Direct Sum Decomposability of Polynomials

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Outline

Direct sum decompositions

Can $F = F(x_1, \dots, x_n)$ be written as a sum of polynomials in separate variables? We allow a linear change of coordinates:

$F = G(t_1, \dots, t_k) + H(t_{k+1}, \dots, t_n)$ where the t_i are linearly independent linear forms.

Example

$xy \neq G(x) + H(y)$, but $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$.

Let $\det_n = \det((x_{ij})_{1 \leq i,j \leq n})$, the $n \times n$ generic determinant.

Problem

$\det_2 = ad - bc$ *is decomposable. Is* $\det_n$ *decomposable for* $n > 2$*?*

Apolarity

- $S = \mathbb{C}[x_1, \ldots, x_n]$, $T = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$ dual ring: α_i acts as $\partial/\partial x_i$.
- For $F \in S$, $F^\perp \subset T$ ideal.

Example

$\det_n^\perp$ is generated by:

- $\alpha_{i,j}^2$
- $\alpha_{i,j_1}\alpha_{i,j_2}$ (two entries from same row)
- $\alpha_{i_1,j}\alpha_{i_2,j}$ (two entries from same column)
- permanents of 2×2 submatrices:
  $\operatorname{per}\begin{pmatrix} \alpha_{i,j} & \alpha_{i,l} \\ \alpha_{k,j} & \alpha_{k,l} \end{pmatrix} = \alpha_{i,j}\alpha_{k,l} + \alpha_{i,l}\alpha_{k,j}$.

[Shafiei, 2012]

Ranestad-Schreyer bound for Waring rank

Waring rank: $r(F)$ = least r such that $F = \ell_1^d + \cdots + \ell_r^d$.

Example

$\det_n$ = sum of $n!$ terms of the form $x_1 \cdots x_n$. Each term has rank $r(x_1 \cdots x_n) = 2^{n-1}$. So $r(\det_n) \leq 2^{n-1} n!$.

Lower bounds for rank:

- Sylvester: $r(F) \geq \max\{\dim(T/F^\perp)_a\}$
- Ranestad-Schreyer: $r(F) \geq \frac{1}{\delta} \sum_a \dim(T/F^\perp)_a$, where δ = maximum degree of a generator of $F^\perp$.

Example

Sylvester's bound gives $r(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2$ $(r(\det_3) \geq 9)$.

The Ranestad-Schreyer bound gives $r(\det_n) \geq \frac{1}{2}\binom{2n}{n}$ $(r(\det_3) \geq 10)$.

Apolar generating degree

What is the generating degree of $F^\perp$?

Say $\deg F = d$. Then $F^\perp$ contains all differential operators of degree $> d$, so no generators of degree $d+2$ or higher (but maybe $d+1$).

Problem

Give conditions for $F^\perp$ to have high-degree or low-degree generators.

Theorem (Casnati–Notari)

$F^\perp$ has a minimal generator of degree $d+1$ if and only if $r(F) = 1$, $F = x^d$.

Theorem

If F is decomposable as a direct sum then $F^\perp$ has a minimal generator of degree d.

Corollary

For $n > 2$, $\det_n$ is not decomposable.

Proof.

Say $F = G(X) - H(Y)$ is a direct sum decomposition.

- For $0 \le a < d$, $F_a^\perp = G_a^\perp \cap H_a^\perp$: If $DG = DH = 0 \in \mathbb{C}[X] \cap \mathbb{C}[Y]$ then $\deg DG = \deg DH = 0$ so $\deg D = d$.
- Let $\Delta^X(G) = 1$, $\Delta^Y(H) = 1$, $\Delta = \Delta^X + \Delta^Y$. Then $F_d^\perp = (G^\perp \cap H^\perp)_d + \langle \Delta \rangle$.

So $F^\perp = (G^\perp \cap H^\perp) + \Delta$ and Δ is a minimal generator. $\square$

Converse

The converse fails.

Example

$F = xy^{d-1}$ has $F^\perp = \langle \alpha^2, \beta^d \rangle$.
But F is indecomposable because $\ell_1^d - \ell_2^d$ has distinct factors.

However xy^{d-1} is a limit of direct sums:

$$xy^{d-1} = \lim_{t \to 0} \frac{1}{dt} \left((y + tx)^d - y^d \right).$$

Theorem

If $F^\perp$ has a minimal generator of degree d then F is a limit of direct sums.

Converse again

Theorem

If $F^\perp$ has a minimal generator of degree d then F is a limit of direct sums.

Once again the converse fails!

Example

- $x^d - ty^d \to x^d$, but $(x^d)^\perp = \langle \alpha^{d+1}, \beta \rangle$.
- $xyz - tw^3 \to xyz$, but $(xyz)^\perp = \langle \alpha^2, \beta^2, \gamma^2, \delta \rangle$.

Theorem

Let $n \geq 2$, $d \geq 3$. If F is a limit of direct sums and F cannot be written using fewer variables, then $F^\perp$ has a minimal generator of degree d.

Chart of inclusions

$$
\begin{array}{ccccc}
\mathrm{DirSum} & \subsetneq & \mathrm{ApoMax} & \subsetneq & \overline{\mathrm{DirSum}} \\
\cup & & \cup & & \cup \\
\mathrm{DirSum} \cap \mathrm{Con} & \subsetneq & \mathrm{ApoMax} \cap \mathrm{Con} & = & \overline{\mathrm{DirSum}} \cap \mathrm{Con}
\end{array}
$$

DirSum: decomposable as a direct sum

ApoMax: $F^{\perp}$ has a minimal generator of degree d

Con: concise, i.e., cannot be written using fewer variables

An idea that doesn't work

Suppose $F_t \to F_0 = F$ and the F_t are direct sums for $t \neq 0$.
Does semicontinuity of graded Betti numbers show that $F^\perp$ has a generator of degree d?

No, because $F^\perp$ is not necessarily the flat limit of the $F_t^\perp$.

Example

- $(x^d - ty^d)^\perp = \langle \alpha\beta, t\alpha^d + \beta^d \rangle$
- $(x^d)^\perp = \langle \alpha^{d+1}, \beta \rangle$

So $(x^d - ty^d)^\perp \not\to (x^d)^\perp$.

Well, it works sometimes

Theorem

- *For* $d = 3$*, if* $F_t \to F$ *and* F *is concise then* $F_t^\perp \to F^\perp$ *is always flat.*
- *For* $n = 3$*, if* $F^\perp$ *has a minimal generator of degree* d *then there exists some family* $F_t \to F$*,* F_t *direct sums, such that* $F_t^\perp \to F^\perp$ *is flat.*

But

- *There exists* F *such that* $F^\perp$ *has a minimal generator of degree* d*, so* F *is a limit of direct sums; but for every family of direct sums* $F_t \to F$*,* $F_t^\perp \to F^\perp$ *is not flat.*

The last item is a consequence of the existence of non-smoothable Gorenstein schemes.

This forces trickier proofs for the previous theorems.

An idea that does work

Theorem

If $F^\perp$ has a minimal generator of degree d then F is a limit of direct sums.

Proof.

Suppose $F^\perp$ has a minimal generator of degree d.

- By Gorenstein duality, $F^\perp$ has a high degree syzygy.
- By Koszul homology, $F^\perp$ contains quadratic generators: the 2×2 minors of a matrix L of linear forms.
- Jordan normal form of L either gives
  - a direct sum decomposition of F,
  - or (if L is nilpotent) a limit of direct sums.

□

Theorem

Let $n \geq 2$, $d \geq 3$. If F is a limit of direct sums and F cannot be written using fewer variables, then $F^\perp$ has a minimal generator of degree d.

Proof.

Suppose F is a concise limit of direct sums, $F_t \to F$.

- Let $J = \lim F_t^\perp$. $J \subseteq F^\perp$.
- Each $F_t^\perp$ has a minimal generator of degree d
- By Gorenstein duality $\beta_{n-1,n}(F_t^\perp) > 0$
- By semicontinuity of graded Betti numbers, $\beta_{n-1,n}(J) > 0$
- This syzygy lies in the minimal-degree strand by conciseness (no linear generators).
- So $F^\perp$ has the same high degree syzygy
- Hence $F^\perp$ also has a minimal generator of degree d.

□

Low-degree generators

Theorem

If F is a homogeneous form of degree d in n variables and δ is the generating degree of $F^\perp$ then $d \leq (\delta - 1)n$.

If $F^\perp$ is generated by quadrics then $d \leq n$.

Question

What are the forms F such that $d = n$ and $F^\perp$ is generated by quadrics?

$F = x_1 \cdots x_n$ has $F^\perp = \langle \alpha_1^2, \dots, \alpha_n^2 \rangle$.