The Geometry of tensor network states

Outline:

I) The Origin of tensor network States (TNS).

II) A question about TNS & the answer.

III) Connections to the Geometric Complexity Theory (GCT) program & Complexity Theory.

I)

EX: n molecules on a lattice

Total Space $= \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

$$= (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$$

pure state = rank 1 tensor

entangled state = tensor of higher rank.

Bad news: Entanglement does occur and the total space is too big to work with.

Good news: entanglement only occurs "locally" (e.g. between neighbors)

$\rightsquigarrow$ Tensor Network States.

II)

Ex: Γ: 

$\dim V_i = 2 \ (i=1,2)$

$\dim E = 2$

$$TNS(\Gamma, (2), V_1 \otimes V_2) := \{ T \in V_1 \otimes V_2 \mid T = \mathrm{Con}(T_1 \otimes T_2), \ T_1 \in V_1 \otimes E^*, \ T_2 \in V_2 \otimes E \}$$

Rmk:

$$TNS(\Gamma, (2), V_1 \otimes V_2) = \hat{\sigma}_2(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2))$$

Ex: Γ: $\begin{tikzcd} V_1 \arrow[r, dash, "E_1"'] & V_2 \arrow[r, dash, "E_2"'] & V_3 \end{tikzcd}$

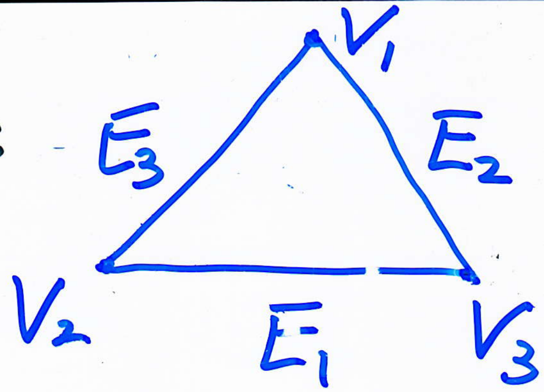
$\dim V_1 = \dim V_3 = 2$
$\dim V_2 = 4$
$\dim E_i = 2 \quad (i=1,2)$

$$\begin{aligned}
&TNS(\Gamma, (2,2), V_1 \otimes V_2 \otimes V_3) \\
&= \{ T \in V_1 \otimes V_2 \otimes V_3 \mid T = \mathrm{Con}(T_1 \otimes T_2 \otimes T_3),\ T_1 \in V_1 \otimes E_1^*,\ T_2 \in V_2 \otimes E_1 \otimes E_2^*, \\
&\quad T_3 \in V_3 \otimes E_2 \}
\end{aligned}$$

Rmk:

$$\begin{aligned}
&TNS(\Gamma, (2,2), V_1 \otimes V_2 \otimes V_3) \\
&= \hat{\sigma}_2(Seg(\mathbb{P}(V_1 \otimes V_2) \times \mathbb{P}V_3)) \\
&\quad \cap \hat{\sigma}_2(Seg(\mathbb{P}V_1 \times \mathbb{P}(V_2 \otimes V_3))).
\end{aligned}$$

Ex: Γ:



$\dim V_i = 4$
$\dim E_i = 2$
$i = 1, 2, 3$

$TNS(\Gamma, (2,2,2), V_1 \otimes V_2 \otimes V_3)$

$$:= \left\{ T \in V_1 \otimes V_2 \otimes V_3 \;\middle|\; \begin{array}{l} T = \mathrm{Con}(T_1 \otimes T_2 \otimes T_3) \\ T_1 \in V_1 \otimes E_2^* \otimes E_3, \\ T_2 \in V_2 \otimes E_3^* \otimes E_1, \\ T_3 \in V_3 \otimes E_1^* \otimes E_2 \end{array} \right\}$$

Question of L. Grasedyck:

Is TNS always Zariski closed?

Answer: Yes, if Γ is a tree

No, if Γ is a triangle

III)

$$\Gamma = \Delta$$

$$\overline{TNS(\Gamma, (2,2,2), V_1 \otimes V_2 \otimes V_3)}$$

$$= \overline{GL(V_1) \times GL(V_2) \times GL(V_3) . M}$$

where $M = Id_{E_3} \otimes Id_{E_2} \otimes Id_{E_1}$

$$\in V_1 \otimes V_2 \otimes V_3$$

$$\cong (E_2^* \otimes E_3) \otimes (E_3^* \otimes E_1) \otimes (E_1^* \otimes E_2)$$

Rmk: M is the tensor of matrix multilplication!!

In GCT program, we want to study the geometry of

$$\overline{GL_{n^2} \cdot [\det_n]} \subseteq \mathbb{P} S^n \mathbb{C}^{n^2}$$

and $$\overline{GL_{n^2} \cdot [\ell^{n-m} \cdot \mathrm{perm}_m]} \subseteq \mathbb{P} S^n \mathbb{C}^{n^2}$$

ℓ: a new variable

and we choose an embedding

$$\mathbb{C} \oplus \mathbb{C}^{m^2} \hookrightarrow \mathbb{C}^{n^2}.$$