Ranks and generalized ranks

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- $\phi \in S^d W = \mathbb{C}[x_0, \dots, x_n]_d$ homogeneous polynomial of degree d
- $\phi = c_1 \ell_1^d + \cdots + c_r \ell_r^d$ power sum decomposition
- (Waring) rank $R(\phi)$ = least r in a power sum decomposition

Problem: Find $R(\phi)$.

$$xy = \frac{1}{4}\Big((x+y)^2 - (x-y)^2\Big)$$

$$R(xy) = 2$$

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Problem: Find $R(\phi)$.

$$xyz = \frac{1}{24}\Big((x+y+z)^3 - (x+y-z)^3 - (x-y+z)^3 + (x-y-z)^3\Big),$$

$$R(xyz) \leq 4$$

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Problem: Find $R(\phi)$.

$$x_1 \cdots x_n = \frac{1}{2^{n-1} n!} \sum \pm (x_1 \pm x_2 \pm \cdots \pm x_n)^n$$

$$R(x_1 \cdots x_n) \leq 2^{n-1}$$

$R(\phi) =$ least r such that $\phi = c_1\ell_1^d + \cdots + c_r\ell_r^d$

Ranks are known:

- $\deg \phi = 2$ ($R(\phi) =$ rank of symmetric matrix)
- $n = 1$ (Sylvester, 1851)
- $\phi \in S^d W$ general (Alexander–Hirschowitz, 1995)
- Plane cubic curves

Some natural questions:

- $R(x_1 \cdots x_n) = 2^{n-1}$?
- Monomials: $R(x_1^{b_1} \cdots x_n^{b_n}) \leq (b_2 + 1) \cdots (b_n + 1)$. Equality?
- What is $R(\det_n)$ ($n \times n$ determinant of $(x_{i,j})_{1 \leq i,j \leq n}$)?
  $14 \leq R(\det_3) \leq 24$ (Landsberg-T.)
  (improved from $9 \leq R(\det_3) \leq 24$, Sylvester's catalecticant bound)

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  (improved from $9 \leq R(\det_3) \leq 24$, Sylvester's catalecticant bound)
  Still wide open.

$\phi \in S^d W$, $0 \leq a \leq d$: a'th **catalecticant**

$$\begin{aligned} \phi_a : S^{d-a} W^* &\to S^a W, \\ D &\mapsto D(\phi) \end{aligned}$$

Sylvester: $R(\phi) \geq \operatorname{rank} \phi_a$

$4 \geq R(xyz) \geq \operatorname{rank}(xyz)_1 = 3.$

$\phi_a : S^{d-a} W^* \to S^a W$

Theorem (Landsberg-T.)

Let $\phi \in S^d W$, $0 \leq a \leq d$.
Assume $\phi \in S^d V$, $V \subseteq W \implies V = W$
(equiv.: $V(\phi)$ *not a cone;* ϕ_1 *surjective).*
Let $\Sigma_a = \{p : \mathrm{mult}_p(\phi) > a\} = V$*(a'th derivatives of* ϕ*). Then*

$$R(\phi) \geq \operatorname{rank} \phi_a + \dim \Sigma_a + 1.$$

- $4 \geq R(xyz) \geq \operatorname{rank}(xyz)_1 + \dim \operatorname{Sing} V(xyz) + 1 = 3 + 0 + 1 = 4.$
- $24 \geq R(\det_3) \geq 14$ (thm above) ≥ 9 (catalecticant)
- $8 \geq R(xyzw) \geq 8$ (ad hoc) ≥ 7 (thm) ≥ 6 (catalecticant)

Proof: Count dimensions of subvarieties in $\mathbb{P} \ker \phi_a$.

Generalize to multihomogeneous polynomials:

- Homogeneous polynomial: rank with respect to Veronese
- Multihomogeneous $\rightsquigarrow$ Segre-Veronese
- Still have catalecticant bound ("symmetric flattening")
- Try to add dimension of some set of singularities

Theorem

Let $M \in \bigotimes S^{d_i} W_i$, $\underline{a} = (a_1, \dots, a_k)$.
$M_{\underline{a}} : \bigotimes S^{d_i - a_i} W_i^* \to \bigotimes S^{a_i} W_i$. *Then* $R(\phi) \geq \operatorname{rank} M_{\underline{a}}$ *(well-known).*
Suppose $M_{\sqrt{\underline{a}}}$ *is surjective,* $\sqrt{\underline{a}}_i = 1$ *if* $a_i \neq 0$, 0 *otherwise.*
Let $\Sigma_{\underline{a}} = \{p : \operatorname{mult}_p(M) > \underline{a}\} = V(\underline{a}$*'th derivatives of* $M)$.
Then

$$R(M) \geq \operatorname{rank} M_{\underline{a}} + \dim \Sigma_{\underline{a}} + 1.$$

- Condition $M_{\sqrt{\underline{a}}}$ surjective analogous to condition ϕ_1 surjective
- Geometric version of this condition?
- $\Sigma_a = V(\text{image of } \phi_a \text{ or } M_{\underline{a}})$

Work in progress: "Young flattenings" (Landsberg–Ottaviani, 2010)
L very ample line bundle on X, $v \in \mathbb{P}H^0(L)^* \hookleftarrow X$; find $R(v) = R_X(v)$

E any vector bundle, $E \otimes E^* \otimes L \longrightarrow L$

$H^0(E) \otimes H^0(L \otimes E^*) \longrightarrow H^0(L) \xrightarrow{v} \mathbb{C}$

$C_v^E : H^0(L \otimes E^*) \longrightarrow H^0(E)^*$, "generalized catalecticant"
Previous results: $L = \mathcal{O}(d)$, $E = \mathcal{O}(a)$; similar for multihomogeneous.

[L-O] $R(v) \geq \operatorname{rank}(C_v^E)/\operatorname{rank} E$
Question: $\ldots + \dim \Sigma + 1$ for some appropriate Σ?

Seems to work if $\exists G$ line bundle, $L \otimes E^* = G^s$
(assume C_v^G surjective, let $\Sigma = V(\text{image of } C_v^G)$)

Thank you!