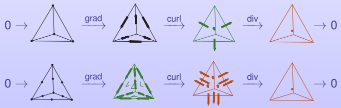
$$0 \longrightarrow V_h^0 \xrightarrow{\mathrm{grad}} V_h^1 \xrightarrow{\mathrm{curl}} V_h^2 \xrightarrow{\mathrm{div}} V_h^3 \longrightarrow 0$$

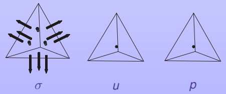
$$0 \longrightarrow \tilde{V}_h^0 \xrightarrow{\mathrm{grad}} \tilde{V}_h^1 \xrightarrow{\mathrm{curl}} \tilde{V}_h^2 \xrightarrow{\mathrm{div}} \tilde{V}_h^3 \longrightarrow 0$$

Suppose that satisfy a *surjectivity hypothesis*. (Roughly, for each DOF of V_h^2 there is a corresponding DOF of $\tilde{V}_h^1$.)

$$\text{Then} \left\{ \begin{array}{ll} \text{stress:} & \tilde{V}_h^2(\mathbb{R}^3) \\ \text{displacement:} & \tilde{V}_h^3(\mathbb{R}^3) \\ \text{rotation:} & V_h^3(\mathbb{K}) \end{array} \right\} \text{is a stable element choice.}$$

The simplest choice





Features of the new mixed elements

- Based on HR formulation with weak symmetry; very natural
- Lowest degree element is very simple: full $\mathcal{P}_1$ for stress, $\mathcal{P}_0$ for displacement and rotation
- Works for every polynomial degree
- Works the same in 2 and 3 (or more) dimensions
- Robust to material constraints like incompressibility
- Provably stable and convergent
- Has been widely implemented. Elastostatics, elastodynamics, viscoelasticity, . . .
- Have opened the way for many other elements: Awanou, Boffi, Brezzi, Cockburn, Demkowicz, Fortin, Gopalakrishnan, Guzman, Qiu, . . .