Stochastic models, tensor rank, and inequalities

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Stochastic model

Three observable random variables, conditionally independent, given state of a fourth unobservable (hidden) variable

All variables have discrete state spaces, with n states

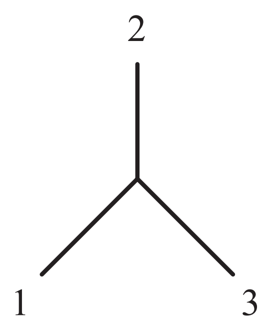
$\Rightarrow$ Probability distribution is given by an

$$n \times n \times n \text{ tensor of (real, non-negative) rank } n.$$

Motivating application:

• Random variables represent a particular site in the DNA sequence for a particular gene for 4 organisms

• $n = 4$, States are A, C, T, G



Observed variable are from 3 extant organisms.

Hidden variable is from their common ancestor

Problem (v1): What $n \times n \times n$ tensors are distributions from this model?

Problem (v2): What semi-algebraic conditions ensure a $n \times n \times n$ tensor is of complex rank n?, real rank n?, non-negative rank n?, etc.?

For $n = 2$, the key to solution is the Cayley hyperdeterminant :

$$\begin{aligned}\Delta(P) = (&p_{000}^2 p_{111}^2 + p_{001}^2 p_{110}^2 + p_{010}^2 p_{101}^2 + p_{011}^2 p_{100}^2) \\ &- 2(p_{000}p_{001}p_{110}p_{111} + p_{000}p_{010}p_{101}p_{111} + p_{000}p_{011}p_{100}p_{111} \\ &+ p_{001}p_{010}p_{101}p_{110} + p_{001}p_{011}p_{110}p_{100} + p_{010}p_{011}p_{101}p_{100}) \\ &+ 4(p_{000}p_{011}p_{101}p_{110} + p_{001}p_{010}p_{100}p_{111}).\end{aligned}$$

$\Delta(P) \neq 0 \Rightarrow$ complex rank 2

P real and $\Delta(P) > 0 \Rightarrow$ real rank 2

P real and $\Delta(P) < 0 \Rightarrow$ real rank 3

$\Delta(P) = 0$ requires a more detailed analysis.

For larger n, what is/are the analog(s) of Δ for this purpose?

Action on $n \times n \times n$ tensors by

$$G = GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$$

Let $D = \mathrm{Diag}(1, 1, 1, \ldots 1)$, and $\mathcal{D}$ denote its G-orbit.

$$\mathcal{D} \subsetneq \ \mathbb{C}\text{-rank } n \text{ tensors } \ \subsetneq \overline{\mathcal{D}} = V$$

where V is the variety of tensors of border rank at most n.

($\mathcal{D}$ is the set of tensors of rank n and multilinear rank (n, n, n).)

For $n = 2$, Δ is an invariant of G:

$$\Delta(P(g_1, g_2, g_3)) = \det(g_1)^2 \det(g_2)^2 \det(g_3)^2 \Delta(P)$$

This shows $\Delta(P) \neq 0$ for $P \in \mathcal{D}$, since $\Delta(\mathrm{Diag}(1,1)) \neq 0$.

Then check $\Delta(P) = 0$ for $P \notin \mathcal{D}$, by determining canonical forms for such tensors, then evaluating the explicit formula for Δ.

For a similar result for $n > 2$ we need

1) a G-invariant space of polynomials with *explicit* formulas,

2) some understanding of canonical forms.

Construction of invariants:

For $n = 2$, $\mathbf{x} = (x_1, x_2)$ indeterminates

$$\Delta(P) = \det(H_{\mathbf{x}}(\det(P *_3 \mathbf{x}))) \qquad \text{(Schläfli)}$$

This gives transformation formula under G, as well as a good way to evaluate at canonical forms.

For arbitrary n, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ indeterminates, let

$$f(P; \mathbf{x}) = \det(H_{\mathbf{x}}(\det(P *_3 \mathbf{x})))$$

a polynomial in entries of P (degree n^2)
and indeterminates $\mathbf{x}$ (degree $n(n-2)$)

($\mathbf{x}$-coefficents are basis of invariant space)

$$f(P(g_1, g_2, g_3)) = \det(g_1)^n \det(g_2)^n \det(g_3)^2 f(P, g_3\mathbf{x})$$

Canonical forms:

Defining polynomials for V are not all known for $n \geq 4$, but some are.

Commutation relations: If $P \in V$, then for any $\mathbf{v}$,

$$(P *_3 \mathbf{e}_i) adj(P *_3 \mathbf{v})(P *_3 \mathbf{e}_j) - (P *_3 \mathbf{e}_j) adj(P *_3 \mathbf{v})(P *_3 \mathbf{e}_i) = 0$$

(the slices of P, adjusted appropriately, must commute)

Proposition: if P satisfies the commutation relations and (– extra condition–), then the G-orbit of P contains a tensor with upper triangular slices.

– If $P \in \mathcal{D}$, such an orbit rep is $\mathrm{Diag}(1, 1, \ldots 1)$.

– Otherwise, there is an orbit rep with one slice **strictly** upper triangular.

Triangular slices, with one strictly triangular, make f easy to evaluate:

$$f(P;\mathbf{x}) = \det(H_{\mathbf{x}}(\det(P *_3 \mathbf{x})))$$

Corollary (Semialgebraic description of $\mathcal{D}$):

$$\left.\begin{array}{c}P \text{ satisfies the commutation relations} \\ \text{and } f(P,\mathbf{x}) \neq 0\end{array}\right\} \Leftrightarrow P \in \mathcal{D}.$$

Applications:

1) Can give a complete semialgebraic description of the statistical model at beginning of talk (slightly restricted) and the only equalities needed are the commutation relations (Allman,Taylor,R)

(Cf. The Salmon Problem)

2) The zero set of f divides $V(\mathbf{R})$ into connected components with tensor decompositions having a constant number of complex-conjugate pairs of rank 1 summands.

3) The canonical form suggests considering tensors of the form

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right],$$

with upper-triangular commuting slices.

Can show the $n \times n \times n$ version has border rank n, but rank $2n - 1$.

Q: Are there $n \times n \times n$ tensors of border rank n with *larger* rank?