Decomposing Tensor Networks

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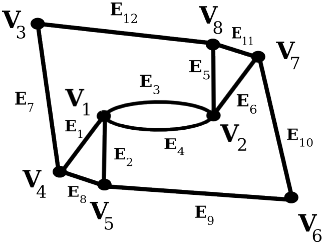
What is a Tensor Network?

A **closed tensor network** can be described as combinatorial object, $(G, \{T_j\}, \{W_i\})$ where

- $G = (V, E) = (\{V_j\}, \{E_i\})$ is a graph.
- For each edge E_i, there is an associated vector space W_i.
- For each vertex V_j, there is an associated tensor

$$T_j \in \bigotimes_{i \in \mathcal{N}(V_j)} W_i$$

where $\mathcal{N}(V_j) = \{i : E_i \text{ is incident to } V_j\}$.



Tensor Contraction

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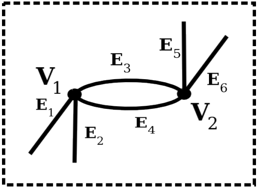
Example



Let

$$T_1 = w_1 \otimes w_2 \otimes w_3 \otimes w_4 \in W_1 \otimes W_2 \otimes W_3 \otimes W_4$$

and

$$T_2 = u_3 \otimes u_4 \otimes u_5 \otimes u_6 \in W_3 \otimes W_4 \otimes W_5 \otimes W_6$$

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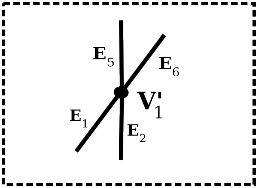

$$T_1' = w_3^*(u_3)w_4^*(u_4) \cdot (w_1 \otimes w_2 \otimes u_4 \otimes u_5)$$

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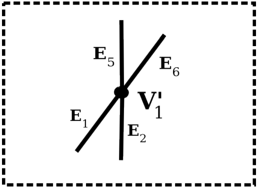

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So this implies that every closed tensor network can be contracted to a scalar.

Why Study These Things?

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- Tensor Networks show up in Physics as they can approximate the ground states of Hamiltonians.
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Complexity Theory

- Finding the scalar a closed tensor network represents is a #P-hard problem.
- Certain families of tensor networks are efficiently computable. Decompositions give a way of deciding if certain tensor networks lie in these families.

Tensor Networks as Polynomials

From here on out, the vector space associated to all edges will be $\mathbb{C}^2$.

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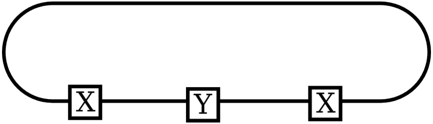

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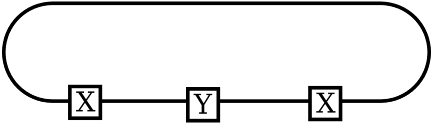

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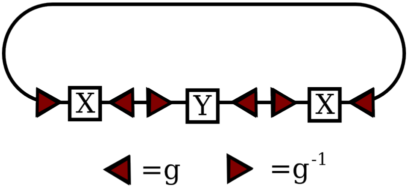

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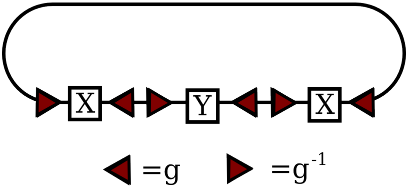

$$= \text{Tr}((g^{-1}Xg)(g^{-1}Yg)(g^{-1}Xg))$$

$$= \text{Tr}(XYX) \in \mathbb{C}[\text{End}(\mathbb{C}^2)^{\oplus 2}]^{GL_2}$$

Generators of $\mathbb{C}[\mathrm{End}(\mathbb{C}^2)^{\oplus 2}]^{GL_2}$

Theorem (?,Classical)

$\mathbb{C}[\mathrm{End}(\mathbb{C}^2)^{\oplus 2}]^{GL_2}$ *is minimally generated by the polynomials* $\mathrm{Tr}(X)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(X^2)$, $\mathrm{Tr}(Y^2)$, *and* $\mathrm{Tr}(XY)$.

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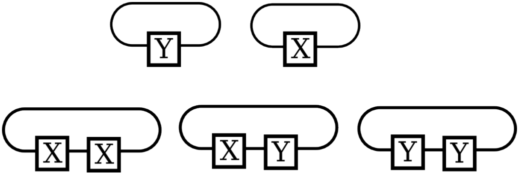
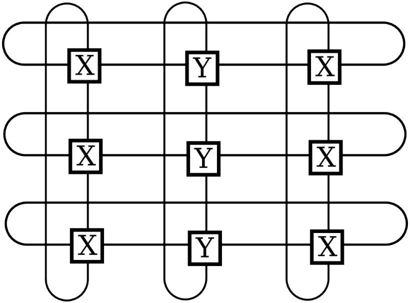
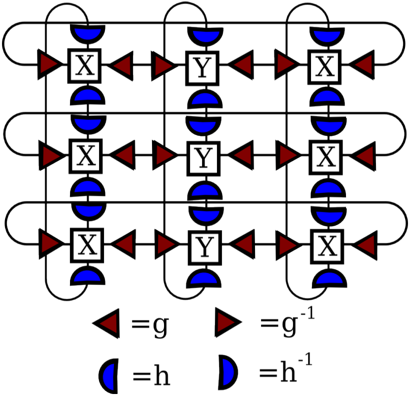


$$X, Y \in \mathrm{End}(\mathbb{C}^4)$$



$$\in \mathbb{C}[\mathrm{End}(\mathbb{C}^4)^{\oplus 2}]$$

Lattice Tensor Networks



$$\in \mathbb{C}[\mathrm{End}(\mathbb{C}^2)^{\oplus 4}]^{GL_2 \times GL_2}$$

Local Conjugation

Suppose that $V = \bigotimes_{i=1}^n V_i$. Let $\mathrm{End}(V) \cong \bigotimes_{i=1}^n \mathrm{End}(V_i)$ be acted on by $G = GL_{t_1} \times \cdots \times GL_{t_n}$ in the following way:

$$(g_1, \ldots, g_n).\left(\sum_{i=1}^{n} A_{1i} \otimes \cdots \otimes A_{ni}\right)$$

$$= \sum_{i=1}^{n} g_1 A_{1i} g_1^{-1} \otimes \cdots \otimes g_n A_{ni} g_n^{-1}$$

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What are the generators of $\mathbb{C}[\text{End}(V)^{\oplus m}]^G$?

The Functions Tr_σ^M

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Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{S}_M^n$. Let $(m_{r_1} \cdots m_{r_k})(m_{s_1} \cdots m_{s_l}) \cdots$ be a disjoint cycle decomposition for σ_i.

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We define the following functions on simple tensors of $\mathrm{End}(V)^{\oplus m}$ and extend multilinearly:

$$T_{\sigma_i}^M(\bigotimes_j A_{j1}, \ldots, \bigotimes_j A_{jm}) = \mathrm{Tr}(A_{im_{r_1}} \cdots A_{im_{r_k}})\mathrm{Tr}(A_{im_{s_1}} \cdots A_{im_{s_l}}) \cdots$$

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Then define

$$\mathrm{Tr}_\sigma^M = \prod_{i=1}^d T_{\sigma_i}^M$$

Example

Let $V = V_1 \otimes V_2$ and $M = \{1, 2, 1\} = \{m_1, m_2, m_3\}$ and $\sigma = ((m_1 m_2)(m_3), (m_1)(m_2 m_3)) \in \mathcal{S}_M^2$.

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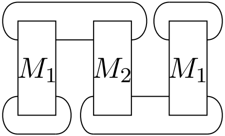

The Generators of $\mathbb{C}[\mathrm{End}(V)^{\oplus m}]^G$

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Unfortunately, this is not useful as it gives us an infinite number of generators.

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Theorem (T.)

Let r be the generic rank of $\mathrm{End}(V)$ as a $4 \times \cdots \times 4 = 4^n$ tensor. Let $\mathfrak{m}$ be the smallest number greater than or equal to $2\lfloor \frac{rm}{2} \rfloor + 1$ that is divisible by 6. Then $K[\mathrm{End}(V)^{\oplus m}]^G$ is generated by Tr_σ^M of degree at most $\mathfrak{m}(\binom{rm}{3}/3 + rm) + 2\lfloor \frac{rm}{2} \rfloor + 1$, $\sigma \in \mathcal{S}_M^{n,\leq 3}$.

THE END

THAT'S ALL FOLKS!